Polylogs of roots of unity: the good, the bad and the ugly

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Abstract: I discuss conjectural enumerations, by weight and by depth, of irreducible multiple polylogarithms of Nth roots of unity.

Quantum field theory already requires us to study $N = 1, 2, 6$.

Mathematically speaking, the good cases are $N = 2, 3, 4, 6, 8$, for all of which I shall give compelling conjectures for concrete sets of irreducibles.

The bad case of multiple zeta values, with $N = 1$, is covered by the Broadhurst-Kreimer conjecture, which enumerates irreducibles but makes it very hard to choose them.

I shall give recent evidence that the cases $N = 5, 7$ are neither good nor bad, but just plain ugly.

Plan: Preface (Boltzmann); Good; Bad; Ugly; QFT; Envoi (Boltzmann)

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Preface

Bring' vor was wahr ist;
Schreib' so, dass es klar ist
Und verficht's, bis es mit Dir gar ist.

Propose what is true;
Write so that it is clear
And defend it to your death.

Ludwig Eduard Boltzmann (20 February 1844 – 5 September 1906)
Vorlesungen über die Prinzipe der Mechanik, 3 August 1897

Die Zeit, 7 September 1906: "He used a short cord from the crossbar of a window casement. His daughter was the first to discover the suicide."

The Stephan-Boltzmann constant involves $\zeta(4)$. The average energy per photon in black-body radiation at temperature T is

$$\overline{E} = \frac{\zeta(4)}{\zeta(3)}\,3kT$$

DB, *Feynman's sunshine numbers*, `arXiv:1004.4238`

1 The good: $N = 2, 3, 4, 6, 8$

7 letter alphabet: let $\lambda = \exp(2\pi i/6) = (1+i\sqrt{3})/2$, $\overline{\lambda} = (1-i\sqrt{3})/2$,

$$\begin{aligned}
A &= d\log(x) \\
B &= -d\log(1-x) \\
C &= -d\log(1+x) \\
D &= -d\log(1-\lambda x) \\
\overline{D} &= -d\log(1-\overline{\lambda} x) \\
E &= -d\log(1-\lambda^2 x) \\
\overline{E} &= -d\log(1-\overline{\lambda}^2 x)
\end{aligned}$$

investigated in `arXiv:hep-th/9803091` and `arXiv:1409.7204`
Subalphabets:
$\{A, B, C\}$, alternating sums, `arXiv:hep-th/9604128`
$\{A, B\}$, Multiple Zeta Values (MZVs), BK, `arXiv:hep-th/9609128`
$\{A, D\}$, Multiple Clausen Values (MCVs), BBK, `arXiv:hep-th/0004153`
$\{A, B, D\}$, Multiple Deligne Values (MDVs) follow MCVs
$\{A, B, E, \overline{E}\}$, solved by Pierre Deligne in
`http://www.math.ias.edu/files/deligne/121108Fondamental.pdf`

Weight, w, is the number of letters in a word.
Depth, d, is the number of letters not equal to A.
Iterated integrals: For example, at weight $w = 3$ and depth $d = 2$,

$$Z(DAB) \equiv \int_0^1 \frac{\lambda\, dx_1}{1 - \lambda x_1} \int_0^{x_1} \frac{dx_2}{x_2} \int_0^{x_2} \frac{dx_3}{1 - x_3}$$

Netsted sums: expand

$$-d \log(1 - \lambda^n x) = \frac{dx}{x} \sum_{k>0} (\lambda^n x)^k$$

to obtained nested sums of the form

$$S\begin{pmatrix} z_1, z_2, \ldots, z_d \\ a_1, a_2, \ldots, a_d \end{pmatrix} \equiv \sum_{k_1>k_2>\ldots>k_d>0} \prod_{j=1}^{d} \frac{z_j^{k_j}}{k_j^{a_j}}$$

where $z_j^6 = 1$ and a_j is a positive integer. Thus, for example,

$$Z(DAB) = S\begin{pmatrix} \lambda, \overline{\lambda} \\ 1, 2 \end{pmatrix} \equiv \sum_{m=1}^{\infty} \frac{\lambda^m}{m} \sum_{n=1}^{m-1} \frac{\overline{\lambda}^n}{n^2}$$

Shuffle product:
$$Z(U)Z(V)=\sum_{W\in\mathcal{S}(U,V)}Z(W)$$
where $\mathcal{S}(U,V)$ is the set of all words W that result from shuffling the words U and V. Thus, for example,
$$\begin{aligned}Z(AB)Z(CD) &= Z(ABCD)+Z(ACBD)+Z(ACDB)\\ &+ Z(CABD)+Z(CADB)+Z(CDAB)\end{aligned}$$
preserves the order of letters in $U=AB$ and $V=CD$.

Stuffle product: the **full** 7-letter alphabet $\{A,B,C,D,\overline{D},E,\overline{E}\}$ has a stuffle algebra, resulting from shuffling the arguments of nested sums, with extra **stuff** when indices of summation coincide. For example
$$\begin{aligned}Z(AB)Z(D) &= S\binom{1}{2}S\binom{\lambda}{1}=S\binom{1,\lambda}{2,1}+S\binom{\lambda,1}{1,2}+S\binom{\lambda}{3}\\ &= Z(ABD)+Z(DAD)+Z(AAD)\end{aligned}$$
Alphabets $\{A,B\}$, $\{A,B,C\}$, $\{A,B,E,\overline{E}\}$ $\{A,B,C,D,\overline{D},E,\overline{E}\}$ have a **double shuffle** algeba, but the Deligne alphabet $\{A,B,D\}$ is **not** closed under stuffles: $Z(AD)Z(D)=Z(ADE)+Z(DAE)+Z(AAE)$.

Enumerations of primitives by weight and depth

A word W is a **primitive** in a **given alphabet** if $Z(W)$ can **not** be expressed as a **Q**-linear combination of terms of **lesser** depth, powers of $(2\pi i)$, or their products.

Example [Broadhurst, 1996]: At weight $w = 12$ and depth $d = 4$,

$$Z(A^3BA^3BABAB) = \zeta(4,4,2,2) = \sum_{j>k>l>m>0} 1/(j^4k^4l^2m^2)$$

is a primitive MZV, but is **not** primitive in the $\{A, B, C\}$ alphabet, because

$$\begin{aligned}
&2^5\cdot 3^3 Z(A^3BA^3BABAB) - 2^{14} Z(A^8CA^2B) = \\
&\quad 2^5\cdot 3^2\,\zeta^4(3) + 2^6\cdot 3^3\cdot 5\cdot 13\,\zeta(9)\,\zeta(3) + 2^6\cdot 3^3\cdot 7\cdot 13\,\zeta(7)\,\zeta(5) \\
&+ 2^7\cdot 3^5\,\zeta(7)\,\zeta(3)\,\zeta(2) + 2^6\cdot 3^5\,\zeta^2(5)\,\zeta(2) - 2^6\cdot 3^3\cdot 5\cdot 7\,\zeta(5)\,\zeta(4)\,\zeta(3) \\
&- 2^8\cdot 3^2\,\zeta(6)\,\zeta^2(3) - \frac{13177\cdot 15991}{691}\,\zeta(12) \\
&+ 2^4\cdot 3^3\cdot 5\cdot 7\,\zeta(6,2)\,\zeta(4) - 2^7\cdot 3^3\,\zeta(8,2)\,\zeta(2) - 2^6\cdot 3^2\cdot 11^2\,\zeta(10,2)
\end{aligned}$$

where $Z(A^8CA^2B) = \sum_{m>n>0}(-1)^{m+n}/(m^9n^3)$ has depth 2.

For a **given** alphabet, let $N_{w,d}$ be the dimension of the space of $\mathbf{Q}$-linearly independent primitives of weight w and depth d. We then seek

$$H(x,y) = \prod_{w>0} \prod_{d>0} (1 - x^w y^d)^{N_{w,d}}$$

The **good** cases are those where everything is determined by the **single** sums, with depth $d = 1$, in the simplest manner imaginable:

$$H(x,y) = 1 - y \sum_{w>0} N_{w,1} x^n$$

$\{A, C\} : 1 - xy/(1 - x^2)$
from $Z(A^{2n}C)$, with $n \geq 0$, and the same for $\{A, B, C\}$.

$\{A, D\} : 1 - x^2 y/(1 - x)$
from $Z(A^n D)$ with $n > 0$, and the same for $\{A, B, D\}$.

$\{A, E\}$: $1 - xy/(1 - x)$
from $Z(A^n E)$, with $n \geq 0$, and the same for $\{A, B, E\}$, for $\{A, B, D, E\}$ and for $\{A, B, E, \overline{E}\}$.

$\{A, B, C, D, \overline{D}, E, \overline{E}\} : 1 - xy - xy/(1 - x)$
from $Z(C)$ and $Z(A^n E)$ with $n \geq 0$.

PS: 4th roots: $1 - xy/(1 - x)$; 8th roots: $1 - 2xy/(1 - x)$.

Lyndon words provide primitives in all cases except for MZVs. A Lyndon word is a word W such that for every splitting $W = UV$ we have U coming before V, in lexicographic ordering.

Alternating sums in the $\{A, B, C\}$ alphabet [Broadhurst, 1997]:
take Lyndon words in the $\{A, C\}$ alphabet and retain those with even powers of A. With $w \leq 5$ this gives C, A^2C, A^2C^2, A^4C, A^2C^3.

MDVs in the $\{A, B, D\}$ alphabet [Deligne, 2010]:
take Lyndon words in the $\{A, D\}$ alphabet and retain those in which D is preceded by A. With $w \leq 5$ this gives AD, A^2D, A^3D, A^4D, A^2DAD.

7-letter alphabet of polylogs of 6th roots of unity [Broadhurst, 2014]:
take Lyndon words in the $\{A, E, C\}$ alphabet, omit A and all words in which C is preceded by A. With $w \leq 5$ this gives E, C, AE, EC, AAE, AEE, AEC, EEC, ECC, $AAAE$, $AAEE$, $AAEC$, $AEEE$, $AEEC$, $AECE$, $AECC$, $EEEC$, $EECC$, $ECCC$, $AAAAE$, $AAAEE$, $AAAEC$, $AAEAE$, $AAEEE$, $AAEEC$, $AAECE$, $AAECC$, $AEAEE$, $AEAEC$, $AEEEE$, $AEEEC$, $AEECE$, $AEECC$, $AECEE$, $AECEC$, $AECCE$, $AECCC$, $EEEEC$, $EEECC$, $EECEC$, $EECCC$, $ECECC$, $ECCCC$.

Cube roots: Lyndon words in $\{A, E\}$
4th roots: Lyndon words in $\{A, -d\log(1-ix)\}$
8th roots: Lyndon words in $\{A, -d\log(1-\sqrt{i}x), -d\log(1+\sqrt{i}x)\}$

Generalized parity conjecture [Broadhurst, 1999]:
the primitives may be taken as real parts of $Z(W)$ for which the parities of weight and depth of W coincide and as imaginary parts if they differ.

A **legal** word does not begin with B or end in A.

Statistics for empirical reductions to conjectured bases:

MDVs of the $\{A, B, D\}$ alphabet: all 118,097 legal words with $w \leq 11$.

$\{A, B, E, \overline{E}\}$: 12,287 words with $w \leq 7$.

$\{A, B, D, E\}$: 12,287 words with $w \leq 7$.

$\{A, B, C, D, \overline{D}, E, \overline{E}\}$: 28,265 words with $w \leq 5$ or $w = 6$ and $d \leq 4$.

PS: 4th roots: 62,499 words; 8th roots: 23,815 words.

MDV datamine with 13,369,520 non-zero rational coefficients:
`http://physics.open.ac.uk/~dbroadhu/cert/MDV.tar.gz`
explained in `http://arxiv.org/pdf/1409.7204v1`

Dimensions of vector spaces

Suppose that we ignore the depth, d. What is the dimension $D(w)$ of the **vector space** of polylogs in one of these good alphabets?
For $N > 2$, how does the generalized parity conjecture **split** it into $D(w) = D_R(w) + D_I(w)$, for the real and imaginary parts?

In 1996, I gave the answer for the $\{A, B, C\}$ alphabet:
$D(w) = F_{w+1}$, where F_n is the n-th **Fibonacci** number.

In 2000, I gave the same answer for the $\{A, D\}$ alphabet. Moreover this also applies for the $\{A, B, D\}$ alphabet. The splits are
$D_R(w) = (F_{w+1} + \chi_3(w+1))/2$,
$D_I(w) = (F_{w+1} - \chi_3(w+1))/2$
where $\chi_3(n) = \chi_3(n+3)$, $\chi_3(0) = 0$, $\chi_3(\pm 1) = \pm 1$.

In 2014, I found the answers for the full 7-letter alphabet:
$D_R(w) = (F_{2w+2} + F_{w+1})/2$,
$D_I(w) = (F_{2w+2} - F_{w+1})/2$.

For 3rd and 4th roots of unity, the answers are $D_R(w) = D_I(w) = 2^{w-1}$.
For 8th roots, $D_R(w) = (3^w + 1)/2$, $D_I(w) = (3^w - 1)/2$.

2 The intriguingly bad case: $N = 1$

According to the Broadhurst-Kreimer conjecture (1997), the answer is **bizarre** for the $\{A, B\}$ alphabet of MZVs:

$$\prod_{w>0}\prod_{d>0}(1 - x^w y^d)^{N_{w,d}} = 1 - \frac{x^3 y}{1 - x^2} + \frac{x^{12} y^2 (1 - y^2)}{(1 - x^4)(1 - x^6)}$$

is not determine by $N_{w,1}$ but has a final term that counts **cuspforms**.

The simpler formula

$$\prod_{w>0}\prod_{d>0}(1 - x^w y^d)^{M_{w,d}} = 1 - \frac{x^3 y}{1 - x^2}$$

counts something different: the numbers $M_{w,d}$ of primitive **alternating** sums of weight w and depth d that furnish an algebra basis for MZVs. So the situation at $w = 12$ and $d = 4$ becomes clearer if we replace the MZV $\zeta(4, 4, 2, 2)$ by the alternating **double** sum $\sum_{m>n>0}(-1)^{m+n}/(m^9 n^3)$.

At $y = 1$ we obtain the vector-space dimensions $D(w)$ of MZVs

$$1 + \sum_{w>0} D(w) x^w = 1/(1 - x^2 - x^3)$$

as **Padovan** numbers enumerating MZVs with arguments 2 or 3.

Conjecture at depth 2

The BK conjecture links the enumeration of primitive MZVs to the enumeration of cuspforms.

Let M_w be the dimensionality of the space of cuspforms of weight w for the full modular group. Then

$$\sum_w M_w x^w = \frac{x^{12}}{(1-x^4)(1-x^6)}$$

Here is a recent conjecture which assigns a set of M_w rational numbers to the set of cuspforms of weight w.

Conjecture:
For even weight w, there exists a *unique* $\mathbf{Q}$-linear combination

$$Y_w = 3^{w-4}\Re Z(A^{w-2}D^2) + \sum_{k=1}^{M_w} Q_{w,k} Z(A^{w-2k-2}CA^{2k}B),$$

with rational coefficients $Q_{w,k}$, such that Y_w reduces to depth-2 MZVs.

These are the reductions up to $w = 10$, where there are no cuspforms:

$$\begin{aligned}
\Re Z(D^2) &= -\frac{1}{3}\zeta_2 \\
\Re Z(A^2 D^2) &= -\frac{23}{216}\zeta_4 \\
\Re Z(A^4 D^2) &= \frac{209}{972}\zeta_6 - \frac{1}{6}\zeta_3^2 \\
\Re Z(A^6 D^2) &= \frac{799331}{1399680}\zeta_8 - \frac{25}{54}\zeta_5\zeta_3 - \frac{7}{270}\zeta_{5,3} \\
\Re Z(A^8 D^2) &= \frac{31013285}{35271936}\zeta_{10} - \frac{535}{2016}\zeta_5^2 - \frac{637}{1296}\zeta_7\zeta_3 - \frac{205}{18144}\zeta_{7,3}
\end{aligned}$$

where $\zeta_{a,b} \equiv \sum_{m>n>0} 1/(m^a n^b)$.

At $w = 14$ we have also have a reduction to depth-2 MZVs:

$$\begin{aligned}
6^{10}\Re Z(A^{12} D^2) &= \frac{45336887777}{594}\zeta_{14} - 30203052\zeta_{11}\zeta_3 - \frac{292990340}{11}\zeta_9\zeta_5 \\
&\quad - \frac{400333213}{33}\zeta_7^2 + \frac{19112030}{33}\zeta_{11,3} - \frac{1938020}{9}\zeta_{9,5}.
\end{aligned}$$

At $w = 12$ and even $w > 14$, alternating sums of depth $d = 2$ are needed.

The story up to weight 36 is as follows:

```
[12, [256]]
[16, [19840]]
[18, [184000]]
[20, [1630720]]
[22, [14728000]]
[24, [165988480, 10183680]]
[26, [51270856000/43]]
[28, [13389295360, 808012800]]
[30, [1573506088000/13, 96652800000/13]]
[32, [1085492600192, 65740846080]]
[34, [3003044404360000/307, 182805638400000/307]]
[36, [95110629053440, 8048874470400, 410097254400]]
```

where the first entry in each line is the weight w and thereafter I give a vector of empirically determined rational numbers, $Q_{w,k}$.

3 The ugly cases: $N = 5, 7$

As for MZVs, with $N = 1$, the cases $N = 5, 7$ are not good: they have fewer depth-2 primitives than would be expected on the basis of the depth-1 primitives.

Ugliness at $N = 5$

The generating function

$$H_5(x, y) = 1 - \frac{2xy}{1-x} + \frac{x^2y^2}{1-x^2} + O(y^3)$$

gives the empirical numbers of primitives for $d < 3$ and $w < 13$.

Recall that BK asserted that

$$H_1(x, y) = 1 - \frac{x^3y}{1-x^2} + \frac{x^{12}y^2(1-y^2)}{(1-x^4)(1-x^6)}$$

Now imagine that there was also no y^3 term in $H_5(x, y)$.
Then the dimensions of the vector spaces for the real and imaginary imaginary parts for $N = 5$ and $d < 4$ would be as follows:
$[10, 12], [23, 28], [50, 39], [58, 84]$, for $w = 3, 4, 5, 6$, respectively.

False hope: I did in fact succeed in reducing all the 5th-root depth-3 polylogs with $w < 7$ to bases of precisely these sizes.

But then, just to be careful, I studied the prediction at $w = 7$, which was that the dimensions would be 122 and 82 for the real and imaginary parts.

This is correct for the imaginary parts, but not the real parts, for which the empirical dimension is 121: there is **one less** primitive than expected at $w = 7$ and $d = 3$. Thus a term x^7y^3 must be **added** to $H_5(x, y)$.

Moreover I found that a term x^4y^4 must be **subtracted** and then that a term x^5y^5 must be **added** to agree with the empirical dimensions at weights $w < 6$.

So I am left with **no viable conjecture** for the generating function.

More ugliness at $N = 7$

Here even the depth-2 fit looks ugly:

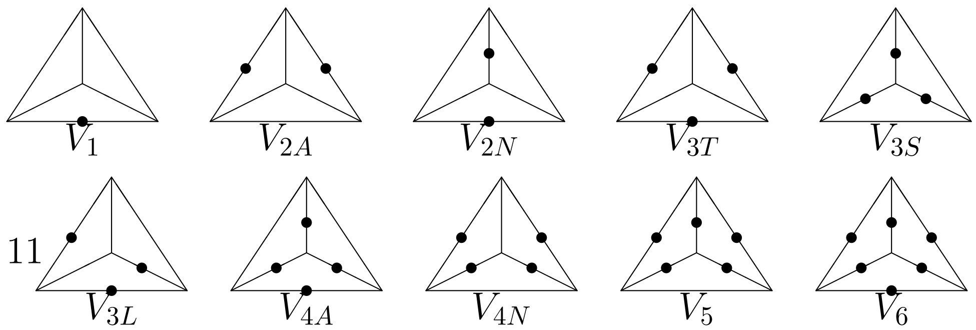
$$H_7(x, y) = 1 - \frac{3xy}{1-x} + \frac{x^2y^2}{1-x^2}\frac{2-x}{1-x} + O(y^3)$$

Moreover a term $3x^4y^4$ must be added. So again I gave up.

4 Sixth roots of unity in QFT

In quantum chromodynamics (QCD) and quantum electrodynamics (QED) we readily find polylogs of Nth roots of unity for $N = 1, 2$. Here I consider $N = 6$.

Colourings of the tetrahedron by mass:



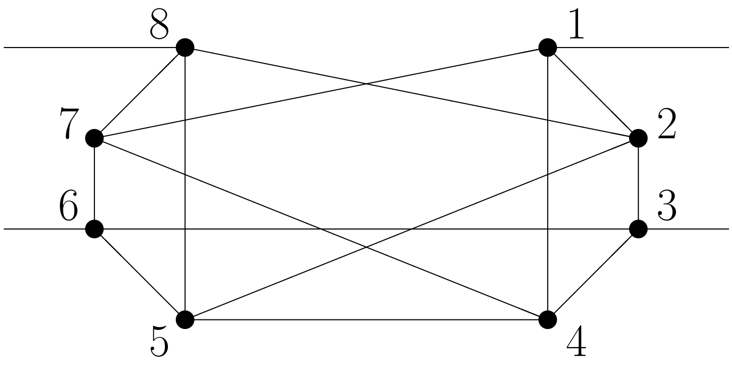
with finite parts in $D = 4 - 2\varepsilon$ dimensions, found in 1999, of the form

$$\begin{aligned}\overline{V}_j &= \lim_{\varepsilon \to 0} \left(V_j - \frac{6\zeta(3)}{3\varepsilon} \right) \\ &= 6\zeta(3) + z_j\,\zeta(4) + u_j\,Z(A^2CB) + s_j\,\left[\Im Z(AD)\right]^2 + v_j\,\Re Z(A^2CD)\end{aligned}$$

with rational coefficients of 4 terms at weight $w = 4$ and depth $d \leq 2$.

V_j	z_j	u_j	s_j	v_j	$\overline{V}_j$
V_1	3				10.4593111200909802869464400586922036529141
V_{2A}	-5				1.8007252504018747548184104863628604307161
V_{2N}	$-\frac{13}{2}$	-8			1.1202483970392420822725165482242095262757
V_{3T}	-9				$-2.5285676844426780112456042998018111803828$
V_{3S}	$-\frac{11}{2}$		-4		$-2.8608622241393273502727845677732419175614$
V_{3L}	$-\frac{15}{4}$		-6		$-3.0270094939876520197863747017589572861507$
V_{4A}	$-\frac{77}{12}$		-6		$-5.9132047838840205304957178925354050268834$
V_{4N}	-14	-16			$-6.0541678585902197393693995691614487948131$
V_5	$-\frac{469}{27}$		$\frac{8}{3}$	-16	$-8.2168598175087380629133983386010858249695$
V_6	-13	-8	-4		$-10.0352784797687891719147006851589002386503$

Comment: The 5-mass case led me to investigate the full 7-letter alphabet at $w = 4$ and $d = 2$, where I found that there are precisely 2 primitives, here taken as $Z(A^2CB)$ and $\Re Z(A^2CD)$.



A 7-loop diagram in ϕ^4 theory

This counterterm for this diagram is the 11th in the 7-loop list of the census of Oliver Schnetz and is there called the period $P_{7,11}$. All other periods of ϕ^4 theory to 7 loops reduce to MZVs; only $P_{7,11}$ requires MDVs. Erik Panzer has reduced $\sqrt{3}P_{7,11}$ to imaginary parts of sums of the form

$$S\binom{z_1, \ldots, z_d}{a_1, \ldots, a_d}$$

with $z_1 = \lambda$, $z_j = \pm 1$, for $j > 1$, and weight $\sum_j a_j = 11$.

These nested sums correspond to 39,366 words in the alphabet $\{A, D, \overline{E}\}$, of which 4,589 were present in the reduction. After evaluating each term to 5,000 decimal digits, he was able to find an empirical reduction to the Lyndon basis for MDVs given by Deligne, which has 72 terms, according to the generalized parity conjecture.

But then a nasty thing emerged. The rational coefficient of $\pi^{11}/\sqrt{3}$ in his result for $P_{7,11}$ was

$$C_{11} = -\frac{964259961464176555529722140887}{2733669078108291387021448260000}$$

$\{A, B, E\}$, whose **denominator** contains 8 primes greater than 11, namely 19, 31, 37, 43, 71, 73, 50909 and 121577.

Schnetz obtained an alternative expression with a coefficient of $\pi^{11}/\sqrt{3}$ that has a 48-digit denominator containing Panzer's 8 primes, above, and four new ones, namely 47, 2111, 14929 and 24137.

My recent work on MDVs concerns the origin of such undesirable denominator-primes and provides an **Aufbau** that has **no** prime greater than 11 in the denominators of the 13,369,520 non-zero rational coefficients of the **datamine** for the 118,097 MDVs with weights $w \leq 11$.

The datamine yielded considerable simplification of Panzer's result. Let

$$W_{m,n} \equiv \sum_{k=0}^{n-1} \frac{\zeta_3^k}{k!} A^{m-2k} D^{n-k}$$

$P_n \equiv (\pi/3)^n/n!$, $I_n \equiv \mathrm{Cl}_n(2\pi/3)$ and $I_{a,b} \equiv \Im Z(A^{b-a-1}DA^{2a-1}B)$. Then

$$\begin{aligned}
\sqrt{3}P_{7,11} &= -10080\Im Z(W_{7,4}+W_{7,2}P_2)+50400\zeta_3\zeta_5P_3\\
&+ \left(35280\Re Z(W_{8,2})+\frac{46130}{9}\zeta_3\zeta_7+17640\zeta_5^2\right)P_1\\
&- 13277952T_{2,9}-7799049T_{3,8}+\frac{6765337}{2}I_{4,7}-\frac{583765}{6}I_{5,6}\\
&- \frac{121905}{4}\zeta_3I_8-93555\zeta_5I_6-102060\zeta_7I_4-141120\zeta_9I_2\\
&+ \frac{42452687872649}{6}P_{11}
\end{aligned}$$

with the datamine transformations

$$\begin{aligned}
I_{2,9} &= 91(11T_{2,9})-898T_{3,8}+11I_{4,7}-292P_{11}\\
I_{3,8} &= 24(11T_{2,9})+841T_{3,8}-190I_{4,7}-255P_{11}
\end{aligned}$$

removing denominator primes greater than 3.

Envoi

1. MZVs seem to be unique in being neither good nor ugly.
2. MDVs are radically different from alternating sums in the $\{A, B, C\}$ alphabet, since the $\{A, B, D\}$ alphabet is not closed under stuffles.
3. Panzer and Schnetz adopted a Deligne basis that generates gratuitously large primes in denominators.
4. Denominator primes greater than 11 are avoided in the MDV datamine.
5. Simplification of Panzer's result for the counterterm $P_{7,11}$ depended crucially on the new datamine, which revealed a notable Taylor-like expansion at depth 4.
6. I conjecture that a single MDV assigns a unique set of rational numbers to a set of cuspforms with the same cardinality. This seems to me to be worthy of further investigation.

On the significance of theory

Ich nannte die Theorie ein rein geistiges inneres Abbild und wir sahen, welch hoher Vollendung dasselbe fähig ist…

So kann es dem Mathematiker geschehen, dass er, fortwährend beschäftigt mit seinen Formeln und geblendet durch ihre innere Vollkommenheit, die Wechselbeziehungen derselben zueinander für das eigentlich Existierende nimmt und sich von der realen Welt abwendet.

I called theory a purely mental inner picture and we saw to what a high degree of perfection this may be brought…

Thus it may happen that a mathematician who is constantly occupied with formulae and blinded by their inner perfection may take those mutual relations as actually existing and turn away from the real world.

Ludwig Boltzmann, 16 July 1890, *Über die Bedeutung von Theorien*, an address delivered at Graz on departing for a chair in Munich.