

$A : \mathbb{Z}$ -algebra

Hochschild homology:

$$HH(A) = \left| \begin{array}{c} \vdash \dashv \\ A \otimes_{\mathbb{Z}} A \otimes_{\mathbb{Z}} A \\ \vdash \dashv \dashv \\ LTLT \\ A \otimes_{\mathbb{Z}} A \\ \vdash \dashv \\ A \end{array} \right|$$

Connes:  $\mathbb{T}$ -spectrum,  $\mathbb{T} = \mathbb{R}/\mathbb{Z}$

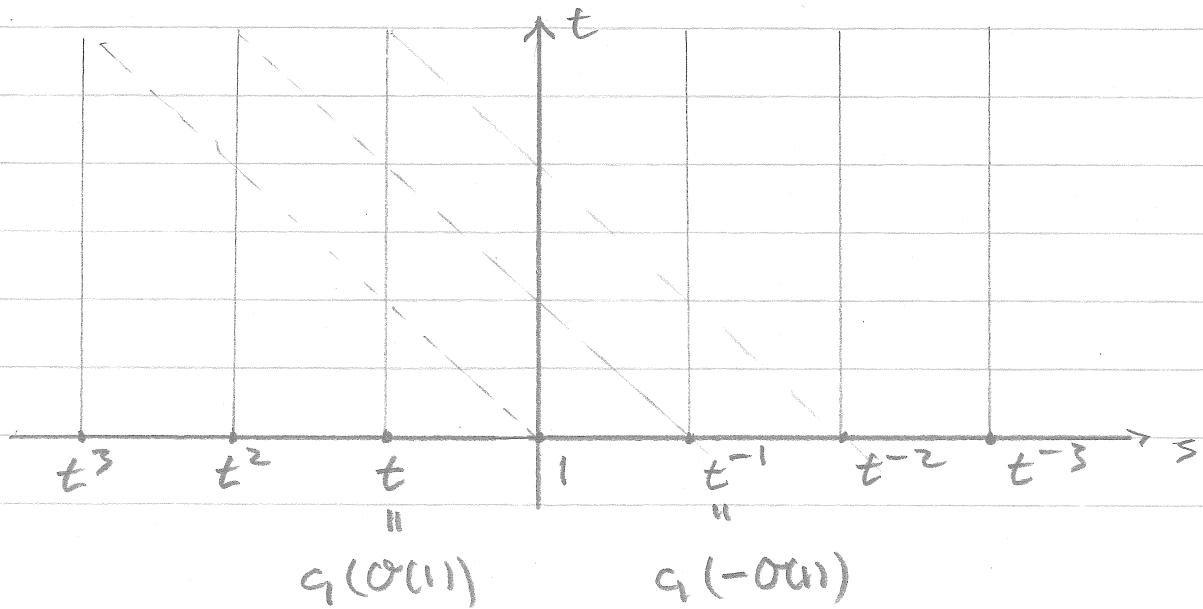
Periodic cyclic homology:

$$HP_*(A) := \hat{H}^{-*}(\mathbb{T}, HH(A))$$

Why periodic? Spectral sequence

$$E^2_{s,t} = H^{-s}(P_{-s}^{\text{co}}(\mathbb{C}), HH_t(A))$$

$$\Rightarrow HP_{s+t}(A)$$



Two easy facts :

1)  $HP_*(A)$  is an  $HP_*(\mathbb{Z})$ -module

2)  $HH_*(\mathbb{Z}) = \mathbb{Z}$ .

By 2),  $HP_*(\mathbb{Z}) = \{ \mathbb{Z} \{ t^{\pm 1} \} \}$ , so by  
1),  $HP_*(A)$  is 2-periodic with

periodicity operator

= mult. by  $t^{-1} \in HP_2(\mathbb{Z})$ .

$A : S$ -algebra

$\mathbb{Z}$

$I \sim Hurewicz$   
 $S$

Topological Hochschild homology

$$THH(A) = \begin{array}{|c|c|c|} \hline & \downarrow \dots \downarrow & \\ & A \otimes_S A \otimes_S A & \\ & \downarrow T \downarrow T \downarrow & \\ & A \otimes_S A & \\ & \downarrow T \downarrow & \\ & A & \\ \hline \end{array}$$

Bökstedt  
Breen

Connes :  $T$ -spectrum,  $T = \mathbb{R}/\mathbb{Z}$

Let us define

$$TP_*(A) = \hat{H}^{-*}(T, THH(A)).$$

3.

Periodic? Spectral sequence

$$E_{s,t}^2 = H^{-s}(TP_{-\infty}^{\otimes t}(A), THH_t(A))$$

$$\Rightarrow TP_{s+t}(A).$$

Again,

- 1)  $TP_*(A)$  is a  $TP_*(S)$ -module
- 2)  $THH(S) = S$ .

But  $THH_*(S) = \pi_*(S) \neq \mathbb{Z}$ ;  
 it does not survive spectral seq;  
 and  $TP_*(S)$  is not periodic, nor  
 is  $TP_*(\mathbb{Z})$ .

Fix prime number  $p$ , consider

$$\begin{array}{ccccccc} \mathbb{Q} & \subset & \mathbb{Q}_p & \subset & \overline{\mathbb{Q}}_p & \subset & \mathbb{C}_p \\ \cup & & \cup & & \cup & & \cup \\ \mathbb{Z} & \subset & \mathbb{Z}_p & \subset & \mathbb{Q}_{\overline{\mathbb{Q}}_p} & \subset & \mathbb{Q}_{\mathbb{C}_p} \end{array} \quad \text{not north.}$$

Suslin:

$$S_{K_0(\mathbb{Q}_{\mathbb{C}_p}, \mathbb{Z}_p)}(T_p K_1(\mathbb{Q}_{\mathbb{C}_p}))$$

$$\begin{array}{ccc} \mathcal{N} & \xrightarrow{\sim} & K_*(\mathbb{Q}_{\mathbb{C}_p}, \mathbb{Z}_p) \\ \text{Symm.} & & \text{algebra} \end{array}$$

$$K_0(\mathcal{O}_{\mathbb{Q}_p}, \mathbb{Z}_p) = \mathbb{Z}_p \cdot 1$$

$$T_p K_1(\mathcal{O}_{\mathbb{Q}_p}) = \mathbb{Z}_p \cdot \varepsilon$$

$$\varepsilon = (\zeta_p, \zeta_{p^2}, \zeta_{p^3}, \dots)$$

H. - Madsen:

$\sim$

Composme  
HTH

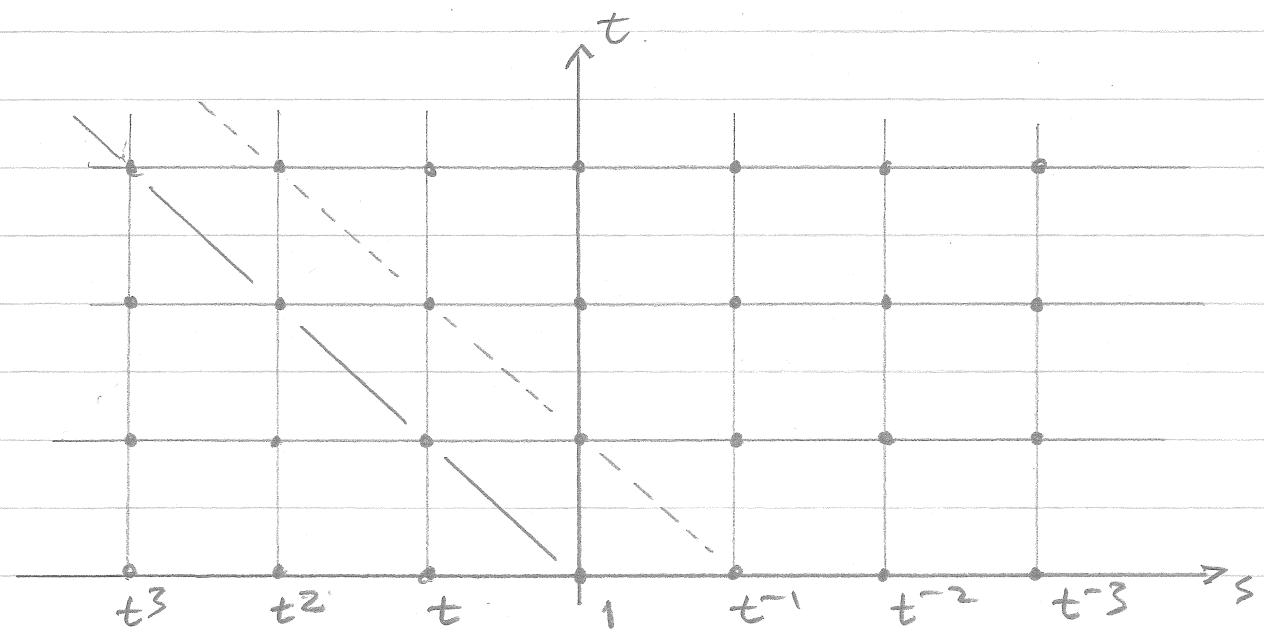
$$S_{THH_0(\mathcal{O}_{\mathbb{Q}_p}, \mathbb{Z}_p)}(T_p THH_1(\mathcal{O}_{\mathbb{Q}_p}))$$

$$\xrightarrow{\sim} THH_* (\mathcal{O}_{\mathbb{Q}_p}, \mathbb{Z}_p)$$

$$THH_0(\mathcal{O}_{\mathbb{Q}_p}, \mathbb{Z}_p) = \mathcal{O}_{\mathbb{Q}_p} \cdot 1$$

$$T_p THH_1(\mathcal{O}_{\mathbb{Q}_p}) = \mathcal{O}_{\mathbb{Q}_p} \cdot (\zeta_p - 1)^{-1} \cdot \text{dlog } \varepsilon$$

So sp. seg. conc. in even total deg.



a ring of  $p$ -adic periods

So for every  $\mathcal{O}_{\mathbb{F}_p}$ -algebra  $A$ ,  
 $TP_*(A, \mathbb{Z}_p)$  is 2-periodic with  
 periodicity operator = mult. by

$$\alpha_{\varepsilon} := c_1^K(-\sigma(1)) \in TP_2(\mathcal{O}_{\mathbb{F}_p}, \mathbb{Z}_p).$$

Question What is won by  
 descending from  $\mathbb{A}$  to  $\mathbb{S}$ ?

Answer An inverse Frobenius  
 operator.

Equivariant homotopy groups

$$TR_g^r(A) = [S^q \wedge (\mathbb{T}/C_r)_+, THH(A)]_+$$

morphisms in homotopy  
category of  $\mathbb{T}$ -spectra

$$S^q \wedge (G/H)_+ = \frac{D^q \times G/H}{\partial D^q \times G/H}$$

If  $r = st$ , then

$$\mathbb{T}/C_r \xleftarrow{\text{pr}_s} \mathbb{T}/C_t$$

induces

$$TR_g^r(A) \xrightleftharpoons{\text{pr}_s^*} TR_g^t(A)$$

$\text{pr}_s^* = F_s$  Frobenius

$\text{pr}_s^* = V_s$  Verschiebung

Additional maps

$$(A^{\otimes_{\mathbb{S}^r}})^C_r \xrightarrow{r_s} (A^{\otimes_{\mathbb{S}^t}})^C_t$$

induce "restriction" maps

$$TR_q^r(A) \xrightarrow{R_s} TR_q^t(A).$$

Consider  $r = p^{n-1}$ ; write

$$TR_q^n(A; p) := TR_q^{p^{n-1}}(A).$$

Thm (H.-Madsen) If  $A$  is a commutative ring, then

$$W_n(A) \xrightarrow{\sim} TR_0^n(A; p),$$

compatible with  $F = F_p$ ,  $V = V_p$ , and  $R = R_p$ . //

Define

$$TF_q(A; p) = \lim_{n, F} TR_q^n(A; p).$$

There is a canonical map

$$TF_q(A; p, \mathbb{Z}_p) \xrightarrow{\hat{F}} TP_q(A, \mathbb{Z}_p)$$

$\cong_R$

no  $R$

Thm (H.-Madsen) The map

$$TF_q(\mathcal{O}_{\mathbb{A}^p}; p, \mathbb{Z}_p) \xrightarrow{\hat{c}} TP_q(\mathcal{O}_{\mathbb{A}^p}, \mathbb{Z}_p)$$

is an isomorphism for  $q \geq 0$ . //

$$\begin{aligned} R & TF_0(\mathcal{O}_{\mathbb{A}^p}; p, \mathbb{Z}_p) \xleftarrow[F]{\sim} \lim W_n(\mathcal{O}_{\mathbb{A}^p}) \\ & \xleftarrow[F]{\sim} \lim W(\mathcal{O}_{\mathbb{A}^p}) \xrightarrow[F]{\sim} \lim W(\mathcal{O}_{\mathbb{A}^p}/p^n \mathcal{O}_{\mathbb{A}^p}) \\ & \xleftarrow[\varphi]{\sim} W(\lim \mathcal{O}_{\mathbb{A}^p}/p^n \mathcal{O}_{\mathbb{A}^p}) \\ & =: W(\mathcal{O}_{\mathbb{A}^p}^\flat) \quad (=: A_{\text{inf}}) \\ & \uparrow \\ & \varphi^{-1} \quad \text{Fontaine} \end{aligned}$$

Edge homomorphism

$$W(\mathcal{O}_{\mathbb{A}^p}^\flat) \xrightarrow{\theta} \mathcal{O}_{\mathbb{A}^p}$$

surjective,

$$\ker(\theta) = W(\mathcal{O}_{\mathbb{A}^p}^\flat) \cdot \frac{[\mathbb{E}^p] - 1}{[\mathbb{E}] - 1}$$

Period isom. for  $X/\mathbb{R}$  sm.

$$H_{\mathbb{R}}^q(X/\mathbb{R}) \otimes \mathbb{C} \xrightarrow[\mathbb{R}]{\sim} H^q(X(\mathbb{C}), \mathbb{C})^{\text{an}}$$

compatible with  $\text{Gal}(\mathbb{C}/\mathbb{R})$ -action. //

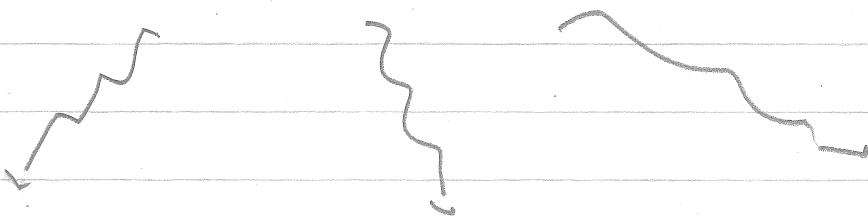
Def (Fargues) A Breuil-Kisin module over  $W(\mathcal{O}_{\mathbb{F}_p}^b)$  is a pair  $(M, \varphi_M)$  of a  $\mathbb{F}_q$ -free  $W(\mathcal{O}_{\mathbb{F}_p}^b)$ -module  $M$  and a  $\mathbb{F}$ -linear isom.

$$M[\frac{1}{\xi}] \xrightarrow{\varphi_M} M[\frac{1}{\xi'}].$$

Ex Tate twist in B-K world:

$$W(\mathcal{O}_{\mathbb{F}_p}^b)(1) = (W(\mathcal{O}_{\mathbb{F}_p}^b), \xi^{-1} \cdot \varphi).$$

B-K world



Crystalline      de Rham      étale

Thm (H.) For  $g = 2m \geq 0$ ,

$$\mathrm{TF}_g(\mathcal{O}_{\mathbb{F}_p}; p, \mathbb{Z}_p) = W(\mathcal{O}_{\mathbb{F}_p}^b)(m),$$

and LHS is zero, otherwise. //

Recently: Motivic theory by  
Bhatt-Morrow-Scholze:

"TF knows everything"