# From equivariant quantization to locally compact quantum groups

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# I Equivariant quantization and its derived products

## The general strategy

**[Equivariant quantization in the differentiable setting]**

- $(M, \omega)$ symplectic manifold
- $\widetilde{G}$ Lie sub-group of the group of symplectomorphisms
- $(\mathcal{H}_\pi, \pi)$ projective irreducible unitary representation of $\widetilde{G}$

**Definition:** A $\widetilde{G}$-equivariant quantization map on $M$ is a continuous linear map

$$\mathbf{\Omega} : C_c^\infty(M) \to \mathcal{B}(\mathcal{H}_\pi)$$

satisfying the covariance property:

$$\pi(g)\,\mathbf{\Omega}(f)\,\pi(g)^* = \mathbf{\Omega}(f^g)\,, \quad f^g := \Big[x \in M \mapsto f(g^{-1}.x)\Big]$$

**Paradigmatic example:** Stratonowich quantization

$$\boldsymbol{\Omega}(f) := \int_M f(x)\,\Omega(x)\,d\mu(x)$$

- $\{\Omega(x)\}_{x\in M} \subset \mathcal{B}(\mathcal{H}_\pi)_{\mathsf{sa}}$ satisfying the covariance property

$$\pi(g)\,\Omega(x)\,\pi(g)^* = \Omega(g.x)$$

- $d\mu$ Liouville measure on $M$

Most of the known examples of quantization are of this form:
Weyl quantization, Berezin quantization, Coherent states quantization, Unterberger's Fuch calculus, BCH quantization of coadjoint orbits of exponential Lie groups.....

To connect equivariant quantization with quantum groups, we need one extra geometric assumption:

**(H1):** $\tilde{G}$ possesses a subgroup $G$ acting simply transitively on $M$ and $\pi|_G$ still irreducible

Under the identification $G \simeq M$:

- $G$ is endowed with a left invariant symplectic structure
- The Liouville measure $d\mu$ becomes a left Haar measure on $G$
- The quantizers read $\Omega(g) = \pi(g) \, \Sigma \, \pi(g)^*$ with $\Sigma := \Omega(e)$

**HENCE:** $G$-equivariant Stratonowich quantization on symplectic Lie group $G$ always of the form

$$\mathbf{\Omega}(f) := \int_G f(g)\, \pi(g)\, \Sigma\, \pi(g)^*\, d^\lambda(g)$$

where $\pi \in \widehat{G}_{\text{proj}}$, $\Sigma \in \mathcal{B}(\mathcal{H}_\pi)$

**BUT:** In the formula above symplectic geometry disappeared from the picture and provides an ansatz to construct equivariant quantization on general locally compact groups

- In bad examples (e.g. Berezin), $\Sigma \in \mathcal{K}(\mathcal{H}_\pi)_+$
- In good example (e.g. Weyl, Unterberger, BCH), $\Sigma \in \mathcal{U}(\mathcal{H}_\pi)_{\text{sa}}$
- Typically, $\mathcal{H}_\pi = L^2(Q, \nu)$, $\sigma$ involution on $Q$

$$\Sigma\varphi(q) = \text{Jac}_\Psi^{1/2}\, \varphi(\sigma(q))$$

**[Equivariant quantization in the general setting]**

• Fix $G$ locally compact group, $\pi$ projective unitary representation, $\Sigma$ self-adjoint operator on $\mathcal{H}_\pi$

**(H2):** $\mathbf{\Omega}$ extends to a unitary operator from $L^2(G)$ to HS($\mathcal{H}_\pi$)

**THEN:**

• Well-defined associative left-equivariant product [star-product]

$$\star : L^2(G) \times L^2(G) \to L^2(G)\,, \quad (f_1, f_2) \mapsto \mathbf{\Omega}^*\Big(\mathbf{\Omega}(f_1)\,\mathbf{\Omega}(f_2)\Big)$$

• $\star$ is associated with a Bruhat distribution $K$ on $G \times G$:

$$f_1 \star f_2 = \int_{G\times G} K(g_1, g_2)\,\rho_{g_1}(f_1)\,\rho_{g_2}(f_2)\,d^\lambda(g_1)\,d^\lambda(g_2)$$

• The tow-point kernel reads [trace in the distributional sense]

$$K(g_1, g_2) = \mathrm{Tr}\left(\Sigma\,\pi(g_1)\,\Sigma\,\pi(g_1^{-1} g_2)\,\Sigma\,\pi(g_2^{-1})\right)$$

## Derived product #1: dual unitary 2-cocycle

**Def:** Let $\mathbb{G}$ a locally compact quantum group. A dual unitary 2-cocycle for $\mathbb{G}$ is an element $F \in \mathcal{U}(\widehat{\mathcal{N}} \bar{\otimes} \widehat{\mathcal{N}})$ satisfying

$$(F \otimes 1)(\hat{\Delta} \otimes \mathrm{Id})(F) = (1 \otimes F)(\mathrm{Id} \otimes \hat{\Delta})(F)$$

- Natural candidate for a unitary 2-cocycle for $\mathbb{G} := W^*(G)$:

$$F_\lambda := \int_{G\times G} \overline{K(g_1, g_2)}\, \lambda_{g_1^{-1}} \otimes \lambda_{g_2^{-1}}\, d^\lambda(g_1)\, d^\lambda(g_2)$$

- 2-cocyclicity is automatic: it is equivalent to associativity of $\star$

- Unitarity has to be checked: matter of domains and computations

## Derived product #2: multiplicative unitary

**Def:** Let $\mathcal{H}$ be a Hilbert space. A multiplicative unitary on $\mathcal{H}$ is an element $W \in \mathcal{U}(\mathcal{H} \otimes \mathcal{H})$ satisfying the pentagonal equation:

$$W_{23}W_{12} = W_{12}W_{13}W_{23}$$

• $W$ is regular (Baaj-Skandalis) if

$$C^*\langle \omega \otimes \mathrm{Id}(W\Sigma)\,,\ \omega \in \mathcal{B}(\mathcal{H})_*\rangle = \mathcal{K}(\mathcal{H})$$

• $W$ is manageable (Woronowicz) if there exists $0 \leq Q$ densely defined with densely defined inverse and a unitary $\widetilde{W}$ on $\overline{\mathcal{H}} \otimes \mathcal{H}$ such that $W^*(Q \otimes Q)W = Q \otimes Q$ and

$$\langle \varphi_1 \otimes \varphi_2, W\varphi_3 \otimes \varphi_4\rangle = \langle \bar{\varphi}_3 \otimes Q\varphi_2, \widetilde{W}(\bar{\varphi}_1 \otimes Q^{-1}\varphi_4)\rangle$$

Both notion leads to a pair of quantum groups in duality

**(H3)**: The operator $F_\lambda$ is invertible (on a suitable domain)

- The doubly star-product (on a suitable domain)

$$f_1 \star_{\lambda,\rho} f_2 := m \circ {F_\lambda}^{-1} \circ F_\rho(f_1 \otimes f_2)$$

- Natural candidate for a multiplicative unitary:

$$W(\varphi_1 \otimes \varphi_2) = \Delta(\varphi_1)\Big( \star_{\lambda,\rho} \otimes \star_{\lambda,\rho} \Big)(1 \otimes \varphi_2)$$

- Pentagonal equation is automatic
- Unitarity on $\mathcal{H}$, the Hilbert space completion of $\mathcal{D}(G)$ for

$$\langle \varphi_1, \varphi_2 \rangle := \int_G \overline{\varphi_1} \star_{\lambda,\rho} \varphi_2(g)\, d^\rho(g)$$

Provided it is an inner product! True when $\delta_G = \delta_G^{1/2} \star \delta_G^{1/2}$

- Manageability/Regularity has to be proven...

# Derived product #3: deformation of $C^*$-algebras

## Derived product #4: scalar Fourier transform on $G$

## II Examples #1: Negatively curved Kählerian Lie groups

Joint with P. Bieliavsky, to appear in MAMS + work in progress with Bieliavsky, Bonneau, D'Andrea

## Structure of Kählerian Lie groups

**Theorem [Pyatetskii-Shapiro]**: The Kählerian Lie groups with negative sectional curvature are of the form:

$$G = \Big(\dots\Big(S_N \ltimes S_{N-1}\Big) \ltimes \cdots \ltimes S_2\Big) \ltimes S_1$$

where $S_j = AN$, Iwasawa of $SU(1,n) = KAN$, $N = H(V, \omega_0)$

- $G$ is non-unimodular, solvable and exponential

- $S_j$ is a symplectic symmetric space and possesses a midpoint map

- $S_j = Q_j \ltimes Y_j$, $Q_j$ also a symmetric space with midpoint, $Y_j$ Abelian

- $S_j$ has two (classes of) UNIRREPs $\pi_\pm$ on $L^2(Q_j)$

**Def:** (1) A symmetric space is a manifold $M$ with $s : M \times M \to M$
(i) $\forall x \in M$, the partial map

$$s_x : M \to M, \quad y \mapsto s(x,y)$$

is a smooth involution of $M$ that admits $x$ as isolated fixed point
(ii) $\forall x, y \in M$, the following identity holds:

$$s_x \circ s_y \circ s_x \; = \; s_{s(x,y)}$$

(2) A symplectic symmetric space is a symmetric space $(M, s)$ with an invariant symplectic form $\omega$:

$$s_x^\star \omega \; = \; \omega \,, \quad \forall x \in M$$

(3) A midpoint map on a symmetric space $(M, s)$:

$$M \times M \to M \,, \quad (x,y) \mapsto \mathrm{mid}(x,y) \quad \text{such that} \quad s_{\mathrm{mid}(x,y)}(x) = y$$

## Quantization of Kähler ian Lie groups

- Set $\Sigma\varphi(q) = \varphi(\underline{s}_e(q))$ on $L^2(Q)$, $S = Q \ltimes Y$

**Theorem**: (i) The map

$$g \in S \mapsto \Omega(g) := \pi(g)\,\Sigma\,\pi(g)^*$$

is a covariant irreducible unitary representation of $(S, \omega, s)$:

$$\pi(g)\,\Omega(g')\,\pi(g)^* = \Omega(gg')$$
$$\mathsf{VN}\Big(\Omega(x),\ x \in S\Big)' = \mathbb{C}$$
$$\Omega(g)^* = \Omega(g), \quad \Omega(g)^2 = \mathsf{Id}, \quad \Omega(g)\,\Omega(g')\,\Omega(g) = \Omega(s(g, g'))$$

(Covariance under $\mathsf{Aut}(S, \omega, s)$, which contains $\mathbb{S}$ and $Sp(V, \omega_0)$)

(ii) The modified quantization map

$$\widetilde{\boldsymbol{\Omega}} : \mathcal{D}(S) \to \mathcal{L}(\mathcal{D}(Q)) \,, \qquad f \mapsto \int_S f(g)\, \widetilde{\Omega}_S(g)\, d^\lambda(g)$$

$$\widetilde{\Omega}_S(g) := \pi(g) \circ \mathsf{Jac}^{1/2}_{\mathsf{mid}_Q} \circ \mathsf{Jac}^{1/2}_{\boldsymbol{\Psi}} \circ \boldsymbol{\Sigma} \circ \pi(g)^*$$

extends to a unitary operator from $L^2(S)$ onto $\mathsf{HS}(L^2(Q))$

(iii) The two-point kernel reads

$$K_S(g, g') = \mathsf{Jac}^{1/2}_{\Phi_S^{-1}}(e, g, g') \exp\left\{ i\theta \mathsf{Area}\left(\Phi^{-1}(e, g, g')\right)\right\}$$

where $\Phi_S \in \mathsf{Diff}(S \times S \times S)$

$$\Phi_S(g, g', g'') = \Big(\mathsf{mid}_S(g, g'), \mathsf{mid}_S(g', g''), \mathsf{mid}_S(g'', g)\Big)$$

• Let $G = S_2 \ltimes S_1$, parametrize $g \in G$ as $g = g_1 g_2$, $g_j \in S_j$ and set

$$\Omega_G(g) := \Omega_2(g_2) \otimes \Omega_1(g_1)$$

• $G$ acts on $L^2(Q_2 \times Q_1)$: $\pi(g_2 g_1) := \pi_2(g_2) \otimes \pi_1(\rho(g_2) g_1)$

**Theorem**: (1) The associated quantization map

$$\boldsymbol{\Omega}_G(f) := \int_G f(g)\, \Omega_G(g)\, d^\lambda(g)$$

is a unitary operator from $L^2(G)$ to $\mathcal{L}^2(L^2(Q_2 \times Q_1))$

(2) It is covariant under the left action of $G$

(3) The two-point kernel reads

$$K_G(g_1 g_2, g_1' g_2') = K_{S_2}(g_2, g_2') K_{S_1}(g_1, g_1')$$

(Restricted covariance)

## Quantum Kählerian groups

**Theorem:** (i) [Neshveyev-Tuset] The following defines a unitary 2-cocycle on $W^*(G \times G)$

$$F_\lambda = \int \overline{K_G(g,g')}\,\lambda_{g^{-1}} \otimes \lambda_{g'^{-1}}\, d^\lambda(g)\, d^\lambda(g')$$

(ii) The paring

$$\langle f, f \rangle := \int_G \overline{f} \star_{\lambda,\rho} f(g) d^\rho(g)$$

is positive definite on $\mathcal{D}(G)$ and the operator

$$\Big(W\, \varphi_1 \otimes \varphi_2\Big) := \Delta(\varphi_1)\Big(\star_{\lambda,\rho} \otimes \star_{\lambda,\rho}\Big)(1 \otimes \varphi_2)$$

extends to a regular multiplicative unitary

**Rem:** However, $W$ seems not to be manageable, which will mean that both constructions do not coincide!

# III Example #2: The affine group of a local field

Joint with D. Jondreville, to appear in JFA + work in progress

- $\mathbf{k}$ be a non-Archimedean local field

- $\mathcal{O}_{\mathbf{k}}$ its ring of integers

- $\Psi$ a fixed non trivial unitary additive character

- $H \subset \mathcal{O}_{\mathbf{k}}$ compact open subgroup of $\mathbf{k}^{\times}$ such that

i) the map $a \mapsto a^2$ is a homeomorphism

ii) $\phi(H)$ is a subgroup of $\mathcal{O}_{\mathbf{k}}$, where $\phi : H \to \mathcal{O}_{\mathbf{k}}$, $a \mapsto a - a^{-1}$

**Example:** $H = 1 + p^n\mathbb{Z}_p$ if $\mathbf{k} = \mathbb{Q}_p$

• Let $G := H \ltimes \mathbf{k}$ (since $H \subset \mathcal{O}_{\mathbf{k}}$, $G$ is unimodular)

• Square integrable irreducible representation $\pi$ of $G$ on $L^2(H)$:

$$\pi(a,t)\varphi(a_0) := \Psi(aa_0^{-1}t)\,\varphi(a^{-1}a_0)$$

• $\Sigma\varphi(a) := \varphi(a^{-1})$

• $P := \mathcal{F}_{\mathbf{k}} \circ \chi_{\phi(H)} \circ \mathcal{F}_{\mathbf{k}}^{-1} \in \mathcal{Z}W^*(G)$

**Remark:** • $PW^*(G)P \simeq W^*(\widehat{G})$ where $\widehat{G} := H \ltimes (\mathbf{k}/\phi(H))$
• If $\mathbf{k} = \mathbb{Q}_p$ and $H = 1 + p^n\mathbb{Z}_p$ then

$$\widehat{G} = (1 + p^n\mathbb{Z}_p) \ltimes (\mathbb{Q}_p/p^{-n}\mathbb{Z}_p) \simeq \mathbb{Z}_p \ltimes \widehat{\mathbb{Z}}_p$$

**Theorem:** i) The quantization map

$$\mathbf{\Omega} : \mathcal{D}(G) \to \mathcal{B}(L^2(H)) \,, \quad f \mapsto \int_G f(g)\, \Omega(g)\, d^\lambda(g)$$

satisfies

$$\mathbf{\Omega} \circ P = \mathbf{\Omega}$$

and extends to unitary operator $PL^2(G) \to \mathsf{HS}(L^2(H))$

ii) The two point kernel reads

$$K((a_1, t_1), (a_2, t_2)) = \Psi\Big(\phi(a_1)t_2 - \phi(a_2)t_1\Big)$$

iii) The element

$$F_\lambda = \int \overline{K(g_1, g_2)} \lambda_{g_1^{-1}} \otimes \lambda_{g_2^{-1}} dg_1 dg_2$$

commutes with $P \otimes P$ and defines a unitary element of $W^*(\widehat{G} \times \widehat{G})$

## IV Examples #3: Playing with involutions

Work in progress with P. Bieliavsky and D. Jondreville

## The BCH quantization of an exponential Lie group

- Let $G$ be an exponential Lie group with $\mathfrak{g}$ its Lie algebra

- Assume $G$ possesses a coadjoint orbit $\mathcal{O} \subset \mathfrak{g}^\star$ on which $G$ acts simply transitively

- Let $\pi_{\mathcal{O}}$ the KKS representation of $G$

- For $f \in C_c^\infty(\mathcal{O}) \subset C_c^\infty(\mathfrak{g}^\star)$ define

$$\mathbf{\Omega}_{\mathcal{O}}(f) := \int_{\mathfrak{g}} \mathcal{F}(f)(X)\, \pi_{\mathcal{O}}\Big(\text{exp}\{X\}\Big)\, dX$$

- After identification $G \simeq \mathcal{O}$, get a $G$-covariant quantization on $G$

• These assumptions are satisfied for $G = \mathbb{R} \ltimes \mathbb{R}$. In this case, two possible orbits $\Pi_{\pm} := \{(x, y) \in \mathfrak{g}^{\star} : \pm y > 0\}$

• The quantization map reads

$$\mathbf{\Omega}_{\pm}(f) = \int_{\mathbb{R} \ltimes \mathbb{R}} f(g)\, \pi_{\pm}(g)\, \mathbf{\Sigma}\, \pi_{\pm}(g) dg,$$

• where, realizing $\mathcal{H}_{\pm}$ as $L^2(\mathbb{R})$, we have

$$\Sigma\varphi(t) = |\gamma'(t)|\, \varphi\Big(\sigma(t)\Big),$$

• $\sigma$ is the involutive diffeomorphism of $\mathbb{R}$ given by

$$\sigma = \mathrm{Id} - \gamma : \mathbb{R} \to \mathbb{R}$$

and $\gamma$ is the inverse diffeomorphism of $\log \circ \lambda : \mathbb{R} \to \mathbb{R}$ where

$$\lambda : \mathbb{R} \to \mathbb{R}^*_+\,, \quad t \mapsto t(1 - e^{-t})^{-1}$$

## Non-geometric variations

• $G = \mathbb{R} \ltimes \mathbb{R}$, $(a,t)(a',t') = (a+a', e^{-a'}t + t')$

• $\pi$ representation on $L^2(\mathbb{R})$

$$\pi(a,t)\varphi(a_0) := e^{ie^{a-a_0}}\varphi(a_0 - a)$$

• Let $\sigma \in \mathrm{Diff}(\mathbb{R})$ be an involution such that
(1) $\gamma := \sigma - \mathrm{Id} \in \mathrm{Diff}(\mathbb{R})$
(2) $\phi := [a \mapsto e^a - e^{\sigma(a)}] \in \mathrm{Diff}(\mathbb{R})$
(3) $\exists!$ solution $\kappa : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ to the functional equation

$$\sigma(\sigma(\sigma(\kappa(a_1,a_2,a_3) - a_1) + a_1 - a_2) + a_2 - a_3) + a_3 - \kappa(a_1,a_2,a_3) = 0$$

Set $\Sigma\varphi(a) := \mathrm{Jac}_\gamma^{1/2}(a)\,\mathrm{Jac}_\phi^{1/2}(a)\,\varphi\Big(\sigma(a)\Big)$

**Theorem:**

(i) The $\mathbb{R} \ltimes \mathbb{R}$-covariant quantization map on $\mathbb{R} \ltimes \mathbb{R}$ associated with $(\pi, \Sigma)$ defines a unitary operator from $L^2(\mathbb{R} \ltimes \mathbb{R})$ to $\mathrm{HS}(L^2(\mathbb{R}))$

(ii) The associated 2-points kernel reads ($\kappa := \kappa(e, q_1, q_2)$)

$$\mathbf{K}_\sigma(a_1, t_1; a_2, t_2) = \frac{\left|\mathrm{Jac}_\gamma^{1/2}\mathrm{Jac}_\phi^{1/2}\right|\big(\kappa\big)\left|\mathrm{Jac}_\gamma^{1/2}\mathrm{Jac}_\phi^{1/2}\right|\big(\sigma(\kappa) - a_1\big)\left|\mathrm{Jac}_\gamma^{1/2}\mathrm{Jac}_\phi^{1/2}\right|\big(\sigma(\kappa - a_2)\big)}{\left|1 - \mathrm{Jac}_\sigma(\kappa)\mathrm{Jac}_\sigma(\sigma(\kappa) - a_1)\mathrm{Jac}_\sigma(\sigma(\kappa - a_2))\right|}$$
$$\times \exp\{\phi(a_1 - \sigma(\kappa))t_1 + \phi(a_2 - \kappa)t_2\}$$

(iii) The 2-cocycle

$$F_\sigma := \int_{(\mathbb{R} \ltimes \mathbb{R})^2} \overline{\mathbf{K}_\sigma(e; g, g')}\, \lambda_{g^{-1}} \otimes \lambda_{g'^{-1}} \, dg dg'$$

is a unitary element of $W^*((\mathbb{R} \ltimes \mathbb{R}) \times (\mathbb{R} \ltimes \mathbb{R}))$ if and only if $\sigma = -\mathrm{Id}$

**Remark:** 1) Point (iii) above has the following interpretation:
Unitarity for the 2-cocycle selects the symplectic symmetric space structure

2) There is good hopes that it also yields a regular multiplicative unitary