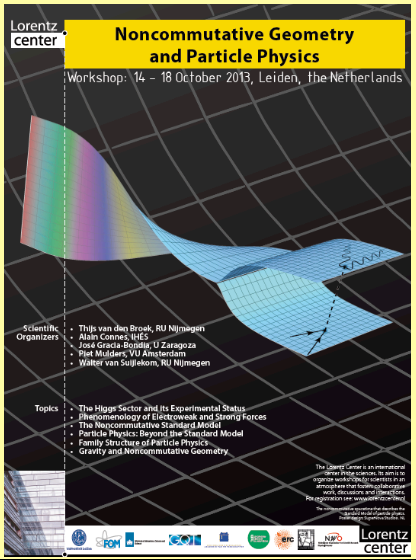
Unimodular Gravity



Instituto de Física Teórica UAM-CSIC

UAM UNIVERSIDAD AUTONOMA DE MADRID

8/26/13 1:44 PM

Linear Theory. Beyond Fierz-Pauli

Massive Spin 1: Three polarizations

$$k = (m, 0, 0, 0)$$

$$\epsilon_\mu^A \; : \; \begin{aligned} \epsilon^1 &\equiv (0,1,0,0) \\ \epsilon^2 &\equiv (0,0,1,0) \\ \epsilon^3 &\equiv (0,0,0,1) \end{aligned}$$

Massless limit

$$k \equiv (1, 0, 0, 1)$$

$$\epsilon^1 \equiv e_1 \quad \epsilon^2 \equiv e_2 \quad \epsilon^3 \equiv k$$

The only known way to stay with only two polarizations is to make the identification

The origin of gauge invariance

$$\epsilon = \epsilon + k \quad \epsilon_\mu = \epsilon_\mu + \partial_\mu \lambda$$

Longitudinal polarizations are pure gauge.

Propagators from unitarity

$$D_{\mu\nu} = \sum_A \epsilon_\mu^A \epsilon_n^A = P_{\mu\nu} \equiv \eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}$$

$$P_{\mu\nu}^{TOS} \equiv \eta_{\mu\nu} - \frac{k_\mu k_\nu}{m^2}$$

Transverse on shell "projector

Lagrangians from propagators

$$L \sim A^\mu \left(P_{TOS}\right)^{-1}_{\mu\nu} A^\nu = \frac{1}{k^2 - m^2} A^\mu \left(\left(k^2 - m^2\right)\eta_{\mu\nu} + k_\mu k_\nu\right) A^\nu$$

Massive photon

$$L = \frac{1}{4} F_{\mu\nu}^2 - \frac{1}{2} m^2 A_\mu^2$$

Massive Spin Two

Five polarizations

$$e_i \otimes e_j + e_j \otimes e_i - \frac{2}{3}\left(\sum_k e_k \otimes e_k\right)\delta_{ij}$$

Massless limit

$$\epsilon_3 \equiv k \otimes e_2 + e_2 \otimes k$$

$$\epsilon_4 = k \otimes e_1 + e_1 \otimes k$$

$$\epsilon_5 \equiv k \otimes k$$

$$\epsilon_1 \equiv e_1 \otimes e_2 + e_2 \otimes e_1$$

$$\epsilon_2 \equiv e_1 \otimes e_1 - e_2 \otimes e_2$$

These two rotate amongst themselves under the little group.

The smallest gauge invariance we need to stay with two polarizations is transverse.

$$\epsilon_{\alpha\beta} \sim \epsilon_{\alpha\beta} + \partial_{(\alpha}\xi_{\beta)}$$

$$k.\xi = 0$$

Unitarity again

$$D_{\mu\nu\lambda\sigma} \equiv \sum_A \epsilon^A_{\mu\nu}\epsilon^A_{\lambda\sigma} = c_1\eta^T_{\mu\nu}\eta^T_{\lambda\sigma} + c_2\eta^T_{\mu\nu}k_\lambda k_\sigma + k_\mu k_\nu \eta^T_{\lambda\sigma}$$
$$+c_3(\eta^T_{\mu\lambda}\eta^T_{\nu\sigma} + \eta^T_{\mu\sigma}\eta^T_{\nu\lambda}) + c_4(k_\mu k_\sigma \eta^T_{\nu\lambda} + k_\mu k_\lambda \eta^T_{\nu\sigma} +$$
$$k_\nu k_\sigma \eta^T_{\mu\lambda} + k_\nu k_\lambda \eta^T_{\mu\sigma} + c_5 k_\mu k_\nu k_\lambda k_\sigma$$

Using transversality and tracelessness

$$D_{\mu\nu\rho\sigma} = c_1\left(P_{\mu\nu}P_{\rho\sigma} - \frac{3}{2}\left(P_{\mu\rho}P_{\nu\sigma} + P_{\mu\sigma}P_{\nu\rho}\right)\right)$$

To find the lagrangian, $P_{\mu\nu} \to P^{TOS}_{\mu\nu}$

$$D_{\mu\nu\rho\sigma} = c_1\left(P^{TOS}_{\mu\nu}P^{TOS}_{\rho\sigma} - \frac{3}{2}\left(P^{TOS}_{\mu\rho}P^{TOS}_{\nu\sigma} + P^{TOS}_{\mu\sigma}P^{TOS}_{\nu\rho}\right)\right)$$

Normalization: $$c_1 = -\frac{4}{3}\frac{1}{k^2 - m^2}$$

Computing the inverse of the propagator

$$\begin{aligned} L = \frac{1}{4}\partial_\mu h^{\nu\rho}\partial^\mu h_{\nu\rho} - \frac{1}{2}\partial_\mu h^{\mu\rho}\partial^\nu h_{\nu\rho} \\ +\frac{1}{2}\partial_\mu h \partial^\sigma h_{\mu\sigma} - \frac{1}{4}\partial_\mu h \partial^\mu h \\ -\frac{m^2}{4}\left(h_{\alpha\beta}h^{\alpha\beta} - h^2\right) \end{aligned}$$

Fierz-Pauli, with the Fierz-Pauli mass term

Is it possible to get only the traceless part of the ME?
(We need Weyl)

$$8K_{FP}^{\mu\nu\rho\sigma} = k^2(\eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho} - 2\eta^{\mu\nu}\eta^{\rho\sigma}) - (k^\mu k^\rho \eta^{\nu\sigma} + k^\nu k^\sigma \eta^{\mu\rho} + k^\mu k^\sigma \eta^{\nu\rho} + k^\nu k^\rho \eta^{\mu\sigma} - 2k^\mu k^\nu \eta^{\rho\sigma} - 2k^\rho k^\sigma \eta^{\mu\nu})$$

$$tr\, K^{\mu\nu} = \frac{n-2}{4}(k^\mu k^\nu - k^2\eta^{\mu\nu})$$

$$tr\, tr\, K = -\frac{(n-1)(n-2)}{4}k^2$$

Traceless part of Fierz-Pauli

$$K_{traceless}^{\mu\nu\rho\sigma} = K^{\mu\nu\rho\sigma} - \frac{1}{n}\eta^{\mu\nu} tr\, K^{\rho\sigma}$$

No lagrangian because it is not symmetric

The most general symmetric lagrangian

$$Q^{\mu\nu\rho\sigma} = K^{\mu\nu\rho\sigma} - \eta^{\mu\nu} M^{\rho\sigma} - M^{\mu\nu}\eta^{\rho\sigma}$$

Asking for tracelessness

$$M^{\mu\nu} = \frac{1}{n}\left(tr\, K^{\mu\nu} - tr\, M \eta^{\mu\nu}\right)$$

$$M^{\mu\nu} = \frac{1}{n}\left(tr\, K^{\mu\nu} - tr\, tr\, K \eta^{\mu\nu}\right)$$

Linear form of Weyl-transverse

$$8K_{WT}^{\mu\nu\rho\sigma} = k^2(\eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho}) - (k^\mu k^\rho \eta^{\nu\sigma} +$$
$$k^\nu k^\sigma \eta^{\mu\rho} + k^\mu k^\sigma \eta^{\nu\rho} + k^\nu k^\rho \eta^{\mu\sigma} - \frac{2(n+2)}{n^2} k^2 \eta^{\mu\nu}\eta^{\rho\sigma} + \frac{4}{n}(k^\mu k^\nu \eta^{\rho\sigma} + k^\rho k^\sigma \eta^{\mu\nu}$$

General linear transverse gauge invariance.

$$\mathcal{L}^{I} = \frac{1}{4}\,\partial_\mu h^{\nu\rho}\partial^\mu h_{\nu\rho}, \quad \mathcal{L}^{II} = -\frac{1}{2}\,\partial_\mu h^{\mu\rho}\partial_\nu h^{\nu}_{\rho},$$
$$\mathcal{L}^{III} = \frac{1}{2}\,\partial^\mu h \partial^\rho h_{\mu\rho}, \quad \mathcal{L}^{IV} = -\frac{1}{4}\,\partial_\mu h \partial^\mu h.$$

There is in general an extra scalar mode.

In order for it not no be a ghost

$$b \geq \frac{1 - 2a + (n-1)a^2}{n-2}$$

$$\mathcal{L}_{\rm TDiff} \equiv \mathcal{L}_A + a\,\mathcal{L}^{III} + b\,\mathcal{L}^{IV}.$$

Transverse

a=b=1

Fierz-Pauli

No extra scalar mode with enhanced symmetry

$$a = \frac{2}{n}, \qquad b = \frac{n+2}{n^2}.$$

Unimodular

A second possibility is to enhance TDiff with an additional Weyl symmetry,

$$\delta h_{\mu\nu} = \frac{2}{n}\phi\eta_{\mu\nu}, \tag{13}$$

by which the action becomes independent of the trace. In the generic transverse Lagrangian $\mathcal{L}_{\mathrm{TDiff}}[h_{\mu\nu}]$ of Eq. (9), replace $h_{\mu\nu}$ with the traceless part

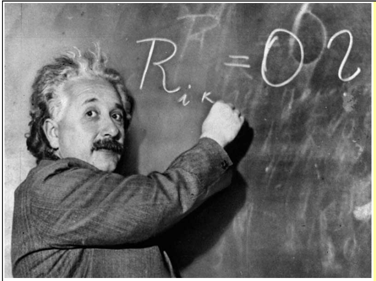
$$h_{\mu\nu} \mapsto \hat{h}_{\mu\nu} \equiv h_{\mu\nu} - (h/n)\eta_{\mu\nu}. \tag{14}$$

This is formally analogous to (10) with $\lambda = -1/n$, but cannot be interpreted as a field redefinition. As such, it would be singular, because the trace h cannot be recovered from $\hat{h}_{\mu\nu}$. The resulting Lagrangian

$$\mathcal{L}_{\mathrm{WTDiff}}[h_{\mu\nu}] \equiv \mathcal{L}_{\mathrm{TDiff}}[\hat{h}_{\mu\nu}], \tag{15}$$

is still invariant under TDiff (the replacement (14) does not change the coefficients in front of the terms $\mathcal{L}^{\mathrm{I}}$ and $\mathcal{L}^{\mathrm{II}}$). Moreover, it is invariant under (13), since $\hat{h}_{\mu\nu}$ is. Using (11) with $\lambda = -1/n$ we immediately find that this "WTDiff" symmetry corresponds to Lagrangian parameters

$$a = \frac{2}{n}, \qquad b = \frac{n+2}{n^2}. \tag{16}$$



Einstein's 1919 theory :"Spielen Gravitationsfelder im Aufber der materiellen Elementarteilchen eine wesentliche Rolle?" (Sitzungsberichte der Prussischen Akad d. Wissenschaften)

$$R_{\mu\nu} - \frac{1}{n} R g_{\mu\nu} = \kappa^2 \left(T_{\mu\nu} - \frac{1}{n} T g_{\mu\nu} \right)$$

$$g^{\alpha\beta} \frac{\delta}{\delta g^{\alpha\beta}} S = 0$$

(Tracefree piece)

$$\nabla_\mu R^{\mu\nu} = \frac{1}{2} \nabla^\nu R$$

Trace recovered through Bianchi

$$(\frac{1}{2} - \frac{1}{n}) \nabla^\nu R = -\frac{\kappa^2}{n} \nabla^\nu T$$

$$\frac{n-2}{2} R + \frac{\kappa^2}{n} T \equiv \lambda$$

$$R_{\mu\nu} - \frac{1}{2} (R - 2\lambda) g_{\mu\nu} = \kappa^2 T_{\mu\nu}$$

Variational principle?

Traceless equations ➔ Scale symmetry?

$$\frac{\delta S}{\delta g_{\alpha\beta}} g_{\alpha\beta} = 0$$

Weyl transformation:

$$g_{\alpha\beta} \to \Omega^2 g_{\alpha\beta}$$

$$g \equiv det\, g_{\alpha\beta} \to \Omega^{2n} g$$

$$S_W \equiv -\frac{1}{2\kappa^2} \int d^n x\, |g|^{\frac{1}{n}} R$$

Not good enough

Einstein frame

$$g_E \equiv \left|\det g^E_{\mu\nu}\right| = 1$$

$$S_E \equiv -\frac{1}{2\kappa^2}\int d^n x\ R_E\left[g_E\right]$$

Jordan frame

$$S_U \equiv -M^{n-2}\int d^n x\ R_E\ +\ S_{matt} =$$

$$-M^{n-2}\int d^n x\ g^{\frac{1}{n}}\left(R + \frac{(n-1)(n-2)}{4n^2}\frac{g^{\mu\nu}\nabla_\mu g\ \nabla_\nu g}{g^2}\right) + S_{matt}$$

We want to study conformal properties in the presence of dynamical gravity

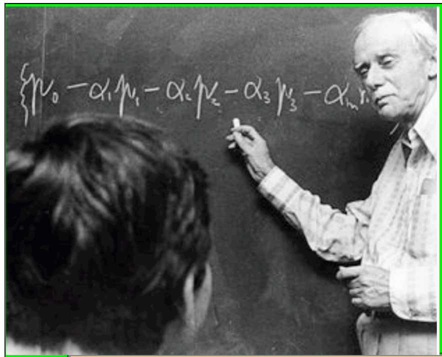
$$\sqrt{|\tilde{g}|}\tilde{R} = \sqrt{|g|}\left[\Omega^{n-2}R + (n-1)(n-2)\Omega^{n-4}(\nabla\Omega)^2\right]$$

Promote the Weyl parameter to a new graviscalar field

$$\Omega \equiv \frac{1}{M_p}\sqrt{\frac{(n-2)}{4(n-1)}}\;\phi_g^{\frac{2}{n-2}}$$

$$M_p^{n-2} \equiv \frac{1}{16\pi G_n}$$

The resulting theory is TWG, with pseudo-Weyl symmetry



(Dirac, Englert et al)

Pseudo Weyl Dilaton Gravity

$$S_{TWG} = \int d(vol) \left(-\frac{n-2}{8(n-1)} R \, \phi_g^2 - \frac{1}{2} g^{\mu\nu} \nabla_\mu \phi_g \nabla_\nu \phi_g \right)$$

$$d(vol) \equiv \sqrt{|g|} d^n x$$

Pseudo Weyl symmetry

$$\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$$
$$\tilde{\phi}_g = \Omega^{\frac{2-n}{2}} \phi_g$$

Why pseudo? Because one of the fields is an spurion than can be eliminated through a field redefinition

HPRW (2013) claim that a similar action is the UV fixed point of a truncated effective action under a functional renormalization group.

In particular the quartic potential vanishes asymptotically

Scalar potential in the Jordan frame = Cosmological constant in the Einstein frame

Conformal invariance (Weyl)

$$\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$$
$$\tilde{\phi}_g = \Omega^{\frac{2-n}{2}}\ \phi_g$$

UG equations of motion are in the gauge fixed sector of TWF

$$\frac{\delta S_U}{\delta g_{\mu\nu}} = \frac{\delta S_{ST}}{\delta g_{\mu\nu}} + \frac{\delta S_{ST}}{\delta \phi_g}\frac{\delta \phi_g}{\delta g_{\mu\nu}}\bigg|_{\phi_g=-2^{\frac{3}{2}}M^{\frac{n-2}{2}}\sqrt{\frac{n-1}{n-2}}g^{-\frac{n-2}{4n}}} .$$

Tautological Weyl Gravity/ Dilaton gravity

$$S = \int d^n x \, \sqrt{|g|} \left(-\frac{n-2}{8(n-1)} \, R\phi_g^2 - \frac{1}{2} \, \left(\nabla \phi_g \right)^2 \right)$$

$$g = \lambda \phi^{\frac{-4n}{n-2}}$$

It contains unimodular gravity in the gauge fixed sector

$$S^{TWG;GF} \equiv -\frac{n-2}{8(n-1)} \lambda^2 \int d^n x \, |g|^{\frac{1}{n}} \left(R + \frac{(n-2)(n-1)}{4n^2} \frac{(\nabla g)^2}{g^2} \right)$$

$$R_{\mu\nu}-\frac{1}{n}Rg_{\mu\nu}=\frac{(n-2)(2n-1)}{4n^2}\left(\frac{\nabla_\mu g\nabla_\nu g}{g^2}-\frac{1}{n}\frac{(\nabla g)^2}{g^2}g_{\mu\nu}\right)-\frac{n-2}{2n}\left(\frac{\nabla_\mu\nabla_\nu g}{g}-\frac{1}{n}\frac{\nabla^2 g}{g}g_{\mu\nu}\right) \tag{1.4}$$

$$g=\lambda\phi^{\frac{-4n}{n-2}}$$

$$R_{\mu\nu}-\frac{1}{n}Rg_{\mu\nu}=-\frac{2n}{n-2}\frac{\nabla_\mu\phi\nabla_\nu\phi}{\phi^2}+2\frac{\nabla_\mu\nabla_\nu\phi}{\phi}+\left(\frac{2}{n-2}\frac{(\nabla\phi)^2}{\phi^2}-\frac{2}{n}\frac{\nabla^2\phi}{\phi}\right)g_{\mu\nu}$$

$$R=4\frac{n-1}{n-2}\frac{\nabla^2\phi}{\phi}$$

$$R=4\frac{n-1}{4n}\left(\frac{5n-2}{4n}g^{-\frac{9n-4}{4n}}(\nabla g)^2-g^{-\frac{5n-2}{4n}}\nabla^2 g\right)$$

It is actually possible to work in the Einstein frame

$$G_{\mu\nu} \equiv \frac{1}{M_p^2}\left(\frac{n-2}{8(n-1)}\right)^{\frac{2}{n-2}} \phi_g^{\frac{4}{n-2}} g_{\mu\nu}$$

The Einstein metric is a singlet
(inert under Weyl transformations)

$$\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$$
$$\tilde{\phi}_g = \Omega^{\frac{2-n}{2}} \phi_g$$

The action then reduces to
EInstein-Hilbert

$$S = -M_p^{n-2} \int \sqrt{G}\, d^n x\, R[G]$$

Phases of Weyl gauge symmetry

There are two Weyl gauge orbits

Isolated point.
Symmetric Phase

$$\boxed{\phi_g = 0}$$

Broken phase

$$\boxed{\phi_g \in \mathcal{F} \setminus \{0\}}$$

Also in GR it is sometimes possible to define a symmetric phase.

Truncation of supergravity superconformal (Kallosh et al)

In the broken phase

TWG reduces classically to GR in the gauge

$$\phi_g = \sqrt{\frac{8(n-1)}{n-2}} M_p^{\frac{n-2}{2}}$$

TWG reduces classically to unimodular gravity (UG) in the gauge

$$\phi_g + 2^{\frac{3}{2}} M_p^{\frac{n-2}{2}} \sqrt{\frac{n-1}{n-2}} g^{-\frac{n-2}{4n}} = 0$$

Our aim now is however to study the much more interesting unbroken phase, in which the vacuum expectation value of the gravitational scalar field vanishes.

Change of variables corresponding to a Weyl transformation

$$0 = \delta Z \equiv \int \mathcal{D}g_{\mu\nu} \prod_i \mathcal{D}\psi_i \int d(vol)_x \omega(x) \Big\{ -2g^{\mu\nu}(x) \frac{\delta S}{\delta g_{\mu\nu}(x)} - \frac{n-2}{2} \phi_g \frac{\delta S}{\delta \phi_g} +$$
$$+2J^{\mu\nu}(x) g_{\mu\nu}(x) - J(x)\phi_g(x) \Big\} \exp \left\{ iS[g_{\mu\nu}\phi_g] + \int d(vol) \left(J^{\mu\nu} g_{\mu\nu} + J\ \phi_g\right) \right\} ($$

Off shell Ward identities

$$\left\langle 0_+ \left| g^{\mu\nu}(x) \frac{\delta S}{\delta g_{\mu\nu}(x)} + \frac{n-2}{4} \phi_g \frac{\delta S}{\delta \phi_g} \right| 0_- \right\rangle = 0$$

(They generalize to the gravitational case the tracelessness of the energy-momentum tensor)

Define the functional integral through the Einstein frame

$$e^{iW\left[\bar{g}_{\mu\nu},\bar{\phi}_g\right]} \equiv \int \mathcal{D}g_{\mu\nu}\ \mathcal{D}\phi_g\ e^{-i\frac{1}{2}\int d^4x\ \sqrt{-g}\left(\partial_\mu\phi_g g^{\mu\nu}\ \partial_\nu\phi_g+\frac{1}{6}\ R\ \phi_g^2\right)}$$

$$e^{iW\left[\bar{G}_{\mu\nu}\left[\bar{g}_{\mu\nu},\bar{\phi}_g\right]\right]} \equiv \int \mathcal{D}G_{\mu\nu}\ e^{\frac{i}{16\pi G}\int d^4x R[G_{\mu\nu}]}$$

‘t Hooft and Veltman effective action

$$S_\infty = \frac{1}{\pi^2(n-4)}\int d^4x\sqrt{|G|}\left(\frac{149}{2880}E_4[G]+\frac{7}{320}W_4[G]+\frac{3}{128}R[G]^2\right)$$

This constructs are point Weyl invariant in four and only in four dimensions

$$\int d(vol)\ W_4\left[\Omega^2 g_{\mu\nu}\right] = \int d(vol)\ \Omega^{n-4}\ W_4\left[g_{\mu\nu}\right]$$

$$\int d(vol)\ E_4\left[\Omega^2 g_{\mu\nu}\right] = \int d(vol)\ \Omega^{n-4}\ E_4\left[g_{\mu\nu}\right]$$

This means that there is a finite residue from the pole in the infinite piece of the effective action when performing a Weyl transformation

$$\frac{1}{n-4} \times (n-4) \rightarrow \text{finite remainder}$$

Specific TWG divergences follow the same pattern

$$S_\infty = \frac{1}{\pi^2(n-4)} \int d(vol) \left\{ \frac{149}{2880} E_4 + \frac{7}{320} W_4 + \left(R - 6 \frac{\nabla^2 \phi_g}{\phi_g} \right)^2 \right\}$$

$$\left(\tilde{\nabla}^2 - \frac{n-2}{4(n-1)} \tilde{R} \right) \left(\Omega^{-\frac{n-2}{2}} \phi \right) = \Omega^{-\frac{n+2}{2}} \left(\nabla^2 - \frac{n-2}{4(n-1)} R \right)$$

$$\left(\tilde{R} - \frac{4(n-1)}{n-2} \frac{\tilde{\nabla}^2 \tilde{\phi}_g}{\tilde{\phi}_g} \right)^2 = \Omega^{-4} \left(R - \frac{4(n-1)}{n-2} \frac{\nabla^2 \phi_g}{\phi_g} \right)^2$$

$\therefore$ There is a conformal anomaly in TWG given by

$$\left\langle 0_+ \left| -2g^{\mu\nu}\frac{\delta S_{TWG}}{\delta g^{\mu\nu}} - \frac{n-2}{2}\phi_g \frac{\delta S_{TWG}}{\delta \phi_g} \right| 0_- \right\rangle \equiv A_{TWG} = \frac{1}{\pi^2}\left\{ \frac{7}{320}W_4 + \left(R - 6\frac{\nabla^2 \phi_g}{\phi_g} \right)^2 \right\}$$

This is at variance with some cherished beliefs

Work is in progress to check this by a direct TWG heat kernel computation

Caveat emptor: Gravitational counterterms in arbitrary gauges

Beta gauge

$$\nabla_\sigma h^{\mu\sigma} = \frac{1+\beta}{2}\nabla^\mu h$$

(Kallosh, Tarasov and Tyutin)

$$W = \frac{1}{48\pi^2}\frac{1}{n-4}\int\sqrt{|g|}\;d^4x\;\left(a\;E_4 + \frac{b}{2}W_4 + \frac{c}{6}R^2\right)$$

$$a \equiv \frac{53}{15} - \frac{b}{2}$$
$$b \equiv 2\gamma^4 - 3\gamma^2 - 4\gamma + \frac{21}{10}$$
$$c \equiv 10\gamma^4 - 3\gamma^2 - 14\gamma + \frac{9}{2}$$

$$\gamma \equiv \frac{\beta}{1-\beta}$$

Gauge dependence of the Conformal Anomaly

The result may seem surprising at first sight, but it is a trivial consequence of

1.- The counterterm must be conformal invariant.

2.-The only pointwise conformal invariant in four dimensions is $\boxed{\sqrt{|g|}W_4}$

The only logical way out would be that either

3.- There is no counterterm, that is the theory is finite

Or else

4.-Give up diffeomorphism invariance.

Then there are pointwise conformal invariants in arbitrary dimension, such as

$$\boxed{(-g)^{\frac{2}{n}}\ W_4.}$$

5.- It is always possible that the formula we have used to compute the conformal anomaly does not hold for some unknown reason?

$$\left\langle g^{\mu\nu} \, \frac{\delta S}{\delta g^{\mu\nu}} \right\rangle = 2 \, \frac{\delta S_{\rm eff}}{\delta \Omega}\bigg|_{\Omega=1}$$

6.- BUT, It is unclear what is the physical meaning of an anomaly that is not gauge-independent

Backup slides

Divergence of the field equation

$$k^{\nu} K_{\mu\nu\rho\sigma} h^{\rho\sigma} = -2m^2 \left(k^{\rho} h_{\rho\mu} - k_{\mu} h\right)$$

$$\therefore$$

$$k^2 h = k_{\rho} k_{\sigma} h^{\rho\sigma}$$

Trace of the field equation

$$\eta^{\mu\nu} K_{\mu\nu\rho\sigma} h^{\rho\sigma} = -2(1-n) m^2 h$$

$$\therefore$$

$$h = k_{\mu} k_{\nu} h^{\mu\nu} = 0$$

$$k^{\mu} h_{\mu\nu} = 0$$

Klein-Gordon

$$\boxed{\left(\Box + m^2\right) h_{\mu\nu} = 0}$$

The canonically normalized field is

$$\phi_g \equiv -2\sqrt{2}M^{\frac{n-2}{2}}\sqrt{\frac{n-1}{n-2}}\, e^{-\frac{1}{2}\sqrt{\frac{n-2}{n-1}}\sigma}. \tag{2.12}$$

The old gauge $C = 1$ now reads

$$\phi_g + 2^{\frac{3}{2}}M^{\frac{n-2}{2}}\sqrt{\frac{n-1}{n-2}}g^{-\frac{n-2}{4n}} = 0. \tag{2.13}$$

In terms of ϕ_g the action is

$$S_{ST} = \int d^n x\sqrt{g}\left\{ -\frac{n-2}{8(n-1)}\, R\, \phi_g^2 - \frac{1}{2}g^{\mu\nu}\nabla_\mu\phi_g\nabla_\nu\phi_g + \right.$$
$$\left. \frac{n-2}{8(n-1)M^{n-2}}\phi_g^2\frac{1}{2}(\nabla\Phi)^2 - (-1)^{\frac{2n}{n-2}}\left(\frac{n-2}{8(n-1)}\right)^{\frac{n}{n-2}}\frac{1}{M^n}\phi_g^{\frac{2n}{n-2}}V(\Phi)\right\}. \tag{2.14}$$

The gauge $C = 1$ means that

$$e^{\frac{2n}{\sqrt{(n-1)(n-2)}}\sigma(x)} = g(x).$$

The new action is then written as

$$S \equiv \int d^n x \, \sqrt{g} \Big\{ e^{-\sqrt{\frac{n-2}{n-1}}\sigma} \Big[-M^{n-2} \big(R + g^{\mu\nu} \nabla_\mu \sigma \, \nabla_\nu \sigma \big) + \frac{1}{2} g^{\mu\nu} \nabla_\mu \Phi \nabla_\nu \Phi \Big]$$
$$- e^{-\frac{n}{\sqrt{(n-1)(n-2)}}\sigma} V(\Phi) \Big\}. \qquad (2.1$$

$$S_U \equiv -M^{n-2} \int d^n x \; R_E \; + \; S_{matt} =$$

$$-M^{n-2} \int d^n x \; g^{\frac{1}{n}} \left(\; R + \frac{(n-1)(n-2)}{4n^2} \frac{g^{\mu\nu} \nabla_\mu g \; \nabla_\nu g}{g^2} \right) + S_{matt}$$

Some comments on gauge fixing

U(1) Lorenz gauge

$$\partial_\mu \tilde{A}^\mu = 0 \qquad \Rightarrow \Box \tilde{A}_\mu = 0$$

How to recover the full covariant and gauge invariant Maxwell equations given four solutions of the Klein Gordon equation?

Perform an arbitrary gauge transformation

$$A_\mu = \tilde{A}_\mu + \partial_\mu \Lambda$$

$$\Box\Lambda = \partial_\mu A^\mu$$

Eliminating the gauge parameter Maxwell is easily recovered

$$\Box A_\mu = \Box\partial_\mu\Lambda = \partial_\mu\partial_\rho A^\rho$$

If the gauge parameter is restricted (harmonic, for example) we DO NOT recover Maxwell.

$\therefore$ Out of the same gauge fixed theory, several gauge invariant theories can be obtained depending on the assumed gauge symmetry.

Belaboring: Weyl gauge

$$\tilde{A}_0 = 0$$

$$\partial_0 \partial_i \tilde{A}^i = 0$$

$$\Box \tilde{A}_i = \partial_i \partial_j \tilde{A}^j$$

$$A_0 = \partial_0 \Lambda \;\Rightarrow\; \partial_i A_0 = \partial_0 \partial_i \Lambda$$

$$A_i = \tilde{A}_i + \partial_i \Lambda$$

$$\partial_0 \partial_i \left(A^i - \partial^i \Lambda \right) = 0 \;\Rightarrow\; \partial_0 \partial_i A^i = \partial_0 \Delta \Lambda$$

$$\Box \left(A_i - \partial_i \Lambda \right) = \partial_i \partial_j \left(A^j - \partial^j \Lambda \right) = \partial_i \partial_j A^j - \partial_i \Delta \Lambda$$

$$\Box A_i = \Box\partial_i\Lambda + \partial_i\partial_j A^j - \partial_i\Delta\Lambda = \partial_0^2\partial_i\Lambda + \partial_i\partial_j A^j = \partial_0\partial_i A_0 + \partial_i\partial_j A^j = \partial_i\partial_\mu A^\mu$$

$$\Box A_0 = \partial_0\Box\Lambda = \partial_0\left(\partial_0^2\Lambda + \Delta\Lambda\right) = \partial_0^3\Lambda + \partial_0\partial_i A^i = \partial_0^2 A_0 + \partial_0\partial_i A^i = \partial_0\partial_\mu A^\mu$$

Covariant form of Maxwell

Back to basics: The linear theory. Fierz-Pauli and beyond

$$k = (m, 0, 0, 0)$$

Three polarizations ϵ_μ^A :

$$\epsilon^1 \equiv (0, 1, 0, 0)$$
$$\epsilon^2 \equiv (0, 0, 1, 0)$$
$$\epsilon^3 \equiv (0, 0, 0, 1)$$

$$k \equiv (1, 0, 0, 1)$$

$$\epsilon^1 \equiv e_1 \quad \epsilon^2 \equiv e_2 \quad \epsilon^3 \equiv k$$

The only known way to stay with only two polarizations is to make the identification

$$\epsilon = \epsilon + k \quad \epsilon_\mu = \epsilon_\mu + \partial_\mu \lambda$$ Longitudinal polarizations are pure gauge.

Propagators from unitarity

$$D_{\mu\nu} = \sum_A \epsilon^A_\mu \epsilon^A_n = P_{\mu\nu} \equiv \eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}$$

Transverse on shell "projector" $$P^{TOS}_{\mu\nu} \equiv \eta_{\mu\nu} - \frac{k_\mu k_\nu}{m^2}$$

Lagrangians from propagators

$$L \sim A^\mu \left(P_{TOS}\right)^{-1}_{\mu\nu} A^\nu = \frac{1}{k^2 - m^2} A^\mu \left(\left(k^2 - m^2\right)\eta_{\mu\nu} + k_\mu k_\nu\right) A^\nu$$

Massive photon

$$\boxed{L = \frac{1}{4} F^2_{\mu\nu} - \frac{1}{2} m^2 A^2_\mu}$$

Massive Spin Two

Five polarizations

$$e_i \otimes e_j + e_j \otimes e_i - \frac{2}{3}\left(\sum_k e_k \otimes e_k\right)\delta_{ij}$$

Massless limit

$$\epsilon_3 \equiv k \otimes e_2 + e_2 \otimes k$$

$$\epsilon_4 = k \otimes e_1 + e_1 \otimes k$$

$$\epsilon_5 \equiv k \otimes k$$

$$\epsilon_1 \equiv e_1 \otimes e_2 + e_2 \otimes e_1$$

$$\epsilon_2 \equiv e_1 \otimes e_1 - e_2 \otimes e_2$$

These two rotate amongst themselves under the little group.

The smallest gauge invariance we need to stay with two polarizations is transverse.

$$\epsilon_{\alpha\beta} \sim \epsilon_{\alpha\beta} + \partial_{(\alpha}\xi_{\beta)}$$

$$k.\xi = 0$$

Unitarity again

$$D_{\mu\nu\lambda\sigma} \equiv \sum_A \epsilon^A_{\mu\nu}\epsilon^A_{\lambda\sigma} = c_1\eta^T_{\mu\nu}\eta^T_{\lambda\sigma} + c_2\eta^T_{\mu\nu}k_\lambda k_\sigma + k_\mu k_\nu \eta^T_{\lambda\sigma}$$

$$+c_3(\eta^T_{\mu\lambda}\eta^T_{\nu\sigma} + \eta^T_{\mu\sigma}\eta^T_{\nu\lambda}) + c_4(k_\mu k_\sigma \eta^T_{\nu\lambda} + k_\mu k_\lambda \eta^T_{\nu\sigma} +$$

$$k_\nu k_\sigma \eta^T_{\mu\lambda} + k_\nu k_\lambda \eta^T_{\mu\sigma} + c_5 k_\mu k_\nu k_\lambda k_\sigma$$

Using transversality and tracelessness

$$D_{\mu\nu\rho\sigma} = c_1\left(P_{\mu\nu}P_{\rho\sigma} - \frac{3}{2}\left(P_{\mu\rho}P_{\nu\sigma} + P_{\mu\sigma}P_{\nu\rho}\right)\right)$$

To find the lagrangian, $\boxed{P_{\mu\nu} \to P^{TOS}_{\mu\nu}}$

$$\boxed{D_{\mu\nu\rho\sigma} = c_1\left(P^{TOS}_{\mu\nu}P^{TOS}_{\rho\sigma} - \frac{3}{2}\left(P^{TOS}_{\mu\rho}P^{TOS}_{\nu\sigma} + P^{TOS}_{\mu\sigma}P^{TOS}_{\nu\rho}\right)\right)}$$

Normalization: $$c_1 = -\frac{4}{3}\frac{1}{k^2 - m^2}$$

Computing the inverse of the propagator

$$L = \frac{1}{4}\partial_\mu h^{\nu\rho}\partial^\mu h_{\nu\rho} - \frac{1}{2}\partial_\mu h^{\mu\rho}\partial^\nu h_{\nu\rho} + \frac{1}{2}\partial_\mu h \partial^\sigma h_{\mu\sigma} - \frac{1}{4}\partial_\mu h \partial^\mu h - \frac{m^2}{4}\left(h_{\alpha\beta}h^{\alpha\beta} - h^2\right)$$

Fierz-Pauli, with the Fierz-Pauli mass term

Divergence of the field equation

$$k^{\nu} K_{\mu\nu\rho\sigma} h^{\rho\sigma} = -2m^2 \left(k^{\rho} h_{\rho\mu} - k_{\mu} h\right)$$

$$\therefore$$

$$k^2 h = k_{\rho} k_{\sigma} h^{\rho\sigma}$$

Trace of the field equation

$$\eta^{\mu\nu} K_{\mu\nu\rho\sigma} h^{\rho\sigma} = -2(1-n) m^2 h$$

$$\therefore$$

$$h = k_{\mu} k_{\nu} h^{\mu\nu} = 0$$

$$k^{\mu} h_{\mu\nu} = 0$$

Klein-Gordon

$$\boxed{\left(\Box + m^2\right) h_{\mu\nu} = 0}$$

Is it possible to get only the traceless part of the ME?
(We need Weyl)

$$8K_{FP}^{\mu\nu\rho\sigma} = k^2(\eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho} - 2\eta^{\mu\nu}\eta^{\rho\sigma}) - (k^\mu k^\rho \eta^{\nu\sigma} + k^\nu k^\sigma \eta^{\mu\rho} + k^\mu k^\sigma \eta^{\nu\rho} + k^\nu k^\rho \eta^{\mu\sigma} - 2k^\mu k^\nu \eta^{\rho\sigma} - 2k^\rho k^\sigma \eta^{\mu\nu})$$

$$tr\, K^{\mu\nu} = \frac{n-2}{4}(k^\mu k^\nu - k^2\eta^{\mu\nu})$$

$$tr\, tr\, K = -\frac{(n-1)(n-2)}{4}k^2$$

Traceless part of Fierz-Pauli

$$K_{traceless}^{\mu\nu\rho\sigma} = K^{\mu\nu\rho\sigma} - \frac{1}{n}\eta^{\mu\nu} tr\, K^{\rho\sigma}$$

No lagrangian because it is not symmetric

The most general symmetric lagrangian

$$Q^{\mu\nu\rho\sigma} = K^{\mu\nu\rho\sigma} - \eta^{\mu\nu} M^{\rho\sigma} - M^{\mu\nu}\eta^{\rho\sigma}$$

Asking for tracelessness

$$M^{\mu\nu} = \frac{1}{n}\left(tr\, K^{\mu\nu} - tr\, M\eta^{\mu\nu}\right)$$

$$M^{\mu\nu} = \frac{1}{n}\left(tr\, K^{\mu\nu} - tr\, tr\, K\eta^{\mu\nu}\right)$$

Linear form of Weyl-transverse

$$8K_{WT}^{\mu\nu\rho\sigma} = k^2(\eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho}) - (k^\mu k^\rho \eta^{\nu\sigma} +$$
$$k^\nu k^\sigma \eta^{\mu\rho} + k^\mu k^\sigma \eta^{\nu\rho} + k^\nu k^\rho \eta^{\mu\sigma} - \frac{2(n+2)}{n^2} k^2 \eta^{\mu\nu}\eta^{\rho\sigma} + \frac{4}{n}(k^\mu k^\nu \eta^{\rho\sigma} + k^\rho k^\sigma \eta^{\mu\nu}$$

General linear transverse gauge invariance.

$$\mathcal{L}^{I} = \frac{1}{4}\,\partial_\mu h^{\nu\rho}\partial^\mu h_{\nu\rho}, \quad \mathcal{L}^{II} = -\frac{1}{2}\,\partial_\mu h^{\mu\rho}\partial_\nu h^{\nu}_{\rho},$$
$$\mathcal{L}^{III} = \frac{1}{2}\,\partial^\mu h \partial^\rho h_{\mu\rho}, \quad \mathcal{L}^{IV} = -\frac{1}{4}\,\partial_\mu h \partial^\mu h.$$

There is in general an extra scalar mode.

In order for it not no be a ghost

$$b \geq \frac{1 - 2a + (n-1)a^2}{n-2}$$

$$\mathcal{L}_{\mathrm{TDiff}} \equiv \mathcal{L}_A + a\,\mathcal{L}^{III} + b\,\mathcal{L}^{IV}.$$

TDiff

No extra scalar mode with enhanced symmetry

a=b=1

Diff

$$a = \frac{2}{n}, \qquad b = \frac{n+2}{n^2}.$$

WTDiff

Coming back to Weyl transverse...

Lowest order effective lagrangian...

No allowed dimension zero operators:

$$L_0 \equiv F(|g|)$$

$$\delta S_0 \equiv \lambda \delta \int d^n x L_0 = 0$$

This (purely gravitational) symmetry is incompatible with a cosmological constant

Dimension two operators:

$$S_2^{(1)} \equiv -\frac{1}{2\kappa_1^2}|g|^{\frac{1-2n}{n}} g^{\alpha\beta}\partial_\alpha|g|\partial_\beta|g|$$

The gravitational equations of motion are now:

$$\frac{\delta S_2^{(1)}}{\delta g^{\alpha\beta}} = |g|^{\frac{1-2n}{n}}\partial_\alpha|g|\partial_\beta|g| - \left(\frac{1-2n}{n}|g|^{\frac{1-2n}{n}} g^{\mu\nu}\partial_\mu|g|\partial_\nu|g| - 2|g|\partial_\nu\left(|g|^{\frac{1-2n}{n}} g^{\mu\nu}\partial_\mu|g|\right)\right) g_\alpha$$

where the gravitational constant has been deleted because it is not important in the absence of matter. These equations are traceless *up to a total derivative*

$$g^{\alpha\beta}\frac{\delta S_2^{(1)}}{\delta g^{\alpha\beta}} = +2n\partial_\nu\left(|g|^{\frac{1-n}{n}} g^{\mu\nu}\partial_\mu|g|\right)$$

This means that the Noether current associated to WTDiff is

$$W^\mu \equiv |g|^{\frac{1-n}{n}} g^{\mu\nu}\partial_\nu|g|$$

Second dimension two operator

$$\delta S_2^{(2)} = \delta\left(-\frac{1}{2\kappa^2}\int d^n x |g|^{1/n} R\right)$$

$$\delta S_2^{(2)} = \int d^n x |g|^{1/n}\, \delta g^{\alpha\beta}\left(\frac{1}{2\kappa^2 n} g_{\alpha\beta} R - \frac{1}{2\kappa^2} R_{\alpha\beta}\right)$$
$$- \int d^n x |g|^{1/n} \frac{1}{2\kappa^2}\left(g_{\alpha\beta}\Delta - \nabla_\alpha \nabla_\beta\right)\delta g^{\alpha\beta}$$

The variation still vanishes for Weyl transformations

$\delta g^{\alpha\beta} = -\Omega^2 g^{\alpha\beta}$ because $\nabla_\alpha g_{\mu\nu} = 0$

EM are traceless up to a total derivative only

$$R_{\mu\nu} - \frac{1}{n} R g_{\mu\nu} = \frac{2-n}{2n}|g|^{-1}\left[\frac{2-3n}{2n} g^{-1}\partial_\mu g \partial_\nu g - \nabla_\nu \partial_\mu g - \left(\frac{1-n}{n} g^{-1}\partial_\alpha g \partial_\beta g g^{\alpha\beta} + \partial_\alpha\left(\partial_\beta g^{\alpha\beta}\right)\right) g_{\mu\nu}\right]$$

Einstein frame: $\sqrt{|g_e|}R[g_e] = |g|^{\frac{1}{n}}R$

$$g^{e}_{\mu\nu} \equiv g^{-\frac{1}{n}} g_{\mu\nu}$$

Einstein metric is unimodular

Unimodular variational principle would yield Einstein's 1919 EM (This is not what he had in mind)

A different viewpoint

$$L = -\frac{1}{2\kappa^2}\sqrt{|g|}\phi R + \sqrt{|g|}\chi\left(\phi - |g|^{\frac{2-n}{2n}}\right)$$

Owing to the auxiliary fields, we can introduce unconstrained Einstein metric:

$$\sqrt{|g_E|}R[g_E] = \sqrt{|g|}\phi R$$

$$g^E_{\mu\nu} = \phi^{\frac{2}{n-2}} g_{\mu\nu}$$

Scalar-tensor lagrangian

$$L = -\frac{1}{2\kappa^2}\phi\sqrt{|g|}R + \sqrt{|g|}\chi(\phi - |g|^{\frac{2-n}{2n}}) = -\frac{1}{2\kappa^2}\sqrt{|g_E|}R_E + \sqrt{|g_E|}\phi^{-\frac{2}{n-2}}\chi(1 - |g_E|^{\frac{2-n}{2n}}) + \\ + \frac{n-1}{2\kappa^2(n-2)}\left(2\partial_\mu\left(\sqrt{g_E}g_E^{\mu\nu}\frac{\partial_\nu\phi}{\phi}\right) - \sqrt{g_E}g_E^{\mu\nu}\frac{\partial_\mu\phi\partial_\nu\phi}{\phi^2}\right) \quad (3.1)$$

Quantum effects give Lagrange multipliers propagators scalar density ϕ. For example, for a scalar field Φ (not to be confused with the scalar density ϕ of gravitational origin),

$$L_I = |g_E|^{\frac{1}{n}} g_E^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi$$

Under conformal transformations in the old frame

$$\phi \rightarrow \Omega^{2-n} \phi$$

and for consistency,

$$\chi \rightarrow \Omega^{-2} \chi$$

whereas the unimodular Einstein metric is inert. What looks like a purely gravitational symmetry in one frame, looks like a *matter* symmetry in another. Potential energy coupled to gravitation is again forbidden, because they appear in the new frame as

$$\phi^{-\frac{2}{n-2}} V(\Phi)$$

$$L_2^{(1)} = \frac{4n^2}{(n-2)^2} \phi^{-2} g_E^{\mu\nu} \partial_\mu \phi \partial_\nu \phi$$

Conclusions

There are no natural models.

Too much arbitrariness?

More work is needed before it can be assessed whether TG is a useful alternative

Newtonian limit

$$S_T \equiv -m_T c \int f(g) ds \sim -m_T c \int f(1+\kappa|h|)\sqrt{1+\kappa h_{00} - \frac{v^2}{c^2}} dt$$

$$f(g) \equiv \frac{f_m(g)}{\sqrt{|g|}}$$

In order for the transverse action to get the correct Newtonian limit

$$S_{NR} = -m \int dt \left(c^2 - \frac{v^2}{2} + \Phi_N + \ldots \right)$$

Potential in terms of the metric

$$\boxed{m_T f(1) = m}$$

$$\boxed{\kappa h_{00} = \frac{2f(1)}{(f(1)+f'(1))c^2} \left(\Phi_N - \kappa c^2 \frac{f'(1)}{f(1)} \sum h^i_i \right)}$$

$$T_{00} = f(g)\rho_T c^2 = \left(f(1) + \kappa f'(1) h\right) \rho_T c^2$$

$$R_{00} \sim \sum \partial_i \Gamma^i_{00} \sim \frac{\kappa}{2} \Delta h_{00}$$

Einstein's equations

$$R_{00} = \frac{c\kappa^2}{2} T_{00} \qquad \Delta \Phi_N \sim \frac{f(1)}{2} c^3 \kappa^2 \rho_T$$

reduce to Poisson's equation provided

$$\kappa^2 = \frac{8\pi G}{c^3}$$

$$T_{00} = f(g)\rho_T c^2 = \left(f(1) + \kappa f'(1)h\right)\rho_T c^2$$

$$R_{00} \sim \sum \partial_i \Gamma^i_{00} \sim \frac{\kappa}{2}\Delta h_{00}$$

Einstein's equations

$$R_{00} = \frac{c\kappa^2}{2} T_{00} \qquad \Delta\Phi_N \sim \frac{f(1)}{2} c^3 \kappa^2 \rho_T$$

reduce to Poisson's equation provided

$$\kappa^2 = \frac{8\pi G}{c^3}$$

The matter part is Diff. invariant

$$\therefore$$

$$\sqrt{|\bar{g}|}\bar{\nabla}_\mu T_{\mu\nu} = \partial_\mu \Lambda$$

Integrability of Einstein's equations

$$\therefore$$

$$\bar{\nabla}_\mu T_{\mu\nu} = 0$$

$$\therefore$$

$$\boxed{\Lambda = const}$$

Just shifts height and position of minima of the potential

The full action, before multiplier condensation, is only TDiff invariant

WTDiff: Einstein's 1919

$$\boxed{R_{\mu\nu} - \frac{1}{n} R g_{\mu\nu} = \chi \left(T_{\mu\nu} - \frac{1}{n} T g_{\mu\nu} \right)}$$

Can't be obtained from an unconstrained variational principle

(Trace-free piece of EE; cc disappears)

Bianchi $\Rightarrow$ $\nabla^\nu R_{\mu\nu} = \frac{1}{2} \nabla_\mu R.$ $\Rightarrow$ $\frac{n-2}{2} \nabla_\mu R = -\frac{\kappa^2}{n} \nabla_\mu T$ $\Rightarrow$

$$\frac{n-2}{2d} R + \frac{2\kappa^2}{d} T = constant \equiv -\lambda \Rightarrow$$

cc reappears as an integration constant

$$\boxed{R_{\mu\nu} - \frac{1}{2}(R + 2\lambda) g_{\mu\nu} = \kappa^2 T_{\mu\nu}}$$

Some simple models:

$$S = \int d^n x \left(-\frac{1}{2\kappa^2} f(g) R + f_m(g) L_m(g_{\mu\nu}, \psi) \right)$$

$f_m(g) = 1$ (Extreme example)

Potential energy does not weigh. Solves the active CC problem (but creates others)

Tdiff viewed as Diff in the unitary gauge C=1

$$S = \int d^n x \frac{1}{C(x)} \left(-\frac{1}{2\kappa^2} f(gC^2) R + f_m(gC^2) L_m(g_{\mu\nu}, \psi) \right)$$

C=Compensator

Masses

$$S_m \equiv \int d^n x \, f_m(g) \sum_i \left(g^{\mu\nu}\partial_\mu \psi_i \partial^\nu \psi_i - V(\psi_i)\right)$$

Eikonal approximation

$$g^{\mu\nu} k_\mu k_\nu = m^2$$
$$\dot{k}_\mu = 0$$

Passive gravitational mass equal to inertial mass

$$\therefore$$

$$\boxed{m_p = m_I \equiv m}$$

Energy-momentum tensors

Active: $$\boxed{T^a_{\mu\nu} \equiv \frac{\delta S_m}{g^{\mu\nu}} = f_m \frac{\delta L_m}{\delta g^{\mu\nu}} - g f'_m L_m g_{\mu\nu}}$$

TDiff ➡ $$\nabla_\nu \left(\frac{T^\nu_\mu}{\sqrt{|g|}} \right) = \frac{1}{\sqrt{|g|}} \partial_\mu \Omega$$ (It is not automatically conserved)

Rosenfeld does not reduce to Belinfante in flat space

$$f_m(g) = 1$$

$$\boxed{T^{Bel}_{\mu\nu} \equiv \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} L_m \eta_{\mu\nu}}$$

(The second piece is missing in the active EMT)

Fluid approximation

$$m_a(GR) = T_{\mu\nu}u^\mu u^\nu \equiv \rho$$

$$m_a = \frac{f_m(g)}{\sqrt{|g|}}\rho + \left(\frac{f_m(g)}{\sqrt{|g|}} - 2\sqrt{|g|}f'_m(g)\right)p$$

$$\delta \equiv \frac{m_a - m_a(GR)}{m_a(GR)} = \frac{p}{\rho}\frac{f_m - 2gf'_m}{\sqrt{|g|}} + \frac{f_m - \sqrt{|g|}}{\sqrt{|g|}}$$

Experimental bound $\boxed{\Delta\delta \equiv \delta_1 - \delta_2 \leq 10^{-13}}$

Dipolar gravitational radiation

One-loop ultraviolet divergences

$$S = -\frac{1}{2\kappa^2}\int d^n x\sqrt{g_*}\left(f(g_*)R^* + 2f_\lambda(g_*)\Lambda + \frac{1}{2}f_\phi(g_*)g_*^{\mu\nu}\partial_\mu g_*\partial_\nu g\right.$$

Several changes of frame and variable

$$\phi_* \equiv g_* C^2$$

$$g_{\mu\nu} \equiv \Omega^2 g^*_{\mu\nu}$$

$$\Omega^{n-2} \equiv f(\phi_*)$$

$$F_\lambda(\Omega) \equiv \Omega^{-n} f_\lambda(f^{-1}(\Omega^{n-2}))$$

$$\left(\frac{2(n-1)(n-2)}{\Omega^2} - \Omega^{2-n} f_\phi(f^{-1}(\Omega^{n-2}))\left(\frac{\partial f^{-1}(\Omega^{n-2})}{\partial\Omega}\right)^2\right) g^{\mu\nu}\partial_\mu\Omega\partial_\nu\Omega \equiv g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$$

"Equivalent" scalar-tensor theory

$$S_g = -\frac{1}{2\kappa^2}\int d^n x\sqrt{g}\,(R + 2F_\lambda(\phi)\Lambda) + \frac{1}{2\kappa^2}\int d^n x\sqrt{g}\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$$

Counterterm:

$$\begin{aligned}\Delta S &= \frac{1}{\epsilon}\frac{1}{(4\pi)^2}\int d^4x\,\sqrt{g^*}\,\Bigg\{\frac{203}{160}\left[3f^{-2}\,f'^2+f^{-1}f_\phi\right]^2(g_*^{\mu\nu}\partial_\mu\varphi^*\partial_\nu\varphi^*)^2\\
&+\frac{57}{20}\Lambda\left[3f^{-3}\,f'^2\,f_\lambda+f^{-2}\,f_\lambda\,f_\phi\right]g_*^{\mu\nu}\partial_\mu\varphi^*\partial_\nu\varphi^*-\frac{57}{5}\Lambda^2\,f^{-2}\,f_\lambda^2\\
&+\frac{1}{3}\Lambda^2\left[f'^{-1}\,f'_\lambda-2f^{-1}\,f_\lambda\right]^2\left[3+f\,f'^{-2}\,f_\phi\right]^{-1}+\frac{1}{2}\Lambda^2\left[3f^{-1}+f'^{-2}\,f_\phi\right]^{-4}\\
&\times\Big[24f^{-3}\,f_\lambda-18f^{-2}\,f'^{-1}\,f'_\lambda-6f^{-1}\,f'^{-3}\,f''\,f'_\lambda+6f^{-1}\,f'^{-2}\,f''_\lambda+10f^{-2}\,f'^{-2}\,f_\lambda\,f_\phi\\
&-7f^{-1}\,f'^{-3}\,f'_\lambda\,f_\phi+2f^{-1}\,f'^{-3}\,f_\lambda\,f'_\phi-4f^{-1}\,f'^{-4}\,f''\,f_\lambda\,f_\phi+2f'^{-4}\,f''_\lambda\,f_\phi-f'^{-4}\,f'_\lambda\,f'_\phi\Big]\\
&\times\Big[12f^{-3}\,f_\lambda-18f^{-2}\,f'^{-1}\,f'_\lambda-6f^{-1}\,f'^{-3}\,f''\,f'_\lambda+6f^{-1}\,f'^{-2}\,f''_\lambda-2f^{-2}\,f'^{-2}\,f_\lambda\,f_\phi\\
&-7f^{-1}\,f'^{-3}\,f'_\lambda\,f_\phi+2f^{-1}\,f'^{-3}\,f_\lambda\,f'_\phi-4f^{-1}\,f'^{-4}\,f''\,f_\lambda\,f_\phi+2f'^{-4}\,f''_\lambda\,f_\phi-f'^{-4}\,f'_\lambda\,f'_\phi\\
&-\frac{4}{3}f^{-1}\,f'^{-4}\,f_\lambda\,f_\phi^2\Big]\Bigg\}\end{aligned}\tag{75}$$

When there is no cosmological constant we recover the old results of 't Hooft -Veltman

$$\Lambda = 0$$

$$f = f_\phi = 1$$

$$\begin{aligned}\Delta S &= \frac{1}{\epsilon}\frac{1}{(4\pi)^2}\int d^4x\,\sqrt{g^*}\,\frac{203}{160}(g_*^{\mu\nu}\partial_\mu\varphi^*\partial_\nu\varphi^*)^2\\
&=\frac{1}{\epsilon}\frac{1}{(4\pi)^2}\int d^4x\,\sqrt{g^*}\,\frac{203}{40}R^2\end{aligned}$$

There are only two cases in which the theory is one-loop finite on-shell (without cc)

$$2(n-1)f^{-1}\left(f'\right)^2-(n-2)f_\phi=0$$

1.-GR

Einstein, 1915

(Diff. invariance)

2.-WTDiff

$$f(g_*)=g_*^{\frac{2-n}{2n}}$$

(Weyl invariance)

Einstein, 1919

$$g(x)C^2 \equiv e^{\frac{2n}{\sqrt{(n-1)(n-2)}}\sigma(x)}$$

$$e^{\frac{2n}{\sqrt{(n-1)(n-2)}}\tilde{\sigma}(x)} = \Omega^{2n} \ e^{\frac{2n}{\sqrt{(n-1)(n-2)}}\sigma(x)} .$$

Scalar fields:

$$|g|^a \frac{1}{2} g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi$$

$$\Phi \to \Omega^{1-na} \Phi$$

No allowed Weyl invariant interactions with the measure

$$|g|^{\frac{1}{n}} d^n x$$

Nonminimal terms are allowed

$$\frac{c_p}{M^{\frac{p(n-2)+4-2n}{2}}} R \Phi^p \qquad m^2 \sim c_2 \bar{R}$$

Also allowed interactions decoupled from gravitation:

$$d^n x \, V(\Phi_i)$$

Potential energy does not weigh (an overkill)