Analyticity results for cumulants in a quartic random matrix model

THOMAS KRAJEWSKI

Centre de Physique Théorique, Marseille

krajew@cpt.univ-mrs.fr

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Random matrix models

Random matrix models (unitarily invariant probability laws on matrices) are ubiquitous in physics (nuclear physics, disordered systems, random surfaces, ...) and mathematics (combinatorics, non commutative probability, knot theory, ...)

Statement of the problem

Derive analyticity in $\lambda \in \mathbb{C}$ for the cumulants defined as

$$K_{a_1 b_1 c_1 d_1, \dots, a_k b_k c_k d_k}(\lambda, N) = \frac{\partial^2}{\partial J^*_{a_1 b_1} \partial J_{c_1 d_1}} \cdots \frac{\partial^2}{\partial J^*_{a_k b_k} \partial J_{c_k d_k}} \log \mathcal{Z}[J, J^\dagger; \lambda, N] \Big|_{J=J^\dagger=0}$$

in a quartic $N \times N$ complex matrix model with

$$\mathcal{Z}[J, J^\dagger; \lambda, N] = \int dM \exp\left\{ -\mathrm{Tr}(MM^\dagger) - \frac{\lambda}{2N}\mathrm{Tr}(MM^\dagger MM^\dagger) + \sqrt{N}\mathrm{Tr}(JM^\dagger) + \sqrt{N}\mathrm{Tr}(MJ^\dagger) \right\}$$

using the Loop Vertex Expansion (LVE) techniques (Rivasseau, 2007).

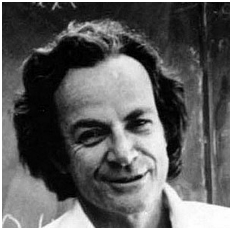
Divergence of perturbative expansion

Perturbative expansion based on Feynman graphs diverges

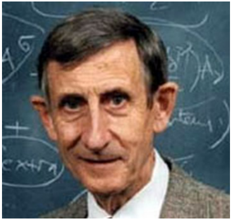
$$\log \mathcal{Z}[J, J^\dagger; \lambda, N] \quad " = " \quad \sum_n a_n \lambda^n$$

a_n = sum of contributions of connected graphs of order n

- Combinatorics: number of order n graphs $\sim n!$
- Analysis: $\lambda = 0$ boundary of analyticity domain
- Physics: instability for $\lambda < 0$



R.P. Feynman



F. Dyson

Perturbative expansion of the path integral

Interacting theory $\Rightarrow$ use of perturbation around free fields

$$\int [D\phi] e^{-S[\phi]} = \sum_n a_n (-\lambda)^n \qquad \text{with } \lambda \ll 1 \text{ coupling constant}$$

$$a_n = \sum_{\text{order } n \text{ graphs}} \underset{n\to\infty}{\sim} C\kappa^n n^a n! \Rightarrow \text{divergence of perturbative series}$$

Exemples:

- Anharmonic oscillator: $H = P^2 + X^2 + \lambda X^4$
  ground state energy $E_0(\lambda) = \sum_n a_n (-\lambda)^n$
  vacuum instability if $\lambda < 0$ (stability for all λ if series convergent)
- Quantum Electrodynamics: $\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + i\overline{\psi}\gamma^\mu(\partial_\mu + ieA_\mu)\psi$
  vacuum instability $\alpha = \frac{e^2}{4\pi\epsilon_0 \hbar c} < 0$ because of pair creation e^+e^-

"Le simple est toujours faux. Ce qui ne l'est pas est inutilisable."

Paul Valery

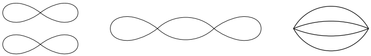
Simple model (functional integral $\to$ ordinary integral)

Expansion based on Feynman graphs

$$\int \frac{d\phi}{\sqrt{2\pi}} \exp -\left\{ \frac{1}{2}\phi^2 + \frac{\lambda}{4!}\phi^4 \right\} = \sum_{\Gamma \text{ 4-valent graph}} \frac{(-\lambda)^{|\Gamma|}}{S(\Gamma)}$$

with $|\Gamma| = \#\{\text{vertices}\}$ (order of perturbation theory)
$S(\Gamma) = \#\{\text{transformations preserving } \Gamma\}$ (symmetry factor)

$$\sum_{|\Gamma|=n} \frac{1}{S(\Gamma)} = \frac{\Gamma(2n+1/2)}{6^n \Gamma(1/2) n!} \underset{n\to\infty}{\sim} \frac{1}{\sqrt{\pi}} \left(\frac{2}{3}\right)^n \frac{n^n}{e^n \sqrt{n}} \qquad \text{(Stirling)}$$

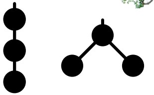
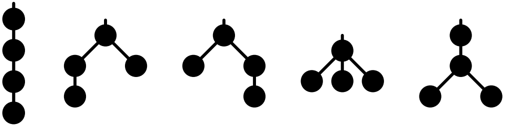

$: \quad \frac{1}{2} \times \left(\frac{1}{2^3}\right)^2 + \frac{1}{2\times 2^3} + \frac{1}{2\times 4!} = \frac{35}{3\times 2^7}$

Remarques:

- Borel summation possible if $a_n > 0$ (sign alternance)
- Other divergence: renromalized amplitudes in $n!$ (renormalons)

What is a (plane,rooted) tree?

Tree = connecter graph without cycle embedded in the plane and with one marked vertex (root)

- 1 tree with 0 vertex: 
- 1 tree with1 vertices: 
- 2 trees with 2 vertices: 
- 5 trees with 3 vertices: 

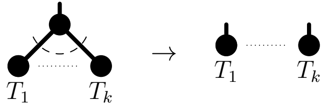
Trees with n vertices counted by Catalan numbers $C_n = \dfrac{2n!}{n!^2(n+1)}$

$C_n = 1, 1, 2, 5, 14, \ldots$ (A000108 On Line Encyclopedia of Integer Sequences)

Combinatorial species

Discrete structure (trees, graphs,...) grouping objects by size (number of vertices,...) equipped with combinatorial operations (insertion,cutting,...)

Generating series $T(z) = \sum_{n \geq 0} C_n z^n$ with $C_n = \# \{\text{trees of size } n\}$



$$C_n = \sum_{k \geq 0} \sum_{n_1 + \cdots + n_k = n+k} C_{n_1} \dots C_{n_k} \quad \Rightarrow \quad T(z) = \frac{1}{1 - zT(z)}$$

$$T(z) = \frac{-1 + \sqrt{1 - 4z}}{2z} = \sum_{n \geq 0} \frac{2n!}{n!^2 (n+1)} z^n \quad \text{(binomial formula)}$$

Convergent expansions over trees

$C_n \underset{n \to \infty}{\sim} \mathrm{C}^{\text{ste}}\, 4^n n^{-3/2} \Rightarrow \sum_T \mathcal{A}_T$ convergent if $|\mathcal{A}_T| < \left(\frac{1}{4}\right)^{\#(\text{vertices of } T)}$

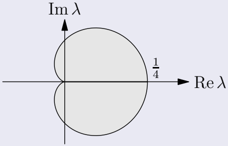
LVE techniques and results

Explicit expansion over **trees** of $\log \mathcal{Z}[J, J^\dagger; \lambda, N]$ based on

- Intermediate field (Hubbard-Stratonovitch transformation)
- Replica trick (one matrix $\to$ n matrices)
- Forest formula (generalisation of fundamental theorem of calculus)
- Bound on the resolvent $\|(1 - \mathrm{i}\sqrt{\frac{\lambda}{N}}A)^{-1}\| \leq \dfrac{1}{\cos^2\left(\frac{1}{2}\arg\lambda\right)}$

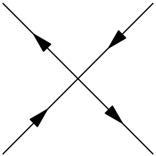
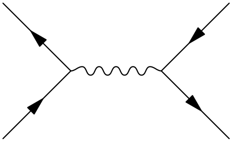
$\Rightarrow$ Analyticity of $\log \mathcal{Z}[J, J^\dagger; \lambda, N]$ and cumulants for λ in a cardioid

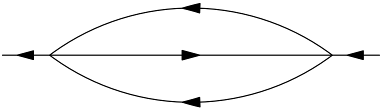
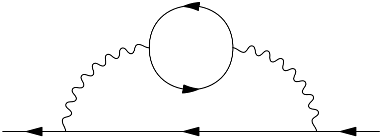
$$4|\lambda|^2 \leq \cos^2\left(\tfrac{1}{2}\arg\lambda\right)$$



Write the quartic interaction using an auxiliary hermitian matrix A

$$\exp -\frac{\lambda}{2N}\mathrm{Tr}(MM^\dagger MM^\dagger) = \int dA \exp -\left\{\frac{1}{2}\mathrm{Tr}(A^2) - \mathrm{i}\sqrt{\frac{\lambda}{N}}\,\mathrm{Tr}(M^\dagger AM)\right\}$$

 $\Leftrightarrow$ 

 $\Leftrightarrow$ 

- Intermediate field (wavy line) on the right following the arrows
- matricial intermediate field also represented as a double line

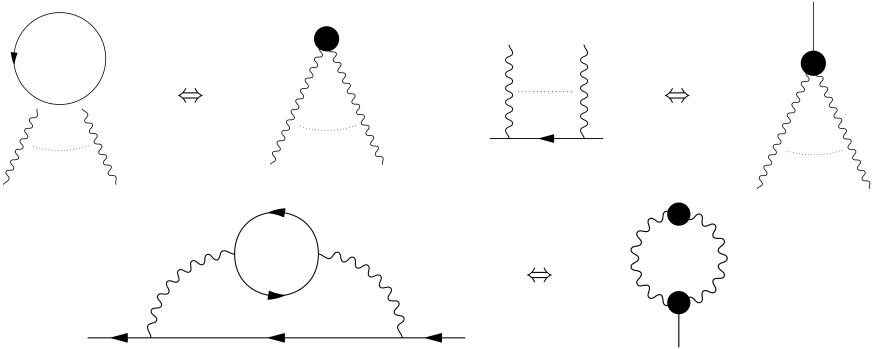
 $\rightarrow$ 

Perform the Gaußian integration over M (with $\log\det = \mathrm{Tr}\log$)

$$\mathcal{Z}[J,J^\dagger;\lambda,N] = \int dMdA$$

$$\exp -\left\{\frac{1}{2}\mathrm{Tr}(A^2)+\mathrm{Tr}\left[M^\dagger\left(1-\mathrm{i}\sqrt{\frac{\lambda}{N}}A\right)M\right]+\sqrt{N}\mathrm{Tr}(JM^\dagger)+\sqrt{N}\mathrm{Tr}(MJ^\dagger)\right\}$$

$$= \int dA\exp -\left\{\frac{1}{2}\mathrm{Tr}(A^2)+N\mathrm{Tr}\log\left(1-\mathrm{i}\sqrt{\frac{\lambda}{N}}A\right)+N\mathrm{Tr}J\left(1-\mathrm{i}\sqrt{\frac{\lambda}{N}}A\right)^{-1}J^\dagger\right\}$$



- Expand the exponential as a power series

$$\mathcal{Z}[J,J^\dagger] =$$
$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int d\mu(A) \left[N\mathrm{Tr}\log\left(1-\mathrm{i}\sqrt{\frac{\lambda}{N}}A\right) + N\mathrm{Tr} J \left(1-\mathrm{i}\sqrt{\frac{\lambda}{N}}A\right)^{-1} J^\dagger \right]^n$$

- Replace the integration over one matrix by an integration over an n-tuple of matrices (replicas) with uniform covariance $C_{ij} = 1$

$$\mathcal{Z}[J,J^\dagger] =$$
$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_{A=(A_1,\dots,A_n)} d\mu_C(A) \prod_{i=1}^{n} \left[N\mathrm{Tr}\log\left(1-\mathrm{i}\sqrt{\frac{\lambda}{N}}A_i\right) + N\mathrm{Tr} J \left(1-\mathrm{i}\sqrt{\frac{\lambda}{N}}A_i\right)^{-1} J^\dagger \right]$$

- Gaußian measure of covariance C_{ij} (positive $n \times n$ matrix)

$$\int d\mu_C(A)\, A_{i|ab} A_{j|cd} = C_{ij}\, \delta_{ad}\delta_{bc}$$

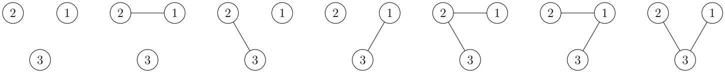
with $A_{i|ab}$ the matrix element in the row a and column b of A_i

Brydges-Kennedy-Abdessalam-Rivasseau forest formula

For any function $\phi : \mathbb{R}^{\frac{n(n-1)}{2}} \to \mathbb{C}$ on complete graph with vertices i, j, ...

$$\phi(1,\dots,1) = \sum_{F \text{ forest}} \int_0^1 \prod_{(ij)\in F} dt_{ij} \frac{\partial^{|E(F)|}\phi}{\prod_{(i,j)\in F} \partial t_{ij}} \left(\inf_{(kl)\in P^F_{i\leftrightarrow j}} t_{kl} \right)$$

with $P^{\mathcal{F}}_{i\leftrightarrow j}$ unique path in $\mathcal{F}$ joining i and j, $\inf_{(kl)\in P^F_{i\leftrightarrow j}} t_{kl} = 0$ if $P^{\mathcal{F}}_{i\leftrightarrow j} = \emptyset$

- n=2:  $\phi(1) = \phi(0) + \int_{[0,1]} dt_{12} \frac{\partial F}{\partial t_{12}}(t_{12})$
- n=3: 

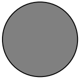
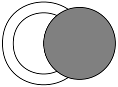
$$\phi(1,1,1) = \phi(0,0,0) + \int_{[0,1]} dt_{12} \frac{\partial \phi}{\partial t_{12}}(t_{12},0,0) + \int_{[0,1]} dt_{23} \frac{\partial \phi}{\partial t_{23}}(0,t_{23},0)$$

$$+ \int_{[0,1]} dt_{13} \frac{\partial \phi}{\partial t_{13}}(0,0,t_{13}) \int_{[0,1]^2} dt_{12}dt_{23} \frac{\partial^2 \phi}{\partial t_{12}\partial t_{23}}(t_{12},t_{23},\inf(t_{12},t_{23})) +$$

$$\int_{[0,1]^2} dt_{12}dt_{13} \frac{\partial^2 \phi}{\partial t_{12}\partial t_{13}}(t_{12},\inf(t_{12},t_{13}),t_{13}) + \int_{[0,1]^2} dt_{23}dt_{13} \frac{\partial^2 \phi}{\partial t_{23}\partial t_{13}}(\inf(t_{23},t_{13}),t_{23},t_{13})$$

- Application of the BKAR forest formula with $C_{ij} \to t_{ij} C_{ij}$ $(i \neq j)$
- Derivative with respect to t_{ij} $\Rightarrow$ edge between vertices i and j

$$\frac{\partial}{\partial t_{ij}} \left(\int d\mu_C(A) V(A) \right) = C_{ij} \int d\mu_C(A) \sum_{a,b} \frac{\partial^2}{\partial A_{i|ab} \partial A_{j|ba}} V(A)$$

$\frac{\partial}{\partial t_{ij}}$  $=$ C_{ij} 

- Derivative of the resolvent $\Rightarrow$ half-edge on vertex i

$$\frac{\partial}{\partial A_{i|ab}} \left(1 - \mathrm{i}\sqrt{\frac{\lambda}{N}} A_j \right)^{-1}_{cd} = \mathrm{i}\sqrt{\frac{\lambda}{N}} \delta_{ij} \left(1 - \mathrm{i}\sqrt{\frac{\lambda}{N}} A_j \right)^{-1}_{ca} \left(1 - \mathrm{i}\sqrt{\frac{\lambda}{N}} A_j \right)^{-1}_{bd}$$

- Cilium on the vertex if there is an insertion of $JJ^\dagger$
- $\mathcal{Z}[J, J^\dagger; \lambda, N]$ sum over forests $\Rightarrow$ $\log \mathcal{Z}[J, J^\dagger; \lambda, N]$ sum over trees since the contribution of a forest factorizes over its connected components

LVE expansion over trees

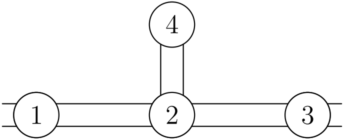
An LVE tree is a plane tree with labelled vertices and at most one cilium per vertex

$$\log \mathcal{Z}[J, J^\dagger, \lambda, N] = \sum_{T \text{ LVE tree}} \mathcal{A}_T[J, J^\dagger; \lambda, N]$$

with

$$\mathcal{A}_T[J, J^\dagger; \lambda, N] = \frac{(-\lambda)^{|E(T)|} N}{|V(T)|!} \int \prod_{e \in E(T)} dt_e \prod_{e=ij \in E(T)} \inf_{e' \in P_{i \leftrightarrow j}} t_{e'}$$
$$\int d\mu_{C_T}(A) \overrightarrow{\prod_{c \in \partial T}} \left(1 - \mathrm{i}\sqrt{\frac{\lambda}{N}} A_{i_c}\right)^{-1} (JJ^\dagger)^{\eta_c}$$

- $(C_T)_{ij} = \inf\left\{ t_e \,\middle|\, e \text{ in the unique path } \mathcal{P}_{i \to j} \text{ in } T \text{ joining } i \text{ and } j \right\}$
- $\overrightarrow{\prod_{c \in \partial f}}$ = oriented product around the corners on the boundary of T.
- Corner = pair of half edges attached to the same vertex
- i_c is the label of the vertex the corner c belongs to.
- $\eta_c = 1, 0$ if c is followed by a cilium (1) or not (0).



$$\mathcal{A} = \frac{N(-\lambda)^3}{4!} \int_0 dt_{12} dt_{23} dt_{24} \int d\mu_C(A)$$

$$\mathrm{Tr}\Big[\Big(1 - \mathrm{i}\sqrt{\frac{\lambda}{N}}A_3\Big)^{-1}\Big(1 - \mathrm{i}\sqrt{\frac{\lambda}{N}}A_2\Big)^{-1}\Big(1 - \mathrm{i}\sqrt{\frac{\lambda}{N}}A_4\Big)^{-1}\Big(1 - \mathrm{i}\sqrt{\frac{\lambda}{N}}A_2\Big)^{-1}$$

$$\Big(1-\mathrm{i}\sqrt{\frac{\lambda}{N}}A_1\Big)^{-1} JJ^\dagger \Big(1-\mathrm{i}\sqrt{\frac{\lambda}{N}}A_1\Big)^{-1}\Big(1-\mathrm{i}\sqrt{\frac{\lambda}{N}}A_2\Big)^{-1}\Big(1-\mathrm{i}\sqrt{\frac{\lambda}{N}}A_3\Big)^{-1} JJ^\dagger\Big]$$

$$C = \begin{pmatrix} 1 & t_{12} & \inf(t_{12}, t_{23}) & \inf(t_{12}, t_{24}) \\ t_{12} & 1 & t_{23} & t_{24} \\ \inf(t_{12}, t_{23}) & t_{23} & 1 & \inf(t_{23}, t_{24}) \\ \inf(t_{12}, t_{24}) & t_{24} & \inf(t_{23}, t_{24}) & 1 \end{pmatrix}$$

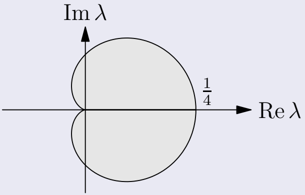
Convergence of the LVE expansion for the generating function

For any disc $\mathcal{D} \subset \mathcal{C}$, there is $\epsilon > 0$ such that for $\lambda \in \mathcal{D}$ and $\|JJ^\dagger\| < \epsilon$

$$\log \mathcal{Z}[J, J^\dagger; \lambda, N] = \sum_{T \text{ LVE tree}} \mathcal{A}_T[J, J^\dagger; \lambda, N]$$

with $\mathcal{C}$ the cardioid

$$\mathcal{C} = \left\{ \lambda \in \mathbb{C} \quad \text{with} \quad 4|\lambda| < \cos^2\left(\frac{\arg \lambda}{2}\right) \right\}$$

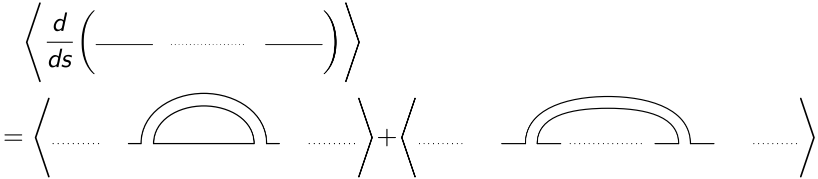


- Bound on the resolvent $\left\| \left(1 - \mathrm{i}\sqrt{\frac{\lambda}{N}} A\right)^{-1} \right\| \leq \frac{1}{\cos \frac{\arg \lambda}{2}}$
- Bound on the tree amplitude $\mathcal{A}[J, J^\dagger; \lambda, N] \leq \frac{N^2 |\lambda|^n \|JJ^\dagger\|^k}{(n+1)! \left(\cos \frac{\arg \lambda}{2}\right)^{2n+k}}$
- Number of LVE trees with n edges and k cilia

$$\frac{(2n+k-1)!\,(n+1)!}{(n+k)!\,(n+1-k)!\,k!} \leq 2^{2n+k-1}\,(n-1)!\,\frac{(n+1)!}{(n+1-k)!\,k!}$$

Recursive generation of loop edges by writing the resolvents as

$$\prod_i \left(1 - \mathrm{i}\sqrt{\frac{\lambda s}{N}} A_i\right)^{-1}_{a_i b_i} = \prod_i \delta_{a_i, b_i} + \int_0^s ds' \frac{d}{ds'} \prod_i \left(1 - \mathrm{i}\sqrt{\frac{\lambda s'}{N}} A_i\right)^{-1}_{a_i b_i}$$



Iteration $\Rightarrow$ addition of loop edges to the tree

LVE graph

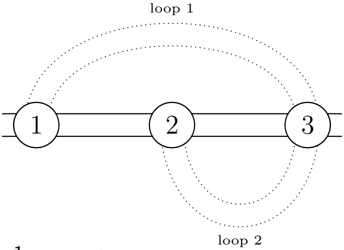
An LVE graph (G, T) is ribbon graph G with at most one cilium per vertex, labels on its vertices, a distinguished spanning tree T and labels on the edges in $E(G) - E(T)$ (loop edges)

The amplitude associated to an LVE graph (G, T) is (for $n > 0$)

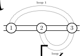
$$\mathcal{A}_{(G,T)}[J, J^\dagger; \lambda, N] = \frac{(-\lambda)^{|E(G)|} N^{|V(G)|-|E(G)|}}{|V(G)|!} \int_{1 \geq s_1 \geq \cdots \geq s_{|E(G)|-|E(T)|} \geq 0} \prod_{e \in E(G)-E(T)} ds_e \int \prod_{e \in E(T)} dt_e \prod_{e=ij \in E(G)-E(T)} \inf_{e' \in P_{i \leftrightarrow j}} t_{e'} \int d\mu_{C_T}(A) \prod_{f \in F(G)} \mathrm{Tr}\Big\{ \overrightarrow{\prod_{c \in \partial f}} \Big(1 - \mathrm{i}\sqrt{\frac{\lambda s_{|E(G)|-|E(T)|}}{N}} A_{i_c}\Big)^{-1} (JJ^\dagger)^{\eta_c} \Big\}$$

- $(C_T)_{ij} = \inf\{ t_e \,|\, e \text{ in the unique path } \mathcal{P}_{i \to j} \text{ in } T \text{ joining } i \text{ and } j\}$
- i_c is the label of the vertex the corner c belongs to.
- $\eta_c = 1, 0$ if c is followed by a cilium (1) or not (0).
- $s_e \in [0, 1]$ associated to every loop edge $e \in E(G) - E(T)$
- $\overrightarrow{\prod_{c \in \partial f}}$ = oriented product around the corners on the boundary of f

Example of a planar LVE graph



$$\mathcal{A}_{\text{[graph]}} = \frac{N^{-1}(-\lambda)^4}{3!} \int_0^1 ds_1 \int_0^{s_1} ds_2 \int_0 dt_{12} dt_{23} dt_{24} \inf(t_{12}, t_{23}) t_{23} \int d\mu_C(A)$$



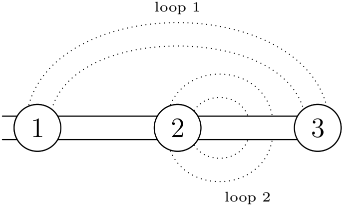
$$\mathrm{Tr}\Bigg[\left(1 - \mathrm{i}\sqrt{\tfrac{s_2\lambda}{N}} A_3\right)^{-1} \left(1 - \mathrm{i}\sqrt{\tfrac{s_2\lambda}{N}} A_1\right)^{-1} JJ^\dagger \left(1 - \mathrm{i}\sqrt{\tfrac{s_2\lambda}{N}} A_1\right)^{-1} \left(1 - \mathrm{i}\sqrt{\tfrac{s_2\lambda}{N}} A_2\right)^{-1}$$

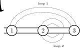
$$\left(1 - \mathrm{i}\sqrt{\tfrac{s_2\lambda}{N}} A_3\right)^{-1} JJ^\dagger\Bigg] \mathrm{Tr}\Bigg[\left(1 - \mathrm{i}\sqrt{\tfrac{s_2\lambda}{N}} A_1\right)^{-1} \left(1 - \mathrm{i}\sqrt{\tfrac{s_2\lambda}{N}} A_2\right)^{-1}$$

$$\left(1 - \mathrm{i}\sqrt{\tfrac{s_2\lambda}{N}} A_3\right)^{-1}\Bigg] \mathrm{Tr}\Bigg[\left(1 - \mathrm{i}\sqrt{\tfrac{s_2\lambda}{N}} A_2\right)^{-1} \left(1 - \mathrm{i}\sqrt{\tfrac{s_2\lambda}{N}} A_3\right)^{-1}\Bigg]$$

$$C = \begin{pmatrix} 1 & t_{12} & \inf(t_{12}, t_{23}) \\ t_{12} & 1 & t_{23} \\ \inf(t_{12}, t_{23}) & t_{23} & 1 \end{pmatrix}$$

Example of a non planar LVE graph



$$\mathcal{A}$$ 

$$= \frac{N^{-1}(-\lambda)^4}{3!} \int_0^1 ds_1 \int_0^{s_1} ds_2 \int_0 dt_{12} dt_{23} dt_{24} \inf(t_{12}, t_{23}) \int d\mu_C(A)$$

$$\mathrm{Tr}\Big[\Big(1 - i\sqrt{\frac{s_2\lambda}{N}} A_1\Big)^{-1} \Big(1 - i\sqrt{\frac{s_2\lambda}{N}} A_2\Big)^{-1} \Big(1 - i\sqrt{\frac{s_2\lambda}{N}} A_2\Big)^{-1} \Big(1 - i\sqrt{\frac{s_2\lambda}{N}} A_1\Big)^{-1}$$

$$\Big(1 - i\sqrt{\frac{s_2\lambda}{N}} A_3\Big)^{-1} \Big(1 - i\sqrt{\frac{s_2\lambda}{N}} A_2\Big)^{-1} \Big(1 - i\sqrt{\frac{s_2\lambda}{N}} A_2\Big)^{-1}$$

$$\Big(1 - i\sqrt{\frac{s_2\lambda}{N}} A_3\Big)^{-1} \Big(1 - i\sqrt{\frac{s_2\lambda}{N}} A_1\Big)^{-1} JJ^\dagger \Big]$$

$$C = \begin{pmatrix} 1 & t_{12} & \inf(t_{12}, t_{23}) \\ t_{12} & 1 & t_{23} \\ \inf(t_{12}, t_{23}) & t_{23} & 1 \end{pmatrix}$$

Perturbative expansion with remainder

For any disc $\mathcal{D} \subset \mathcal{C}$, there is $\epsilon > 0$ such that for $\lambda \in \mathcal{D}$ and $\|JJ^\dagger\| < \epsilon$

$$\log \mathcal{Z}[J, J^\dagger; \lambda, N] = \sum_{\substack{G \text{ ciliated ribbon graph} \\ |E(G)| \leq n}} \frac{(-\lambda)^{|E(G)|} N^{|V(G)|-|E(G)|+|F(G)|-|B(G)|}}{|\mathrm{Aut}(G)|} \prod_{f \in B(G)} \mathrm{Tr}\Big[\big(JJ^\dagger\big)^{c(f)}\Big] + \mathcal{R}_n[J, J^\dagger; \lambda, N]$$

where $\chi(G) = |V(G)| - |E(G)| + |F(G)| - |B(G)|$ is the Euler characteristic ($B(G)$ = set of faces containing cilia) and $c(f)$is the number of cilia in the broken face. The order n perturbative remainder can be expressed as a convergent sum over LVE graphs with at least $n+1$ edges and at most $n+1$ loop edges

$$\mathcal{R}_n[J, J^\dagger; \lambda, N] = \sum_{\substack{(G,T) \text{ LVE graph} \\ |E(G)|=n+1}} \mathcal{A}_{(G,T)}[J, J^\dagger; \lambda, N] + \sum_{\substack{T \text{ LVE tree} \\ |E(T)| \geq n+2}} \mathcal{A}_T[J, J^\dagger; \lambda, N]$$

and is analytic in the cardioid $\mathcal{C}$.

- Contributions of LVE graphs with $s = 0$ reconstruct perturbative expansion in terms of Feynman graphs

$$\sum_{T \subset G \text{ spaning tree}} \int \prod_{e \in E(T)} dt_e \prod_{e=(ij) \in E(G)-E(T)} \inf_{e' \in P_{i\leftrightarrow}} t_{e'} = 1$$

where $P_{i\leftrightarrow j}$ is the unique path on T joining the vertices labelled i and j.

- Counting LVE graphs with n vertices, k cilia and l loop edges

$$\mathcal{N}(n,k,l) = \frac{(2n+2l+k-3)!\, n!}{2^l\,(n+k-1)!\,(n-k)!\,k!}$$

- Bound on the contribution of each LVE graph

$$\left|\mathcal{A}_{(G,T)}[J,J^\dagger;\lambda,N]\right| \leq \int \prod_{e \in E(T)} dt_e \prod_{e=ij \in E(G)-E(T)} \inf_{e' \in P_{i\leftrightarrow j}} t_{e'}$$

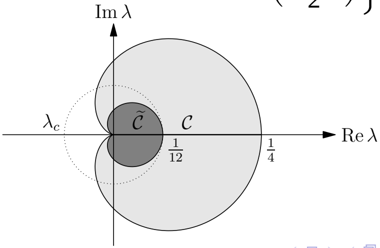
$$\frac{N^{|F(G)|+|V(G)|-|E(G)|}|\lambda|^{|E(G)|}}{|V(G)|!(|E(G)|-|E(T)|)!} \left(\frac{1}{\cos\frac{\arg\lambda}{2}}\right)^{2|E(G)|+k} \|JJ^\dagger\|^k$$

Topological expansion

Matrix models admit a topological expansion in N^{2-2g} with g the minimal genus of the surface in which the ribbon graph G is embedded

- Generate loops up to genus g (genus $g+1 \rightarrow$ remainder)
  $\Rightarrow$ remainder made of LVE graphs such that last added loop edge increases g to $g+1$
- Series based on graphs of fixed genus have radius of convergence $\frac{1}{12}$
  $\Rightarrow$ perturbative expansion convergent for $|\lambda| < \frac{1}{12}$ and remainder convergent for λ in the cardioid

$$\widetilde{\mathcal{C}} = \left\{ \lambda \in \mathbb{C} \quad \text{with} \quad 12|\lambda| < \cos^2\left(\frac{\arg\lambda}{2}\right) \right\}$$



Topological expansion with remainder

For any disc $\mathcal{D} \subset \widetilde{\mathcal{C}}$, there is $\epsilon > 0$ such that for $\lambda \in \mathcal{D}$ and $\|JJ^\dagger\| < \epsilon$

$$\log \mathcal{Z}[J, J^\dagger; \lambda, N] = \left(\sum_{\substack{G \text{ ciliated ribbon graph} \\ g(G) \leq g}} \frac{(-\lambda)^{|E(G)|} N^{2-2g(G)-|B(G)|}}{|\mathrm{Aut}(G)|} \prod_{f \in B(G)} \mathrm{Tr}\left[(JJ^\dagger)^{c(f)} \right] \right) + \widetilde{\mathcal{R}}_g[J, J^\dagger; \lambda, N]$$

with $|V(G)| - |E(G)| + |F(G)| = 2 - 2g(G)$ and topological remainder

$$\widetilde{\mathcal{R}}_g[J, J^\dagger; \lambda, N] = \sum_{\substack{(G,T) \text{ LVE graph with} \\ g(G) = g+1 \text{ and } g(G - e_{L(G,T)}) = g}} \mathcal{A}_{(G,T)}[J, J^\dagger, \lambda, N]$$

with $G - e_{L(G,T)}$ = graph with last added loop edge removed

- Bound on amplitude and number of genus g graphs

- Any homogenous degree k unitarily invariant polynomial expanded as

$$P(JJ^{\dagger}) = \sum_{\pi \in \Pi_k} P_{\pi} \mathrm{Tr}\big(JJ^{\dagger}\big)^{k_1} \cdots \mathrm{Tr}\big(JJ^{\dagger}\big)^{k_p}$$

with trace invariants indexed by partitions $\pi = k_1 \leq \cdots \leq k_{|\pi|}$ such that $k_1 + \cdots + \cdots k_{|\pi|} = k$.
- Integration formula over unitary group $\mathrm{U}(N)$ involving Weingarten functions $\mathrm{Wg}(\sigma, N) \Rightarrow$ expression of P_{π} in terms of P and Wg

$$\int dU \quad U_{a_1 b_1} \ldots U_{a_k b_k} U^*_{c_1 d_1} \ldots U^*_{c_l d_l} = \sum_{\sigma, \tau \in \mathfrak{S}_k} \delta_{a_{\sigma(1)} c_1} \ldots \delta_{a_{\sigma(k)} c_k} \delta_{b_{\tau(1)} d_1} \ldots \delta_{b_{\tau(k)} d_k} \mathrm{Wg}(\tau\sigma^{-1}, N)$$

Scalar cumulants indexed by integer partitions

$$K_{a_1 b_1 c_1 d_1, \ldots, a_k b_k c_k d_k}(\lambda, N) = \sum_{\pi \in \Pi_k} K_{\pi}(\lambda, N) \sum_{\rho, \sigma \in \mathfrak{S}_k} \prod_{1 \leq l \leq k} \delta_{c_l, a_{\rho\tau\sigma^{-1}(l)}} \delta_{d_l, b_{\rho\xi\sigma^{-1}(l)}}$$

with $\tau, \xi \in \mathfrak{S}_k$ such that cycle decomposition of $\tau\xi^{-1}$ corresponds to π

Analytic expansion for scalar cumulants

Scalar cumulants can be written as a convergent series for $\lambda \in \mathcal{C}$

$$K_\pi(\lambda, N) = \sum_{\text{LVE trees with } k \text{ cilia}} K_{\pi,T}(\lambda, N)$$

and is analytic for $\lambda \in \mathcal{C}$. Moreover,

$$\Big| K_{\pi,T}(\lambda, N) \Big| \leq \frac{2^{2k}(k!)^2 |\lambda|^{|E(T)|} N^{2-|\pi|}}{\left(\cos \frac{\arg \lambda}{2}\right)^{2|E(T)|+k} |V(T)|!}$$

- Express $\mathcal{A}_T[J, J^\dagger; \lambda, N]$ in terms of trace invariants
- Asymptotic behaviour $\Rightarrow$ $\mathrm{Wg}(\sigma, N) \leq \frac{2^{2k}}{N^{2k-|C(\sigma)|}}$
- Bound the resolvent $\Big\| \Big(1 - \mathrm{i}\sqrt{\frac{\lambda}{N}}\Big)^{-1} \Big\| \leq \frac{1}{\cos \frac{\arg \lambda}{2}}$
- Bound the number of cycles in permutations $\rho, \sigma \in \mathfrak{S}_k$ as

$$|C(\rho)| + |C(\sigma)| \leq k + |C(\rho\sigma)|$$

Perturbative expansion with remainder

The perturbative expansion of the cumulants is analytic for $\lambda \in \mathcal{C}$

$$K_\pi(\lambda, N) = \sum_{\substack{G \text{ ribbon graph} \\ \text{with broken faces corresponding to } \pi \text{ and } |E(G)| \leq n}} \frac{(-\lambda)^{|E(G)|} N^{\chi(G)}}{|\mathrm{Aut}(G)|} + \mathcal{R}_{\pi,n}(N, \lambda)$$

$\mathcal{R}_{\pi,n}(N, \lambda)$ is a sum over LVE graphs with k cilia, at least $n+1$ edges and at most $n+1$ loop edges. For all $0 \leq \alpha < \pi$ and $0 \leq \rho < \frac{1}{4}$, there exist $C_{k,\alpha,\rho}$ and σ_α such that for $|\arg \lambda| < \alpha$ and $|\lambda| < \rho \cos^2 \frac{\arg \lambda}{2}$

$$\left| R_{\pi,n}(\lambda, N) \right| \leq N^{2-|\pi|} C_{k,\alpha,\rho} (\sigma_\alpha)^{n+1} |\lambda|^{n+1} (n+1)!$$

Borel summability of cumulants

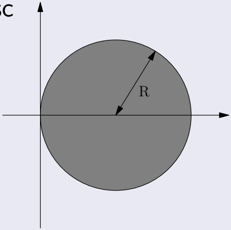
Scalar cumulants are Borel summable at the origin, uniformly in N

$$K_\pi(\lambda, N) = \int_0^\infty ds \left(\sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\substack{G \text{ ribbon graph} \\ \text{with } |E(G)| \leq n \text{ and broken faces corresponding to } \pi}} \frac{(-s)^{|E(G)|} N^{\chi(G)}}{|\mathrm{Aut}(G)|} \right) \exp\left\{ -\frac{s}{\lambda} \right\}$$

Borel summability (Nevanlinna-Sokal theorem)

$\big[F(\lambda)\big]_{\omega\in\Omega}$ = family of analytic functions in the disc

$$\mathcal{D}_R = \left\{ \lambda \in \mathbb{C} \,\middle|\, \mathrm{Re}\left(\frac{1}{\lambda}\right) > \frac{1}{R} \right\}$$



If there are $\sigma > 0$ and $C > 0$ such that for $\lambda \in \mathcal{D}_R$ and $\omega \in \Omega$

$$\Big| F_\omega(\lambda) - \sum_{m=0}^{n} a_m(\omega)\lambda^m \Big| < \sigma^{n+1}|\lambda|^{n+1}(n+1)!$$

then F_ω can be recovered from its perturbative series as

$$F_\omega(\lambda) = \int_0^\infty ds \left(\sum_{n=0}^{\infty} \frac{a_n(\omega)}{n!} s^n \right) \exp\left\{ -\frac{s}{\lambda} \right\}$$

.

$\Rightarrow$ perturbative series $\sum_n a_n(\omega)\lambda^n$ sufficient to construct $F_\omega(\lambda)$

Topological expansion for scalar cumulants

Cumulants $K_\pi(\lambda, N)$ are expanded in inverse powers of N for $\lambda \in \widetilde{\mathcal{C}}$. as

$$K_\pi(\lambda, N) = \sum_{h=0}^{g} N^{2-2g-|\pi|} K_{\pi,h}(\lambda) + \widetilde{R}_{\pi,g}(N, \lambda)$$

where $K_{\pi,h}(\lambda)$ is a convergent series for $|\lambda| < \frac{1}{12}$ over ciliated ribbon graphs of genus $\leq g$ whose broken faces correspond to the partition π

$$K_{\pi,h}(\lambda) = \sum_{\substack{G \text{ ribbon graph with} \\ \text{genus } g \text{ and } \pi \text{ broken faces}}} \frac{(-\lambda)^{|E(G)|}}{|\mathrm{Aut}\, G|}$$

$\widetilde{R}_{\pi,g}(N, \lambda)$ is a sum over LVE graphs with $|\pi|$ broken faces, genus $g+1$ and such that, if we remove the loop edge of highest label, we get a genus g graph. For all $0 \leq \alpha < \pi$ and $0 \leq \rho < \frac{1}{12}$, there exists a constant $\widetilde{C}_{g,k,\alpha,\rho}$ such that

$$\left|\widetilde{R}_{\pi,n}(\lambda, N)\right| \leq N^{2-2(g+1)-|\pi|} |\lambda|^{2(g+1)}\, \widetilde{C}_{k,\alpha,\rho}$$

for $|\arg \lambda| < \alpha$ and $|\lambda| < \frac{1}{12}$.

Summary of results

- Convergent expansion over trees $\Rightarrow$ analyticity of cumulants
- Perturbative and topological expansion with controlled remainder

Techniques used

- LVE expansion over trees
- Recursive generation of loop edges (LVE graphs)
- Bound on resolvent
- Expansion on trace invariants (Weingarten functions)

Other possible applications (not discussed here)

- Non commutative field theory (Rivasseau, 2007)
- Two dimensional models (Rivasseau & Wang, 2010)
- Random tensoris (Gurau, 2013)
- Other matrix models (Kontsevitch, multimatrix) (...)
- Other quantum field theories (Gross-Neveu, QCD) (...)