Model Building in Almost-Commutative Geometry

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NCG and Particle Physics
Leiden 2013

Overview

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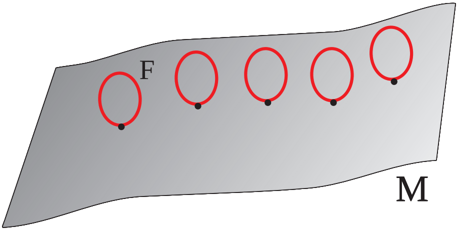
Analogy: Almost-comm. geometry $\leftrightarrow$ Kaluza-Klein space

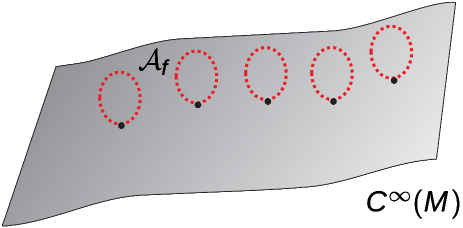


Idea:

$M \longrightarrow C^\infty(M)$, $F \longrightarrow$ some "finite space",

differential geometry $\longrightarrow$ spectral triple

Almost-commutative Geometry



Replacing manifolds by algebras

extra dimension: $F \to \mathcal{A}_f = M_1(\mathbb{K}) \oplus M_2(\mathbb{K}) \oplus \dots$

Kaluza-Klein space: $M \times F \to \mathcal{A} = C^\infty(M) \otimes \mathcal{A}_f$

The almost-commutative standard model automatically produces:

- The combined Einstein-Hilbert and standard model action
- A cosmological constant
- The Higgs boson with the correct quartic Higgs potential

The Dirac operator plays a multiple role:

$$\boxed{D = \partial\!\!\!/ \otimes 1_f + A\!\!\!/ + \gamma_5 \otimes \Phi}$$

Higgs & Gauge Bosons

Particle Dynamics, Ferm. Mass Matrix

Metric of M, Internal Metric

An even, real spectral triple $(\mathcal{A}, \mathcal{H}, D)$

The ingredients (A. Connes):

- A real, associative, unital pre-C^*-algebra $\mathcal{A}$
- A Hilbert space $\mathcal{H}$ on which the algebra $\mathcal{A}$ is faithfully represented via a representation ρ
- A self-adjoint operator D with compact resolvent, the Dirac operator
- An anti-unitary operator J on $\mathcal{H}$, the real structure or charge conjugation
- A unitary operator γ on $\mathcal{H}$, the chirality or volume element

The axioms of noncommutative geometry (A. Connes):

Axiom 1: Classical Dimension n (we assume n even)

Axiom 2: Regularity

Axiom 3: Finiteness

Axiom 4: First Order of the Dirac Operator

Axiom 5: Reality

Axiom 6: Orientability

Axiom 7: Poincaré Duality

The Spectral Action (A. Connes, A. Chamseddine 1996)

$$(\Psi, \mathcal{D}\Psi) + S_{\mathcal{D}}(\Lambda) \quad \text{with } \Psi \in \mathcal{H}$$

- $(\Psi, \mathcal{D}\Psi)$ = fermionic action
  includes Yukawa couplings
  & fermion–gauge boson interactions
  + constraints at Λ
- $S_{\mathcal{D}}(\Lambda)$ = $\sharp$ eigenvalues of $\mathcal{D}$ up to cut-off $\pm\Lambda$
  = Einstein-Hilbert action + Cosm. Const.
  + full bosonic SM action
  + constraints at Λ
- constraints => less free parameters than classical SM

Overview

The standard model (A. Chamseddine, A. Connes 1996):

- Discrete space: $\mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})(\oplus\mathbb{C})$
- Symmetries of discrete space: $SU(2) \times U(3)$
- Hilbert space of minimal standard model fermion multiplets
- Dirac operator: ordinary Dirac op. + fermionic mass matrix (CKM/PMNS matrix)
- Majorana masses and SeeSaw mechanism for right-handed neutrinos (J. Barrett & A. Connes 2006)

Constraints on the SM parameters at the cut-off Λ:

$$\frac{5}{3}g_1(\Lambda)^2 = g_2(\Lambda)^2 = g_3(\Lambda)^2$$

$$g_2(\Lambda)^2 = \frac{Y_2(\Lambda)^2}{H(\Lambda)}\frac{\lambda(\Lambda)}{24} = \frac{1}{4}Y_2(\Lambda)$$

- g_1, g_2, g_3: $U(1)_Y$, $SU(2)_w$, $SU(3)_c$ gauge couplings
- λ: quartic SMS coupling
- Y_2: trace of the Yukawa matrix squared
- H: trace of the Yukawa matrix to the fourth power
- 3 standard model generations

Consequences from the SM constraints:

Input:

- Big Desert
- $g_1(m_Z) = 0.3575, \ g_2(m_Z) = 0.6514, \ g_3(m_Z) = 1.221$
- renormalisation group equations
- $(m_{top} = 171.2 \pm 2.1 \text{ GeV})$

Output:

- $g_2^2(\Lambda) = g_3^2(\Lambda)$ at $\Lambda = 1.1 \times 10^{17}$ GeV
- $m_{top} < 190$ GeV
- no 4^{th} SM generation

Excluded by Tevatron & LHC since:

- $m_{SMS} \neq 168.3 \pm 2.5$ GeV
- $\frac{5}{3} g_1(\Lambda)^2 \neq g_2(\Lambda)^2$

Overview

A Classification of the internal spaces

classify $\mathcal{A}_f = M_1(\mathbb{K}) \oplus M_2(\mathbb{K}) \oplus \ldots$

- with respect to the number of summands in the algebra
- with respect to physical criteria

Physicist's "shopping list" (B. Iochum, T. Schücker, C.S. 2003)

The physical models are required to

- be irreducible i.e. to have the smallest possible internal Hilbert space (minimal approach)
- have a non-degenerate Fermionic mass spectrum
- be free of harmful anomalies
- have unbroken colour groups
- possess no uncharged massless Fermions

Classification Results (B.Iochum, J.-H. Jureit, T.Schücker, C.S. 2003-2008):

# sum. in $\mathcal{A}_f$	KO 0	KO 6
1	no model	no model
2	no model	no model
3	SM[2]	no model
4	SM[2], el.-str.[1]	SM[2], el.-str.[1]
6		SM[2] + el.-str.[1], $2 \times$ el.-str.[1]

[1] Electro-Strong Model: "electron+proton", no Higgs,

$$\mathcal{A}_f = \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \oplus M_C(\mathbb{C}),$$
$$G_{gauge} = U(1) \times SU(C)/SO(C)/Sp(C)$$

[2] first family, colour group = $SU(C)/SO(C)/Sp(C)$

Going beyond the Standard Model

Idea: Use models from classification as basic building blocks

The geometric setup imposes constraints:

- mathematical axioms
  $\longrightarrow$ Restrictions on particle content
- symmetries of finite space
  $\longrightarrow$ determines gauge group
- representation of matrix algebra
  $\longrightarrow$ representation of non-abelian gauge sub-group
- Dirac operator $\longrightarrow$ allowed mass terms / scalar fields

General requirements for Particle Models:

- "minimality" requirement for internal space
  $\longrightarrow$ Standard Model is a sub-model
- no harmful (Yang-Mills) anomalies
  $\longrightarrow$ representation of abelian gauge sub-group
- Dirac-Yang-Mills-Scalar action from Spectral Action
  $\longrightarrow$ high energy effective action + constraints
- low energy physics by renormalisation group flow

Experimental requirements at m_Z:

- $g_1(m_Z) = 0.3575, \; g_2(m_Z) = 0.6514, \; g_3(m_Z) = 1.221$
- $m_{W^\pm} = 80.39$ GeV
- $m_{top} = 172.9 \pm 1.5$ GeV
- $m_{SMS} = 125.5 \pm 1.1$ GeV

The Spectral Action (A. Connes, A. Chamseddine 1996)

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Beyond SM: the general strategy (bottom-up approach)

- find finite geometry that has SM as sub-model (tricky)
  => particle content, gauge group & representation
- make sure everything is anomaly free
- compute the spectral action => constraints on parameters
- determine the cut-off scale Λ with suitable sub-set of the constraints
- use renorm. group equations to obtain low energy values of (hopefully) interesting parameters
  (scalar couplings, Yukawa couplings)
- check with experiment!

Excluded models:

Extensions of the Standard Model:

- AC-model (C.S. '05):
  new particles (A and C) with opposite hypercharge
  dark matter as bound AC-states (Fargion,Khlopov,C.S. '05)
- θ-model (C.S. '07): new particles with $SU_c(D)$-colour
- Vector-Doublet Model (Squellari, C.S. '07):
  new $SU_w(2)$-vector doublets

Problem for models: • $m_{SMS} \geq 170$ GeV
• constraints on g_1, g_2, g_3 at Λ.

Saving these models?

Some of these models, e.g. the AC-model may perhaps be extended to comply with experimental data!

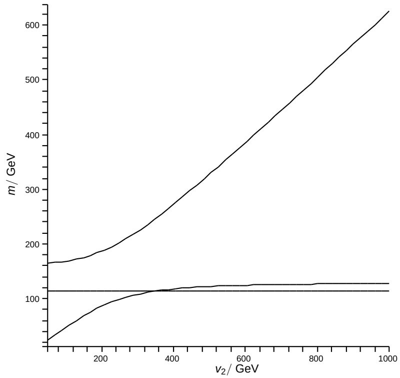
SM + $U(1)_X$ scaler field + $U(1)_X$ fermion singlet (C.S. 2009):

- Discrete space: $\mathbb{C} \oplus M_2(\mathbb{C}) \oplus M_3(\mathbb{C}) \oplus \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}$
- Gauge group: $U(1)_Y \times SU(2)_w \times SU(3)_c \times U(1)_X$
- New fermions: $U(1)_X$-vector singlets (X-particles)
  neutral w.r.t SM gauge group , $M_X \sim \Lambda$
- New scalar: $U(1)_X$ singlet σ, neutral w.r.t SM gauge group
- $\mathcal{L}_{scalar} = -\mu_1^2|H|^2 + \frac{\lambda_1}{6}|H|^4 - \mu_2^2|\sigma|^2 + \frac{\lambda_2}{6}|\sigma|^4 + \frac{\lambda_3}{3}|H|^2|\sigma|^2$
- $U(1)_Y \times SU(2)_w \times SU(3)_c \times U(1)_X \to U(1)_{el.} \times SU(3)_c$
- $\mathcal{L}_{ferm+gauge} = \bar{X}_L M_X X_R + g_{\nu,X} \bar{\nu}_R \sigma X_L + h.c. + 1/g_4^2 F_X^{\mu\nu} F_{X,\mu\nu}$

The constraints at Λ:

- only top-quark & ν_τ
- valid at $g_2 = g_3$
  => $\Lambda = 1.1 \times 10^{17}$ GeV
- $g_2^2 = \frac{\lambda_1}{24} \frac{(3g_t^2+g_\nu^2)^2}{3g_t^4+g_\nu^4}$
- $g_2^2 = \frac{\lambda_2}{24}$
- $g_2^2 = \frac{\lambda_3}{24} \frac{3g_t^2+g_\nu^2}{g_\nu^2}$
- $g_2^2 = \frac{1}{4}\left(3g_t^2 + g_\nu^2\right)$
- free parameters: $|\langle\sigma\rangle|$, g_4

- Problem: $\sqrt{5/3}g_1 \neq g_2 = g_3$



Mass EVs of scalar fields for
$v_2 = \sqrt{2}\,|\langle\sigma\rangle|$,
$\sqrt{2}|\langle H\rangle| = 246$ GeV, $g_4(m_Z) = 0.3$

SM + $U(1)_X$ scalar field + new fermions (C.S. '13):

- SM as a sub-model: comme il faut!
- gauge group: $U(1)_Y \times SU(2)_w \times SU(3)_c \times U(1)_X$
- new fermions in each SM-generation:

  $X_l^1 \oplus X_l^2 \oplus X_l^3 : (0,1,1,+1) \oplus (0,1,1,+1) \oplus (0,1,1,0)$
  $X_r^1 \oplus X_r^2 \oplus X_r^3 : (0,1,1,+1) \oplus (0,1,1,0) \oplus (0,1,1,+1)$
  $V_\ell^w,\, V_r^w \,:\, (0,\bar{2},1,0)$
  $V_\ell^c,\, V_r^c \,:\, (-1/6,1,\bar{3},0)$

- new scalar: $\sigma : (0,1,1,+1)$

The Lagrangian (scalar potential & new terms):

- $\mathcal{L}_{scalar} = -\mu_1^2|H|^2 - \mu_2^2|\sigma|^2 + \frac{\lambda_1}{6}|H|^4 + \frac{\lambda_2}{6}|\sigma|^4 + \frac{\lambda_3}{3}|H|^2|\sigma|^2$
- $\mathcal{L}_{ferm} = g_{\nu,X^1}\bar{\nu}_r \sigma X_\ell^1 + \bar{X}_\ell^1 m_X X_r^1 + g_{X^2}\bar{X}_\ell^2 \sigma X_r^2$
  $+ g_{X^3}\bar{X}_\ell^3 \sigma X_r^3 + \bar{V}_\ell^c m_c V_r^c + \bar{V}_\ell^w m_w V_r^w + \; h.c.$
- $\mathcal{L}_{gauge} = \frac{1}{g_4^2} F_X^{\mu\nu} F_{X,\mu\nu}$
- Symmetry breaking:
  $U(1)_Y \times SU(2)_w \times SU(3)_c \times U(1)_X \to U(1)_{e\ell.} \times SU(3)_c \times \mathbb{Z}_2$

The constraints at Λ:

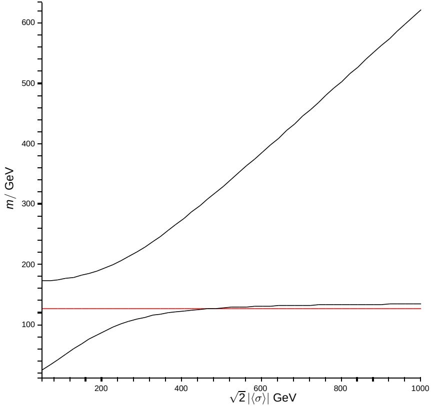
- $g_2(\Lambda) = g_3(\Lambda) = \sqrt{\frac{7}{6}}\, g_1(\Lambda) = \sqrt{\frac{4}{3}}\, g_4(\Lambda)$
- $\lambda_1(\Lambda) = 36\, \frac{H}{Y_2}\, g_2(\Lambda)^2, \;\; \lambda_2(\Lambda) = 36\, \frac{tr(g^4_{\nu,X^1})}{tr(g^2_{\nu,X^1})^2}\, g_2(\Lambda)^2$
- $\lambda_3(\Lambda) = 36\, \frac{tr(g^2_\nu)}{Y_2}\, g_2(\Lambda)^2$
- $Y_2(\Lambda) = tr(g^2_{\nu,X^1})(\Lambda) + tr(g^2_{X^1})(\Lambda) + tr(g^2_{X^2})(\Lambda) = 6\, g_2(\Lambda)^2$

Some simplifications:

- $Y_2 \approx 3g_{top} + g_{\nu_\tau}$
- $tr(g^2_{X^1})(\Lambda) \approx tr(g^2_{X^2})(\Lambda) \approx 0$
- $tr(g^2_{\nu,X^1})(\Lambda) \approx g_{\nu,X}(\Lambda)^2 = 6\, g_2(\Lambda)^2$
- $(m_w)_{ij} \approx \Lambda$, $(m_c)_{ij} \approx 10^{15}$ GeV

Results for 1-loop renormalisation groups:

- Constraints
  => $\Lambda \approx 2 \times 10^{18}$ GeV
- $m_{top} \approx 172.9 \pm 1.5$ GeV
- $m_{\sigma_{1,SMS}} \approx 125 \pm 1.1$ GeV
- $m_{\sigma_2} \approx 445 \pm 139$ GeV
- $m_{Z_X} \approx 254 \pm 87$ GeV
- $g_4(m_Z) \approx 0.36$
- $m_{X_2,X_3} \precsim 50$ GeV
- free parameter: $|\langle\sigma\rangle|$



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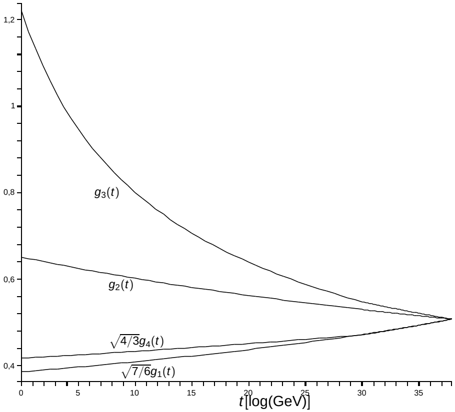

Abbildung: Running of the gauge couplings with normalisation factors.

Overview

Questions & to-do-list

- Is the SM + scalar model compatible with LHC-data?
- Does the SM + scalar model contain viable dark matter candidates?
- Explore parameter space (g_{ν,X^1}, $g^2_{X^2}$, $g^2_{X^3}$, m_{X^1}, m_{V^w}, m_{V^c})
- Extend renormalisation group analysis to n-loop, $n \geq 2$
- Is the geometry a "sub-geometry" of a Connes-Chamseddine-type geometry?