DECOMPOSITIONS AND RESIDUE OF MEROMORPHIC FUNCTIONS WITH LINEAR POLES (BASED ON JOINT WORK WITH LI GUO AND BIN ZHANG)

Abstract. Germs of meromorphic functions with linear poles at zero naturally arise in various contexts in mathematics and physics. We provide a decomposition of the algebra of such germs into the holomorphic part and a linear complement by means of an inner product using our results on cones and associated fractions in an essential way. Using this decomposition, we generalize the graded residue on germs of meromorphic functions in one variable to a graded residue on germs of meromorphic fractions in several variables with linear poles at zero and prove that it is independent of the chosen inner product. When this residue is applied to exponential discrete sums on lattice cones, we obtain exponential integrals, giving a first relation between exponential sums and exponential integrals on lattice cones. On the other hand, this decomposition of meromorphic germs also provides a key ingredient in the Birkhoff-Hopf type factorization through which we revisited Berline and Vergne's Euler-Maclaurin formula on lattice cones, establishing another relation between exponential sums and integrals.

This is an abridged version of [15].

Contents

Date: April 14, 2015.

1991 *Mathematics Subject Classification.* 32A20, 32A27, 52A20, 52C07, 65B15.

Key words and phrases. meromorphic functions, lattice cones, Birkhoff decomposition, residues, Euler-Maclaurin formula.

1. Introduction

Meromorphic functions play a fundamental role in complex geometry, where they relate to divisors, bundles and sheaves. A special class of meromorphic functions, namely meromorphic functions with linear poles, arise naturally in various contexts, in perturbative quantum field theory when computing Feynman integrals by means of dimensional regularization (see e.g. [8]) or by means of analytic regularization[1] à la Speer [23, 24], in number theory with multiple zeta functions [17, 26] (see also [16, 20, 21, 27], in the combinatorics on cones when evaluating exponential integrals or sums on cones following Berline and Vergne [2] (see also [11]).

We study these meromorphic functions locally, that is, we study germs of meromorphic functions (or meromorphic germs in short) with linear poles, with the aim in mind to extend to this context results which are known for meromorphic germs in one variable. Let us recall some basic results in one variable of direct interest to us for future generalizations to meromorphic germs in several variables. The space of germs of meromorphic functions at a point–say 0– which we denote by $\mathcal{M}_0(\mathbb{C})$, coincides with the space of convergent Laurent series at 0 denoted by $\mathbb{C}\{\varepsilon^{-1}, \varepsilon\}\}$. A meromorphic germ in one variable has a unique Laurent expansion, so our first task is to establish a generalized Laurent expansion for meromorphic germs in several variables with linear poles.

The space $\mathbb{C}\{\varepsilon^{-1}, \varepsilon\}\}$ or its formal version $\mathbb{C}[\varepsilon^{-1}, \varepsilon]]$, or its twisted version $\mathfrak{g}[\varepsilon^{-1}, \varepsilon]]$ for a Lie algebra $\mathfrak{g}$, play important roles in various mathematical fields, such as representation theory, algebraic geometry, mathematical physics. Here are some of its key features:

- It is filtered by the order, say r, of the pole at zero;
- It has a projection: $\pi_+ : \mathbb{C}\{\varepsilon^{-1}, \varepsilon\}\} \to \mathbb{C}\{\{\varepsilon\}\}$ giving a decomposition

$$\mathbb{C}\{\varepsilon^{-1}, \varepsilon\}\} = \varepsilon^{-1}\mathbb{C}[\varepsilon^{-1}] \oplus \mathbb{C}\{\{\varepsilon\}\}; \tag{1}$$

- The corresponding filtered algebra
$$\mathcal{F} := \bigcup_{r=0}^{\infty} \mathcal{F}_r, \quad \mathcal{F}_r := \varepsilon^{-r}\mathbb{C}\{\{\varepsilon\}\}$$
has a residue
$$\mathrm{Res}_0^r(f) = \lim_{z\to 0}(z^r f(z)) \quad \text{for all } f \in \varepsilon^{-r}\mathbb{C}\{\{\varepsilon\}\}$$
which is a graded residue in the sense that it induces a map on the corresponding graded algebra $\mathrm{Gr}\,\mathcal{F} = \oplus_{r=0}^{\infty}\mathcal{F}_r/\mathcal{F}_{r-1}$, which is compatible with the graded product.

The decomposition (1) commonly used in physics is called the minimal subtraction scheme and follows from the fact that π_+ is a Rota-Baxter operator [10, 12], one of the fundamental algebraic concepts used for the algebraic Birkhoff factorization in the Connes-Kreimer approach to renormalization [6]. Given the importance of the decomposition and residue of meromorphic germs in one variable, it is interesting to find their generalizations in the case of several variables.

We provide a decomposition[2] (Theorem 5.3) of the algebra of germs of meromorphic functions in several variables with linear poles at zero into the holomorphic part and its linear complement

[1] see also recent work by N.V. Dang [9]

[2] This decomposition compares with a decomposition [3, Theorem 7.3] of the algebra $\mathcal{R}_\Delta$ of rational functions with linear poles in a finite set Δ of (linear) hyperplanes. Their result can be compared with the decomposition derived in in [7, Theorem 8.16] for affine hyperplanes.

by means of an inner product using previous results on cones and associated fractions [13] in in an essential way.

This decomposition leads to interesting applications:

- an algebraic Birkhoff factorization for characters with range in meromorphic functions in several variables [14], which justifies calling *renormalization map* the projection map π_+ which assigns to a function its holomorphic part in the above decomposition
- the factorization property (9) of the projection map π_+ which corresponds to locality in the context of quantum field theory,
- a generalization of the residue (Definition 6.9) by means of which the exponential integral on a cone can be viewed as the residue of the corresponding exponential sum on the discrete points of the cone (Corollary 6.19).

In order to generalize the fraction $\frac{\mathrm{Res}_0^r(f)}{z^r}$ containing the highest order residue $\mathrm{Res}_0^r(f)$ to meromorphic germs with linear poles, we need a filtration. Using the decomposition of the algebra of meromorphic germs at zero with linear poles of Theorem 5.3, we first filter the algebra of meromorphic germs with linear poles by what we call the p-order (p for polar), which would be r in the above one variable example. Using again the decomposition of the algebra of meromorphic germs, we then pick (see Definition 6.9) the highest polar order fractions in the decomposition, to build the highest polar order residue, or the **p-residue** in short, which boils down to $\frac{\mathrm{Res}_0^r(f)}{z^r}$ in the above example. The p-residue is uniquely defined and we show (Proposition 6.11) that it is independent of the decomposition of f induced by the chosen splitting, and hence is an intrinsic invariant of f. Had we instead picked out for the residue the homogeneous part of p-order 1, it would have depended on the choice of splitting for germs with p-order larger than 1. The p-residue, which is actually a graded residue, is compatible with the product on perpendicular fractions (Proposition 6.13). Had we picked instead of the highest p-order term, the term of p-order 1 as a residue, this compatibility would not have held true.

We then apply this p-residue to the exponential sums on lattice cones, which actually was our original motivation to study the p-residue. Let us recall the one-dimensional Euler-Maclaurin formula for exponential sums and the generalization to higher dimensional cones by Berline and Vergne. On the (closed) cone $[0, +\infty)$, the exponential sum on the lattice points of the cone (the lattice given by the natural integer points) $S(\varepsilon) := \sum_{k=0}^{\infty} e^{\varepsilon k} = \frac{1}{1-e^{\varepsilon}}$ defined for negative ε, relates to the integral $I(\varepsilon) := \int_0^{\infty} e^{\varepsilon x}\, dx = -\frac{1}{\varepsilon}$ by means of the Euler-Maclaurin formula,

$$S(\varepsilon) = I(\varepsilon) + \mu(\varepsilon).$$

Here $\mu(\varepsilon) = -\sum_{n=0}^{\infty} \frac{B_{n+1}}{(n+1)!}\, \varepsilon^n$ is holomorphic at $\varepsilon = 0$, so the integral $I(\varepsilon)$ corresponds to the pole part of the Laurent expansion of the sum $S(\varepsilon)$ at zero.

Similarly, to a lattice cone in a linear space, namely a convex cone equipped with a lattice, one can assign two meromorphic functions, the exponential integral I on the cone and the exponential sum S on the lattice points of the cone, which Berline and Vergne could relate by a generalized Euler-Maclaurin formula ([2, Theorem 19]). One of the motivations for the present paper is to generalize to higher dimensions, the fact that in the one-dimensional case, I arises as the pole part of S; we indeed show that the p-residue of the exponential sum S on a lattice cone is the exponential integral I. The integrals being p-residues of the discrete sums, our result, like the Euler-Maclaurin formula, relates discrete sums and the corresponding integrals.

To conclude, we introduce here a new type of residue on meromorphic functions in several variables with linear poles; those include the ones usually associated with a hyperplane arrangement.

As previously mentioned, meromorphic functions with linear poles and their generalizations to meromorphic functions with affine poles, play a role in

- number theory where they arise while regularizing multiple zeta functions; for given complex numbers $s_i, i = 1, \dots, k$ such that $s_1 + \cdots + s_i = i$ for some $i \in \{1, \dots, k\}$, the function
$$(\varepsilon_1, \dots, \varepsilon_k) \longmapsto \zeta(s_1 + \varepsilon_1, \dots, s_k + \varepsilon_k) := \sum_{0 < n_k < \cdots < n_1} n_1^{-s_1 - \varepsilon_1} \cdots n_k^{-\varepsilon_k}$$
extends to a meromorphic function with poles corresponding to the linear forms $L_i : \vec{\varepsilon} \longmapsto \varepsilon_1 + \cdots + \varepsilon_i, i = 1, \dots, k$ (see e.g. [20]),
- quantum field theory, while regularizing Feynman integrals analytically [24], see Theorem 3.1. In this context, the factorization property (9) of the projection map π_+ over orthogonal meromorphic functions corresponds to the locality of the underlying theory [9],
- but also and maybe more surprisingly, in queuing networks [4].

Our residue does not extend the classical residue of a Laurent expansion and to our knowledge therefore differs in nature from other generalizations of the classical residue such as Grothendieck's residue symbol (see e.g. [18]) (also referred to as the multidimensional residue) which, as the classical residue does, has an integral realization [25] (see also [22]). Similarly to our residue, the Grothendieck residue arises in the combinatorics on polytopes since it can be used to count integer points on certain dilated polytopes [1], leading to a derivation of Erhardt's polynomial. It further arises together with its integral representation in the context of queuing networks [4].firs

Our residue uses a decomposition of the space of meromorphic functions with linear poles induced by an inner product on the underlying space, into the space of holomorphic functions and its (orthogonal) complement. It differs from the Jeffrey-Kirwan residue which applies to rational functions with poles in a hyperplane arrangement. Our decomposition of meromorphic germs with linear poles (see Theorem 5.3) and geometric criterion for non-holomorphicity (see Theorem 4.9) lead to another proof [15] of the decomposition of the space of rational functions with poles in a given hyperplane arrangement derived by Brion and Vergne [5].

2. Germs of meromorphic functions

In this section, we provide some terminology to describe the class of meromorphic functions under consideration. Here F denotes a subfield of $\mathbb{R}$ which will often be chosen to be $\mathbb{R}$.

Definition 2.1. (a) A **rational (vector) space** is a pair (V, Λ_V) where V is a finite dimensional real vector space and Λ_V is a lattice in V, that is, a finitely generated abelian subgroup of V whose $\mathbb{R}$-linear span is V;

(b) A **filtered space** is a real vector space V with a filtration $V_1 \subset V_2 \subset \cdots$ of real vector subspaces such that $V = \cup_{k\geq 1} V_k$. Let $j_k : V_k \to V_{k+1}$ denote the inclusion;

(c) A **filtered rational space** is a filtered space $V = \cup_k V_k$ with lattices $\Lambda_k := \Lambda_{V_k}$ of V_k such that $\Lambda_{k+1}|_{V_k} = \Lambda_k, k \geq 1$. Then we denote the filtered rational space by $(V, \Lambda_V) = \cup_k (V_k, \Lambda_{V_k})$ where $\Lambda_V = \cup_k \Lambda_{V_k}$;

(d) An **inner product** Q on a filtered space $V = \cup_{k\geq 1} V_k$ is a sequence of inner products

$$Q_k(\cdot,\cdot) = (\cdot,\cdot)_k : V_k \otimes V_k \to \mathbb{R}, \quad k \geq 1,$$

that is compatible with the inclusions $j_k, k \geq 1$;

(e) An F**-inner product** on a filtered rational space (V, Λ_V) is an inner product $\{Q_k\}_{k\geq 1}$ on the filtered space $V = \cup_{k\geq 1} V_k$ such that the restriction of Q_k to $\Lambda_{V_k} \otimes F$ and hence Λ_{V_k} takes values in F. A filtered rational space together with an F-inner product is called a **filtered rational F-Euclidean space.**

We now assume that $V = \cup V_k$ is a filtered rational F-Euclidean space. Let V_k^* be the dual space of V_k, then a vector v in V_k can be viewed as a linear functional on V_k^*

$$v : V_k^* \to \mathbb{R}, \quad f \mapsto f(v).$$

In particular, for a basis $\{e_i\}$ of V_k with dual basis $\{e_i^*\}$ of V_k^*, we have the linear functionals

$$e_i : V_k^* \to \mathbb{R}, \quad u = \sum_i \varepsilon_i e_i^* \mapsto \varepsilon_i.$$

Thus we also denote e_i by ε_i, seen as a function on V_k^*.

Definition 2.2. Let $\cup_k (V_k, \Lambda_k)$ be a filtered rational space.

(a) A **germ of meromorphic functions at** 0 or **meromorphic germ** in short on $V_k^* \otimes \mathbb{C}$ is the quotient of two holomorphic functions in a neighborhood of 0 inside $V_k^* \otimes \mathbb{C}$ with respect to the canonical complex structure on $V_k^* \otimes \mathbb{C}$. So a function on $V_k^* \otimes \mathbb{C}$ is holomorphic if it is holomorphic in the complex coordinates in any dual basis of V_k^*.

(b) A germ of meromorphic functions $f(\vec{\varepsilon})$ on $V_k^* \otimes \mathbb{C}$ is said to have **linear poles at zero with coefficients in** F if there exist vectors $L_1, \cdots, L_n \in \Lambda_{V_k} \otimes F$ (possibly with repetitions) such that $f \Pi_{i=1}^n L_i$ is a holomorphic germ at zero whose Taylor expansion for coordinates in the dual basis $\{e_1^*, \cdots, e_k^*\}$ of a given (and hence every) basis $\{e_1, \cdots, e_k\}$ of Λ_k has coefficients in F.

(c) A germ of meromorphic functions of the form $\frac{1}{L_1^{s_1} \cdots L_n^{s_n}}$ with linearly independent vectors $L_1, \cdots, L_n$ in $\Lambda_k \otimes F$ and $s_1, \cdots, s_n \geq 1$ is called a **simplicial fraction** with coefficient in F. Such a fraction is called **simple** if all $s_1 = \cdots = s_n = 1$ and **multiple** otherwise. The linear space generated by simple simplicial fractions with coefficient in F is denoted by $S(F)$.

Let $\mathcal{M}_F(V_k^*\otimes\mathbb{C})$ be the set of germs of meromorphic functions on $V_k^*\otimes\mathbb{C}$ with linear poles at zero and with coefficients in F, which defines a linear space over F.

The F-inner product $Q_k: V_k\otimes V_k\to\mathbb{R}$ induces an isomorphism $Q_k^*: V_k\to V_k^*$. This yields an embedding $V_k^*\hookrightarrow V_{k+1}^*$ induced from $j_k: V_k\to V_{k+1}$. We refer to the direct limit $V^{\circledast}:=\bigcup_{k=0}^\infty V_k^*=\varinjlim V_k^*$ as the **filtered dual space** of V and set

$$\mathcal{M}_F(V^{\circledast}\otimes\mathbb{C}):=\varinjlim \mathcal{M}_F(V_k^*\otimes\mathbb{C})=\bigcup_{k=1}^\infty \mathcal{M}_F(V_k^*\otimes\mathbb{C}).$$

Let $\mathcal{M}_{F,+}(V_k^*\otimes\mathbb{C})$ denote the space of germs of holomorphic functions at zero in $V_k^*\otimes\mathbb{C}$ whose Taylor expansions at zero have coefficients in F under the dual basis of a basis of Λ_k. We set $\mathcal{M}_{F,+}(V^{\circledast}\otimes\mathbb{C}):=\bigcup_{k=1}^\infty \mathcal{M}_{F,+}(V_k^*\otimes\mathbb{C})$.

For $V=\mathbb{R}^\infty$ equipped with the filtration $V_k:=\mathbb{R}^k$ with its standard lattice $\mathbb{Z}^k$ and standard inner product, the dual rational space V_k^* is identified with $\mathbb{R}^k$ equipped with the standard lattice. Thus the space $\mathcal{M}_{F,+}(\mathbb{C}^k):=\mathcal{M}_{F,+}(V_k^*\otimes\mathbb{C})$ corresponds to the space of germs of holomorphic functions at zero in $\mathbb{C}^k$ whose Taylor expansions at zero have coefficients in F with respect to the canonical basis of $\mathbb{R}^k$.

We next identify a class of polar germs (that is, non-holomorphic meromorphic germs) that will be shown to give a linear complement of the subspace $\mathcal{M}_{F,+}(V_k^*\otimes\mathbb{C})$ (Theorem 5.3). Thus they can be regarded as purely polar germs. For notational simplicity, we will call them polar germs.

Definition 2.3. A **polar germ with F-coefficients** in $V_k^*\otimes\mathbb{C}$ is a germ of meromorphic functions at zero of the form

$$\frac{h(\ell_1,\cdots,\ell_m)}{L_1^{s_1}\cdots L_n^{s_n}},$$

where

(a) $\ell_1,\cdots,\ell_m,L_1,\cdots,L_n$ lie in $\Lambda_k\otimes F$, with $L_1,\cdots,L_n$ linearly independent, such that

$$Q(\ell_i,L_j)=0\qquad\text{for all } (i,j)\in[m]\times[n],$$

where for a positive integer p, we have set $[p]=\{1,\cdots,p\}$,

(b) h lies in $\mathcal{M}_{F,+}(\mathbb{C}^m)$,

(c) $s_1,\cdots,s_n$ are positive integers.

Remark 2.4. Without loss of generality we can assume that $\ell_1,\cdots,\ell_m$ are linearly independent.

Definition 2.5. We let $\mathcal{M}^Q_{F,-}(V_k^*\otimes\mathbb{C})$ denote the F-span of polar germs in $\mathcal{M}_F(V_k^*\otimes\mathbb{C})$ and set

$$\mathcal{M}^Q_{F,-}(V^{\circledast}\otimes\mathbb{C}):=\bigcup_{k=1}^\infty \mathcal{M}^Q_{F,-}(V_k^*\otimes\mathbb{C}).$$

Remark 2.6. Whereas the space $\mathcal{M}^Q_{F,-}(V_k^*\otimes\mathbb{C})$ depends on the choice of the inner product Q, the space $\mathcal{M}_{F,+}(V_k^*\otimes\mathbb{C})$ does not.

Example 2.7. (a) For linearly independent vectors $L_1,\cdots,L_k\in\Lambda_k\otimes F$ and $s_1,\cdots,s_k>0$, $\dfrac{1}{L_1^{s_1}\cdots L_k^{s_k}}$ lies in $\mathcal{M}^Q_{F,-}(V_k^*\otimes\mathbb{C})$ for any inner product Q.

(b) Let $Q:=(\cdot,\cdot)$ be the canonical Euclidean inner product on $\mathbb{R}^\infty$. Then the functions $f(\varepsilon_1e_1^*+\varepsilon_2e_2^*)=\dfrac{(\varepsilon_1-\varepsilon_2)^t}{(\varepsilon_1+\varepsilon_2)^s}$, $s>0,t\geq 0$, lie in $\mathcal{M}^Q_{\mathbb{Q},-}((\mathbb{R}^2)^*\otimes\mathbb{C})$.

3. From cones to fractions

In this section we recall from [13] how, by means of Laplace type transforms, the geometry of cones can be used for the decomposition of fractions. Propostion 3.7 provides a geometric criterion for the linear independence of certain fractions arising from expopential integrals on cones.

Consider a filtered rational space $V = \cup_{k\geq 1} V_k$, with a fixed ordered basis $(e_1, e_2, \cdots)$ such that $(e_1, e_2, \cdots) \cap V_k = (e_1, \cdots, e_k)$ is a basis of Λ_k. A **(closed convex polyhedral) cone** in V_k is the set

$$\langle v_1, \cdots, v_n\rangle := \mathbb{R}_{\geq 0} v_1 + \cdots + \mathbb{R}_{\geq 0} v_n, \tag{2}$$

where $v_i \in V_k$, $i = 1, \cdots, n$. It is called an F-cone if the v_i's in Eq. (2) are in $\Lambda_k \otimes F$. If $F = \mathbb{Q}$, then it is called rational. The dimension is defined as the dimension of the linear subspace spanned by the v_i's. A cone is called **strongly convex** if it does not contain any nonzero linear subspace. A cone is **simplicial** if it is generated by $\mathbb{R}$-linearly independent vectors, so a simplicial cone is strongly convex. A rational cone is **smooth** if it is generated by part of a basis of Λ_k.

A **subdivision of a cone** C is a set $\{C_1, \cdots, C_r\}$ of cones such that

(i) $C = \cup_{i=1}^r C_i$,
(ii) $C_1, \cdots, C_r$ have the same dimension as C, and
(iii) $C_1, \cdots, C_r$ intersect along their faces,
i.e., for $1 \leq i, j \leq r$, $C_i \cap C_j$ is a face of both C_i and C_j.

A subdivision is called simplicial (resp. smooth, in the case when C is rational) if all C_i's are simplicial (resp. smooth). An F-subdivision of an F-cone is a subdivision such that every C_i is an F-cone.

A **proper subdivision of a family of cones** $\{C_i\}$ is a set $\{D_1, \cdots, D_r\}$ of cones such that

(a) $D_1, \cdots, D_r$ intersect along their faces,
(b) for any i, there is $I_i \subset [r]$ such that $\{D_l\}_{l\in I_i}$ is a subdivision of C_i, and
(c) $\cup_i I_i = [r]$.

A proper F-subdivision of a family of F-cones is a proper subdivision such that every D_i is an F-cone.

We now rephrase results in [13].

Lemma 3.1. [13, Lemma 2.3]

(a) *Any finite family $\{C_i\}$ of cones in V_k has a simplicial proper subdivision.*
(b) *Any finite family of rational cones in V_k has a smooth proper subdivision.*

As a subfield of $\mathbb{R}$, F contains $\mathbb{Q}$, so $\Lambda_k \otimes F$ is dense in V_k. Adapting the proof of [13, Lemma 2.3(a)], we obtain

Lemma 3.2. *Any finite family $\{C_i\}_1^m$ of F-cones has a simplicial proper F-subdivision.*

We now assign to each cone a fraction by means of the Laplace transform of the characteristic function of the cone.

Let C be a simplicial cone in V_k with $\mathbb{R}$-linearly independent generators $v_1, \cdots v_n$ expressed in the fixed basis $\{e_1, \cdots, e_k\}$ as $v_i = \sum\limits_{j=1}^{k} a_{ji} e_j$, for $1 \leq i \leq n$. Define linear functions $L_i(\vec{\varepsilon}) :=$

$L_{v_i}(\vec{\varepsilon}) := \sum_{j=1}^{k} a_{ji}\varepsilon_j$, where $\vec{\varepsilon} := \sum_{i=1}^{k} \varepsilon_j e_j^* \in V_k^* \otimes \mathbb{C}$ and $\{e_1^*, \cdots, e_k^*\}$ is the dual basis in V_k^*. Let $A_C = [a_{ij}]$ denote the associated matrix in $M_{k\times n}(\mathbb{R})$ with v_i as column vectors. Let $w(v_1, \cdots, v_n)$ or $w(C)$ denote the sum of absolute values of the determinants of all minors of A_C of rank n. As in [13], define , using the symbol I (for "integral")

$$I(C) := \frac{w(v_1, \cdots, v_n)}{L_1 \cdots L_n}, \tag{3}$$

which is in $S(\mathbb{R})$ introduced in Definition 2.2 (c). For a simplicial F-cone C, $I(C)$ is in $S(F)$.

Further for any cone C, define $I(C) := \sum_i I(C_i)$ where $\{C_i\}$ is a simplicial subdivision of C. As shown in [13], thanks to the following lemma, I is well-defined, independently of the choice of the chosen simplicial subdivision.

Lemma 3.3. [13, Lemma 3.2] *Let C be a simplicial cone and $\{C_1, \cdots, C_r\}$ be a simplicial subdivision of C, then $I(C) = \sum_{i=1}^{r} I(C_i)$.*

Let $\mathfrak{C}$ be the set of cones in V, and $\mathbb{R}\mathfrak{C}$ be the vector space with basis $\mathfrak{C}$. We obtain a linear map

$$I : \mathbb{R}\mathfrak{C} \to S(\mathbb{R}).$$

Definition 3.4. A family of cones is said to be **properly positioned** if the cones meet along faces and the union does not contain any nonzero linear subspace.

Example 3.5. Any subdivision of a strongly convex cones yields a properly positioned family of cones.

Adapting the proof of [13, Lemma 2.3 (a)] we get the following result.

Proposition 3.6. *A family of F-cones whose union does not contain any nonzero linear subspace, can be subdivided into a properly positioned set of simplicial F-cones.*

We have the following geometric criterion for the linear independence of fractions which follows from a slight reformulation of Lemma 3.5 in [13].

Proposition 3.7. *Let $\{C_i\}$ be a set of properly positioned simplicial cones each of whose elements C_i spans the same linear subspace. Then the set $\{I(C_i)\}$ of fractions is linearly independent.*

4. From polar germs to cones

In this section, we start from fractions to which we assign (under certain technical conditions) cones called supporting cones. Proposition 4.7 gives for a family of polar germs, sufficient conditions on their supporting cones for the linear independence of the germs, thus providing a geometric criterion for the linear independence. Theorem 4.9 gives sufficient conditions on the supporting cones for the non holomorphicity of polar germs.

Partial differentiation on the space of fractions carries over to cones. For the fixed basis $\{e_1, e_2, \cdots\}$, let $\{e_1^*, e_2^*, \cdots\}$ be the dual basis, and $\vec{\varepsilon} = \sum \varepsilon_i e_i^*$ be an element in $V^{\circledast} \otimes \mathbb{C}$. We define the differential operators

$$\partial_i = -\frac{\partial}{\partial \varepsilon_i} : \mathcal{M}_F(V^{\circledast} \otimes \mathbb{C}) \to \mathcal{M}_F(V^{\circledast} \otimes \mathbb{C}).$$

The following proposition follows from straightforward computations.

Proposition 4.1. ([13, Proposition 4.8]) *For a fraction $\frac{1}{L_1^{s_1}\cdots L_k^{s_k}}$, let $\{L_i^* = \sum_j c_{ij}e_j^*\}_i$ be dual to $\{L_i\}_i$ in the sense that $(L_i, L_j^*) = \delta_{ij}, 1 \le i, j \le k$. Define $\partial_{L_i^*} = \sum_j c_{ij}\partial_j$. Then we have*

$$\frac{1}{L_1^{s_1}\cdots L_k^{s_k}} = \frac{1}{(s_1-1)!\cdots(s_k-1)!}\partial_{L_1^*}^{s_1-1}\cdots\partial_{L_k^*}^{s_k-1}\frac{1}{L_1\cdots L_k}. \tag{4}$$

The following lemma is proved by induction on the total order $s := s_1 + \cdots + s_n$ of the poles.

Lemma 4.2. *If a polar germ can be written as $\frac{h(\ell_1,\cdots,\ell_m)}{L_1^{s_1}\cdots L_n^{s_n}}$ and $\frac{g(\ell_1',\cdots,\ell_k')}{(L_1')^{t_1}\cdots(L_\ell')^{t_\ell}}$, both in a form satisfying the conditions in Definition 2.3, then $k = \ell$, and $L_1', \cdots, L_\ell'$ can be rearranged in such a way that L_i is a multiple of L_i' and $s_i = t_i$ for $1 \le i \le k$.*

The following definition will be justified with the subsequent lemma.

Definition 4.3. (a) A vector v of V is called **pseudo-positive** (with respect to the chosen basis) if the first nonzero coefficient of v under the ordered basis $(e_1, e_2, \cdots)$ is 1.

(b) Let
$$\frac{h(\ell_1,\cdots,\ell_m)}{L_1^{s_1}\cdots L_n^{s_n}},$$
be a polar germ such that the vectors $L_1, \cdots, L_n$ are pseudo-positive. Then the cone $\langle L_1, \cdots, L_n\rangle$ is called the **supporting cone** of the germ.

Example 4.4. The supporting cone of the germ $\dfrac{1}{(\varepsilon_1+\varepsilon_2)^2(\varepsilon_2-\varepsilon_1)}$ is the cone $\langle e_1 + e_2, e_1 - e_2\rangle$.

Example 4.5. Let C be a simplicial cone. The supporting cone of the germ $I(C)$ clearly is the cone C.

The following lemma justifies the introduction of the notion of pseudo-positive vectors.

Lemma 4.6. *If $\{L_1, \cdots, L_n\}$ is a set of pseudo-positive vectors, then the cone $\langle L_1, \cdots, L_n\rangle$ is strongly convex.*

With the concept of supporting cone at hand, we can now restate Proposition 3.7 in terms of a geometric criterion for the linear independence of fractions the proof of which is similar to that of Lemma 4.9 in [13].

Proposition 4.7. (a) *Simple simplicial fractions not pairwise proportional whose supporting cones are properly positioned and span the same linear subspace are linearly independent.*

(b) *More generally any set of simplicial fractions not pairwise proportional whose supporting cones are properly positioned and span the same linear subspace are linearly independent.*

Example 4.8. Given two $\mathbb{R}$-linearly independent linear forms L_1, L_2, then the fractions $\frac{1}{L_1^{s_1}(L_1+L_2)^{s_2}}$ and $\frac{1}{(L_1+L_2)^{r_1}L_2^{r_2}}$ are $\mathbb{R}$-linearly independent for any (r_1, r_2) and (s_1, s_2) in $\mathbb{N}^2$ since their supporting cones $\langle L_1, L_1 + L_2\rangle$ and $\langle L_2, L_1 + L_2\rangle$ span the same linear space and are properly positioned.

Based on this, we prove the following non-holomorphicity of polar germs.

Theorem 4.9. *Let $\left\{\frac{h_i}{L_{i1}^{s_{i1}}\cdots L_{in_i}^{s_{in_i}}}\right\}_{1\le i\le p}$ be a set of polar germs with coefficients in F such that*

- *for any* $1 \leq i \neq j \leq p$*, the two functions* $L_{i1}^{s_{i1}} \cdots L_{in_i}^{s_{in_i}}$ *and* $L_{j1}^{s_{j1}} \cdots L_{jn_j}^{s_{jn_j}}$ *are not proportional to each other,*
- *the supporting cones for* $\frac{h_i}{L_{i1}^{s_{i1}} \cdots L_{in_i}^{s_{in_i}}}$*,* $1 \leq i \leq p$*, are properly positioned.*

If a linear combination

$$\sum_{i=1}^{p} a_i \frac{h_i}{L_{i1}^{s_{i1}} \cdots L_{in_i}^{s_{in_i}}}, \quad a_i \in F, 1 \leq i \leq p, \tag{5}$$

is holomorphic, then $a_i = 0$ *for* $1 \leq i \leq p$*.*

Proof. The proof is carried out ad absurdum. Suppose that the theorem does not hold. Then there is a linear combination of the form described in Eq. (5) with nonzero real coefficients a_i, $1 \leq i \leq p$, and a germ of holomorphic functions h_0 such that the germ

$$\sum_{i=1}^{p} a_i \frac{h_i}{L_{i1}^{s_{i1}} \cdots L_{in_i}^{s_{in_i}}} - h_0 = 0 \tag{6}$$

vanishes in a neighborhood of zero. From there one extracts a family of fractions $\frac{1}{L_{i1}^{s_{i1}} \cdots L_{in_i}^{s_{in_i}}}$, $1 \leq i \leq t$, that satisfies the conditions in Proposition b which implies that the coefficients a_i, $1 \leq i \leq t$, are zero leading to a contradiction. □

The non-holomorphicity theorem 4.9 is used to prove the uniqueness of the decomposition.

5. A decomposition of the space of meromorphic germs at zero with linear poles

In this section, we establish our decomposition of the space of meromorphic germs. Again fix a subfield F of $\mathbb{R}$.

We begin with a reduction of a general fraction to a linear combination of simplicial fractions.

Lemma 5.1. *Let* $L_1, \cdots, L_n$ *be vectors in* $\Lambda_k \otimes F$ *and* $s_1, \cdots, s_k$ *be positive integers. Then the fraction* $\frac{1}{L_1^{s_1} \cdots L_n^{s_n}}$ *can be rewritten as a linear combination*

$$\sum_i \frac{a_i}{M_{i1}^{t_{i1}} \cdots M_{in_i}^{t_{in_i}}},$$

with $a_i \in F$ *and a subset* $\{M_{i1}, \cdots, M_{in_i}\}$ *of linearly independent subset of* $\{L_1, \cdots, L_k\}$*.*

We leave out the proof which can be carried out by induction on the difference $d := n - \dim(\mathrm{lin}\{L_1, \cdots, L_n\})$ and instead choose to illustrate the lemma with an example.

Example 5.2. Given two linearly independent linear forms L_1, L_2, since $\frac{1}{L_1 L_2 (L_1+L_2)} = \frac{1}{L_1(L_1+L_2)^2} + \frac{1}{L_2(L_1+L_2)^2}$ we have the following decomposition of $\frac{1}{L_1^2 L_2^2 (L_1+L_2)}$ into simple simplicial fractions:

$$\begin{aligned} \frac{1}{L_1^2 L_2^2 (L_1+L_2)} &= \left(\frac{1}{L_1(L_1+L_2)} + \frac{1}{L_2(L_1+L_2)}\right)^2 \frac{1}{L_1+L_2} \\ &= \frac{1}{L_1^2(L_1+L_2)^3} + \frac{2}{L_1 L_2 (L_1+L_2)^3} + \frac{1}{L_2^2(L_1+L_2)^3} \\ &= \frac{1}{L_1^2(L_1+L_2)^3} + \frac{2}{L_1(L_1+L_2)^4} + \frac{2}{L_2(L_1+L_2)^4} + \frac{1}{L_2^2(L_1+L_2)^3}. \end{aligned}$$

The following decomposition, which is our first main result, generalizes the Laurent expansion of meromorphic functions in one variable and the Rota-Baxter decomposition $\mathcal{M}(\mathbb{C}) = \mathcal{M}_-(\mathbb{C}) \oplus \mathcal{M}_+(\mathbb{C})$ of the space of Laurent series at 0 given by the projection onto the holomorphic part of the Laurent series.

Theorem 5.3. *Let (V, Λ_V) be a filtered rational F-Euclidean space, we have*

(a) *For $1 \leq k < \infty$, we have the direct sum decomposition*

$$\mathcal{M}_F(V_k^* \otimes \mathbb{C}) = \mathcal{M}_{F,-}(V_k^* \otimes \mathbb{C}) \oplus \mathcal{M}_{F,+}(V_k^* \otimes \mathbb{C}).$$

In particular, an element $f = \frac{h}{L_1 \cdots L_n}$ in $\mathcal{M}_F(V_k^* \otimes \mathbb{C})$ can be written as a sum

$$f = \sum_i \left(\frac{h_i(\vec{\ell}_i)}{\vec{L}_i^{\vec{s}_i}} + \phi_i(\vec{\ell}_i, \vec{L}_i) \right) \tag{7}$$

where for each i,

(i) $|\vec{s}_i| > 0$,

(ii) $\vec{L}_i = (L_{i1}, \cdots, L_{im_i})$ *where* $\{L_{i1}, \cdots, L_{im_i}\}$ *is a linear independent subset of* $\{L_1, \cdots, L_k\}$,

(iii) $\vec{\ell}_i = (\ell_{i(m_i+1)}, \cdots \ell_{ik})$ *where* $\{\ell_{i(m_i+1)}, \cdots \ell_{ik}\}$ *is a basis with F-coefficients of the orthogonal complement of the subspace spanned by $\vec{L}_i$ with respect to the inner product Q,*

(iv) $h_i(\vec{\ell}_i)$ *is holomorphic in the independent variables $\vec{\ell}_i$ (reduced to constant when $k = 1$) whose Taylor series expansion has coefficients in F, so that $\frac{h_i(\vec{\ell}_i)}{\vec{L}_i^{\vec{s}_i}}$ lies in $\mathcal{M}_{F,-}(V_k^* \otimes \mathbb{C})$,*

(v) ϕ_i *is a germ of holomorphic function in the independent variables $\vec{\ell}_i$ and $\vec{L}_i$.*

(b) *Taking a direct limit yields*

$$\mathcal{M}_F(V^\circledast \otimes \mathbb{C}) = \mathcal{M}_{F,-}(V^\circledast \otimes \mathbb{C}) \oplus \mathcal{M}_{F,+}(V^\circledast \otimes \mathbb{C}).$$

(c) *The projection map*

$$\pi_+ : \mathcal{M}_F(V^* \otimes \mathbb{C}) \to \mathcal{M}_{F,+}(V^* \otimes \mathbb{C}) \tag{8}$$

onto $\mathcal{M}_{F,+}(V^\circledast \otimes \mathbb{C})$ along $\mathcal{M}_{F,-}(V^\circledast \otimes \mathbb{C})$ factorizes on perpendicular functions. More precisely, with the notation of Eq. (7), let

$$f = \sum_i \left(\frac{h_i(\vec{\ell}_i)}{\vec{L}_i^{\vec{s}_i}} + \phi_i(\vec{\ell}_i, \vec{L}_i) \right) \quad \text{and } g = \sum_j \left(\frac{k_j(\vec{m}_j)}{\vec{M}_j^{\vec{t}_j}} + \psi_j(\vec{m}_j, \vec{M}_j) \right) \tag{9}$$

be two germs of functions in $\mathcal{M}_F(V^\circledast \otimes \mathbb{C})$ with the property that the linear forms in $\{\vec{L}_i, \vec{\ell}_i\}$ are perpendicular to those in $\{\vec{M}_j, \vec{m}_j\}$ for any i, j. We have

$$\pi_+(f\, g) = \pi_+(f)\, \pi_+(g). \tag{10}$$

Remark 5.4. This factorization property applied to analytically regularized Feynman integrals [23] says that integrals corresponding to a concatenation of Feynman diagrams factorize so that independent events can be observed independently (see also the recent work by N.V. Dang [9]).

Before giving the proof, let us illustrate the statement with an example.

Example 5.5. Take the standard inner product. Since $\frac{1}{L_1L_2(L_1+L_2)} = \frac{1}{L_1(L_1+L_2)^2} + \frac{1}{L_2(L_1+L_2)^2}$, for any real number c we have

$$\begin{aligned} f_c := \frac{(L_1+L_2+c)^3}{L_1L_2(L_1+L_2)} &= \frac{(L_1+L_2)^3 + 3c\,(L_1+L_2)^2 + 3c^2\,(L_1+L_2) + c^3}{L_1L_2(L_1+L_2)} \\ &= 2 + \frac{L_2}{L_1} + \frac{3c}{L_1} + \frac{3c^2}{L_1(L_1+L_2)} + \frac{c^3}{L_1(L_1+L_2)^2} \\ &+ \frac{L_1}{L_2} + \frac{3c}{L_2} + \frac{3c^2}{L_2(L_1+L_2)} + \frac{c^3}{L_2(L_1+L_2)^2}. \end{aligned}$$

Hence $\pi_+(f_c) = 2$.

Remark 5.6. The holomorphic functions h_i and ϕ_i in the decomposition (7) depend on the choice of inner product Q.

Proof. (a) We prove the statement for $1 \le k < \infty$. Then taking the direct limit gives the result for any filtered space.

Let $1 \le k < \infty$ be given. We first verify the decomposition

$$\mathcal{M}_F(V_k^* \otimes \mathbb{C}) = \mathcal{M}_{F,-}(V_k^* \otimes \mathbb{C}) + \mathcal{M}_{F,+}(V_k^* \otimes \mathbb{C}).$$

Thanks to Lemma 5.1, without loss of generality we can reduce the proof to germs of functions of the type

$$f = \frac{h}{L_1^{s_1} \cdots L_m^{s_m}}$$

with $h \in \mathcal{M}_{F,+}(V_k^* \otimes \mathbb{C})$, linearly independent linear forms $L_1, \cdots, L_m \in \Lambda_k \otimes F$ and $s_1, \cdots, s_m$ positive integers. Then we extend $\{L_1, \cdots, L_m\}$ to a basis $\{L_1, \cdots, L_m, \ell_1, \cdots, \ell_{k-m}\}$ of $\Lambda_k \otimes F$ with the additional property that

$$Q(L_i, \ell_j) = 0, \quad 1 \le i \le m, 1 \le j \le k-m.$$

Note that for $k = 1$, this decomposition corresponds to the minimal subtraction scheme decomposition $\frac{h(L_1)}{L_1} = \frac{h(0)}{L_1} + \frac{h(L_1)-h(0)}{L_1}$. To prove the general case we proceed by induction on the sum $s = s_1 + \cdots + s_m$. If $s = 1$, then $m = 1$, and $s_1 = 1$. We write

$$\frac{h(L_1, \ell_1, \cdots, \ell_{k-1})}{L_1} = \frac{h(0, \ell_1, \cdots, \ell_{k-1})}{L_1} + \frac{h(L_1, \ell_1, \cdots, \ell_{k-1}) - h(0, \ell_1, \cdots, \ell_{k-1})}{L_1}.$$

The first term lies in $\mathcal{M}_{F,-}(\mathbb{C}^k)$ as a consequence of the orthogonality of L_1 with the ℓ_i's. The second term

$$\begin{aligned} &\frac{h(L_1, \ell_1, \cdots, \ell_{k-1}) - h(0, \ell_1, \cdots, \ell_{k-1})}{L_1} \\ = \quad &\frac{h(L_1, \ell_1, \cdots, \ell_{k-1})(\varepsilon_1 L_1^* + \varepsilon_2 \ell_1^* + \cdots + \varepsilon_k \ell_{k-1}^*) - h(L_1, \ell_1, \cdots, \ell_{k-1})(\varepsilon_2 \ell_1^* + \cdots + \varepsilon_k \ell_{k-1}^*)}{\varepsilon_1} \end{aligned}$$

is holomorphic at 0. This yields the required decomposition.

Assume that the decomposition exists for all element with $s \le t$ where $t \ge 1$ and consider $f = \frac{h}{L_1^{s_1} \cdots L_m^{s_m}}$ with $s = s_1 + \cdots + s_m = t + 1$. We note that

$$f := g_0 + g_1 + \cdots + g_m,$$

where we have set

$$g_i := \begin{cases} \dfrac{h(0,\cdots,0,\ell_1,\cdots,\ell_{k-m})}{L_1^{s_1}\cdots L_m^{s_m}}, & i=0,\\ \dfrac{f_i}{L_1^{s_1}\cdots L_i^{s_i-1}\cdots L_m^{s_m}}, & 1\le i\le m,\end{cases}$$

with

$$f_i := \frac{h(L_1,\cdots,L_i,0,\cdots,0,\ell_1,\cdots,\ell_{k-m}) - h(L_1,\cdots,L_{i-1},0,\cdots,0,\ell_1,\cdots,\ell_{k-m})}{L_i}, \quad 1\le i\le m.$$

Then g_0 lies in $\mathcal{M}_{F,-}(V_k^*\otimes\mathbb{C})$. Further, for $i=1,\cdots,m$, f_i is holomorphic at 0. Thus it follows from the induction hypothesis that g_i lies in $\mathcal{M}_{F,-}(V_k^*\otimes\mathbb{C})+\mathcal{M}_{F,+}(V_k^*\otimes\mathbb{C})$. Hence f lies in $\mathcal{M}_{F,-}(V_k^*\otimes\mathbb{C})+\mathcal{M}_{F,+}(V_k^*\otimes\mathbb{C})$. This completes the proof for the sum decomposition $\mathcal{M}_F(V_k^*\otimes\mathbb{C}) = \mathcal{M}_{F,-}(V_k^*\otimes\mathbb{C})+\mathcal{M}_{F,+}(V_k^*\otimes\mathbb{C})$.

We next show that $\mathcal{M}_{F,-}(V_k^*\otimes\mathbb{C})\cap\mathcal{M}_{F,+}(V_k^*\otimes\mathbb{C}) = \{0\}$. Suppose that there is $0\neq f\in\mathcal{M}_{F,-}(V_k^*\otimes\mathbb{C})\cap\mathcal{M}_{F,+}(V_k^*\otimes\mathbb{C})$. Then f is holomorphic. On the other hand, by Lemma **??**, the element $f\in\mathcal{M}_{F,-}(V_k^*\otimes\mathbb{C})$ can be written as a linear combination $\sum_i a_iS_i$ of polar germs S_i with supporting cones satisfying the condition in Theorem 4.9. Then applying Theorem 4.9 to $\sum_i a_iS_i = f$, we get $a_i=0$ for all i. This is a contradiction.

(b) The statement follows from the compatibility of the decomposition with the filtration.

(c) Let f and g be as in Eq. (9). We have

$$fg = \sum_{i,j}\left(\frac{h_i(\vec{\ell}_i)}{\vec{L}_i^{\vec{s}_i}}\frac{k_j(\vec{m}_j)}{\vec{M}_j^{\vec{t}_j}} + \phi_i(\vec{\ell}_i,\vec{L}_i)\frac{k_j(\vec{m}_j)}{\vec{M}_j^{\vec{t}_j}} + \frac{h_i(\vec{\ell}_i)}{\vec{L}_i^{\vec{s}_i}}\psi_j(\vec{m}_j,\vec{M}_j) + \phi_i(\vec{\ell}_i,\vec{L}_i)\psi_j(\vec{m}_j,\vec{M}_j)\right).$$

The first three terms in each sum lie in $\mathcal{M}_{F,-}(V^{\circledast}\otimes\mathbb{C})$ and the last term lies in $\mathcal{M}_{F,+}(V^{\circledast}\otimes\mathbb{C})$. Hence

$$\pi_+(fg) = \sum_{i,j}\phi_i(\vec{\ell}_i,\vec{L}_i)\psi_j(\vec{m}_j,\vec{M}_j) = \left(\sum_i\phi_i(\vec{\ell}_i,\vec{L}_i)\right)\left(\sum_j\psi_j(\vec{m}_j,\vec{M}_j)\right) = \pi_+(f)\pi_+(g).$$

□

Combining the previous results leads to the following existence and uniqueness result. The existence is a direct consequence of Theorem 5.3.

Corollary 5.7. *Any element f of $\mathcal{M}_F(V^{\circledast}\otimes\mathbb{C}))$ can be written as*

$$f = \sum_i S_i + h,$$

where h is holomorphic and the S_i's are polar germs satisfying the following requirements

- *their supporting cones are properly positioned,*
- *their denominators are pairwise not proportional,*
- *the linear forms in their denominators are pseudo-positive.*

If two such decompositions have the same properly positioned supporting cones, then they are the same decomposition.

6. Residue for germs of meromorphic functions at zero with linear poles

Based on the decomposition in Section 5, we now introduce a filtered structure on $\mathcal{M}_F(V^{\circledast}\otimes\mathbb{C})$, define p-orders and p-residues for germs of meromorphic functions at zero with linear poles. We only deal with real coefficients since the results for coefficients in a subfield F immediately follow from the case $F = \mathbb{R}$.

Let us first define the p-order of germs of meromorphic functions with linear poles.

Definition 6.1. The **polar order**, or **p-order** in short, of the germ $\frac{h(\ell_1,\cdots,\ell_m)}{L_1^{s_1}\cdots L_n^{s_n}}$ is defined to be

$$\text{p-ord}\frac{h(\ell_1,\cdots,\ell_m)}{L_1^{s_1}\cdots L_n^{s_n}} := \sum_{i=1}^{n} s_i.$$

Definition 6.2. Let $f \in \mathcal{M}(V^{\circledast}\otimes\mathbb{C})$. By Corollary 5.7 f decomposes as

$$f = \sum_{i=1}^{n} S_i + h. \tag{11}$$

In particular, h is a holomorphic germ and $S_i, i = 1,\cdots,n$, are polar germs whose denominators are not proportional to each other and whose supporting cones are properly positioned. We define the **polar order**, or **p-order** in short, of f to be

$$\text{p-ord}(f) := \text{Max}(\text{p-ord}(S_i)).$$

Example 6.3. The p-order of f_c in Example 5.5 is 1 if $c = 0$ and 3 otherwise.

Lemma 6.4. *Any two sets of cones with pseudo-positive generators have a common subdivision that is properly positioned.*

Proof. Since the generators of any cone in the family are pseudo-positive, the union of any set of faces can not contain a nonzero linear subspace. Thus the proof is the same as that of Lemma 3.1. □

Lemma 6.4 is one of the ingredients used to prove that the p-order is well-defined.

Proposition 6.5. *The p-order of a germ of meromorphic function with linear poles does not depend on any choice of a decomposition of the germ in Eq. (11).*

We next establish the independence of the p-order on the choice of inner product.

Proposition 6.6. *The p-order of a meromorphic germ with linear poles does not depend on the choice of an inner product in the decomposition of* $\mathcal{M}(V^{\circledast}\otimes\mathbb{C})$ *in Theorem 5.3.*

Proof. For an inner product Q in V, and $f \in \mathcal{M}(V^{\circledast}\otimes\mathbb{C})$ with p-ord$(f) = p$, let

$$f = \sum_{i=1}^{r} S_i + \sum_{i=r+1}^{n} S_i + h$$

be a decomposition of f into polar germs S_i and a holomorphic function h as in Definition 6.4, with polar germs with the highest order in the first sum and those with lesser order in the second sum.

Now consider a different inner product R on V. For this inner product, an S_i might not be a polar germ. Set

$$S_i = \frac{h_i(\ell_{i1},\cdots,\ell_{im_i})}{L_{i1}^{s_{i1}}\cdots L_{in_i}^{s_{in_i}}},$$

with $Q(\ell_{ip}, L_{iq}) = 0$. For $j = 1, \cdots, m_i$, we have

$$\ell_{ij} = \ell'_{ij} - \sum_{k=1}^{n_i} a_{ij}^k L_{ik},$$

where $R(\ell'_{ij}, L_{ik}) = 0$ for $k = 1, \cdots, n_i$. Then

$$S_i = \frac{h_i(\ell'_{i1}, \cdots, \ell'_{im_i})}{L_{i1}^{s_{i1}} \cdots L_{in_i}^{s_{in_i}}} + \text{terms of lower denominator degrees.}$$

Thus

$$f = \sum_{i=1}^{r} \frac{h_i(\ell'_{i1}, \cdots, \ell'_{im_i})}{L_{i1}^{s_{i1}} \cdots L_{in_i}^{s_{in_i}}} + \text{terms of lower denominator degrees.} \tag{12}$$

This gives a decomposition of f into a linear combination of polar germs for the inner product R.

Note the supporting cones in the above sum are faces of the supporting cones in the decomposition of f under the inner product Q. So they remain properly positioned. Since $h_i(\ell_{i1}, \cdots, \ell_{im_i}) \neq 0$, we also have $h_i(\ell'_{i1}, \cdots, \ell'_{im_i}) \neq 0$. Therefore under the inner product R, the p-order of f is again p. $\square$

Using the p-order, we define a filtration in $\mathcal{M}(V^{\circledast} \otimes \mathbb{C})$.

Definition 6.7. For $n \geq 0$, define

$$\mathcal{M}_n := \{f \in \mathcal{M}(V^{\circledast} \otimes \mathbb{C}) \mid \text{p-ord}(f) \leq n\}.$$

Clearly, we have

$$\mathcal{M} = \bigcup_{i=0}^{\infty} \mathcal{M}_i, \quad \text{and} \quad \mathcal{M}_i \mathcal{M}_j \subset \mathcal{M}_{i+j}$$

for the usual product of functions. Therefore,

Proposition 6.8. *$\mathcal{M}_0 := \mathcal{M}_+(V^{\circledast} \otimes \mathbb{C}) \subset \mathcal{M}_1 \subset \mathcal{M}_2 \subset \cdots$ equips $\mathcal{M}(V^{\circledast} \otimes \mathbb{C})$ with the structure of a filtered algebra.*

Thanks to this filtration, we can now define a residue on $\mathcal{M}(V^{\circledast} \otimes \mathbb{C})$.

Definition 6.9. Let $f \in \mathcal{M}(V^{\circledast} \otimes \mathbb{C})$ with the decomposition

$$f = \sum_i S_i + h$$

as in Definition 6.4 and with p-ord$(f) = r$. Let $S_1, \cdots, S_t$ the polar germs with p-ord$(S_i) = r$ and let $S_i = \frac{h_i}{L_{i1}^{s_{i1}} \cdots L_{in_i}^{s_{in_i}}}$. We define the **highest polar order residue**, or the **p-residue** in short, of f to be

$$\text{p-res}(f) = \sum_{i=1}^{t} \frac{h_i(0)}{L_{i1}^{s_{i1}} \cdots L_{in_i}^{s_{in_i}}}.$$

Example 6.10. The function f_c as in Example 5.5 has p-residue given by p-res$(f_c) = \frac{L_2}{L_1} + \frac{L_1}{L_2}$ if $c = 0$ and p-res$(f_c) = \frac{c^3}{L_1(L_1+L_2)^2} + \frac{c^3}{L_2(L_1+L_2)^2}$ if $c \neq 0$.

Proposition 6.11. *The p-residue of a germ of meromorphic function with linear poles is well-defined in so far as it does not depend on a choice of the decomposition in Eq. (11) or the inner product used in the decomposition of $\mathcal{M}(V^{\circledast} \otimes \mathbb{C})$ in Theorem 5.3.*

Proof. We first prove that the p-residue does not depend on a particular choice of the decomposition of f; as in the the proof of Proposition 6.5, we only need to prove that the p-residue does not change under subdivision.

For $f \in \mathcal{M}(V^{\circledast} \otimes \mathbb{C})$ with a given decomposition as in Eq. (11), let

$$S = \frac{h(\ell_1, \cdots, \ell_m)}{L_1^{s_1} \cdots L_n^{s_n}}$$

be one of the polar germ in the decomposition with p-ord(S) = p-ord(f). Then the supporting cone is $\langle L_1, \cdots, L_n\rangle$. For a fixed subdivision of the set of supporting cones, let the subdivision for $\langle L_1, \cdots, L_n\rangle$ be $\{D_j := \langle L_{j1}, \cdots, L_{jn}\rangle\}$. Then

$$\frac{1}{L_1 \cdots L_n} = \sum_j \frac{b_j}{L_{j1} \cdots L_{jn}},$$

where b_j's are constants. Assume

$$\frac{1}{(s_1-1)! \cdots (s_k-1)!} \partial_{L_1^*}^{s_1-1} \cdots \partial_{L_k^*}^{s_k-1} \frac{1}{L_{j1}, \cdots, L_{jn}} = \sum_{r_{j1}+\cdots+r_{jn}=s_1+\cdots+s_n} \frac{c_{r_{j1}\cdots r_{jn}}}{L_{j1}^{r_{j1}}, \cdots, L_{jn}^{r_{j1}}},$$

where $c_{r_{j1}\cdots r_{jn}}$'s are constants. Then

$$S = h(\ell_1, \cdots, \ell_m) \sum_j b_j \sum_{r_{j1}+\cdots+r_{jn}=s_1+\cdots+s_n} \frac{c_{r_{j1}\cdots r_{jn}}}{L_{j1}^{r_{j1}}, \cdots, L_{jn}^{r_{j1}}}$$

is the new decomposition of S with supporting cones $\{D_j\}$.

The contribution of the polar fraction S to the p-residue of f is $\frac{h(0)}{L_1^{s_1} \cdots L_n^{s_n}}$. The contribution from the new decomposition of S is

$$h(0) \sum_j b_j \sum_{r_{j1}+\cdots+r_{jn}=s_1+\cdots+s_n} \frac{c_{r_{j1}\cdots r_{jn}}}{L_{j1}^{r_{j1}}, \cdots, L_{jn}^{r_{j1}}}$$

which agrees with the first contribution. Therefore the p-residue does not depend on the decomposition.

For a different choice of inner product, Eq. (12) tells us how the polar germs of p-order p-ord(f) change. In particular, the constant terms of the numerators remain the same. This exactly means that the p-residue does not depend on the choice of inner products. $\square$

To simplify the notation, we set

$$S(0) := \frac{h(0)}{L_1^{s_1} \cdots L_k^{s_k}}$$

for a polar germ $S = \frac{h(\ell_1, \cdots, \ell_m)}{L_1^{s_1} \cdots L_k^{s_k}}$.

Proposition 6.12. *Let $f = \sum S_i + \sum T_j + h$, with S_i, T_j polar germs, h holomorphic, and* p-ord(S_i)*'s all equal to k, $\sum S_i \neq 0$,* p-ord$(T_j) < k$. *Then* p-ord$(f) = k$ *and*

$$\text{p-res}(f) = \sum S_i(0).$$

Proof. Taking a subdivision of the set of supporting cones of the germs S_i's and T_j's, we have $S_i = \sum S_{il}$ and $T_j = \sum T_{jm}$. Then

$$f = \sum S_{il} + \sum T_{jm} + h.$$

Combining terms that are proportional to one another, we can assume that this decomposition satisfies the conditions in Definition 6.4. In the decomposition

$$\sum_{i,l} S_{il} = \sum_i S_i \neq 0,$$

there is some non zero polar germ of p-order k, and this is the maximal of p-orders of polar germs, so p-ord$(f) = k$.

Therefore

$$\text{p-res}(f) = \sum S_{il}(0) = \sum S_i(0).$$

□

By construction, this p-residue has a weak multiplicative property.

Proposition 6.13. *Let $f = f(L_1, \cdots L_k)$ and $g = g(L'_1, \cdots, L'_n) \in \mathcal{M}(V^{\circledast} \otimes \mathbb{C})$, with $Q(L_i, L'_j) = 0$, then*

$$\text{p-res}(fg) = \text{p-res}(f)\,\text{p-res}(g).$$

As in [14], we can reinterpret the constructions of [2, 11, 19] in terms of lattice cones, so to a lattice cone (C, Λ_C) we can assign two meromorphic functions, the exponential discrete sum $S(C, \Lambda_C)$ (corresponding to $S^c(C, \Lambda_C)$ in [14]) and the map $I(C, \Lambda_C)$. A direct computation of the p-residue of the discrete sum on a smooth lattice cones yields the corresponding integral given by $I(C, \Lambda_C)$.

Lemma 6.14. *For a smooth lattice cone (C, Λ_C), we have*

$$\text{p-res}(S(C, \Lambda_C)) = I(C, \Lambda_C).$$

In fact, we have

$$S(C, \Lambda_C) = I(C, \Lambda_C) + (\text{terms of p} - \text{order} < \dim(C)).$$

Proof. Let $v_1, \cdots, v_d(C)$ (where $d = \dim C$) be a basis of Λ_C that generates C as a cone. Then

$$S(C, \Lambda_C)(\vec{\varepsilon}) = \prod_{i=1}^{d} \frac{1}{1 - e^{\langle v_i, \vec{\varepsilon}\rangle}} = \prod_{i=1}^{d} \left(-\frac{1}{\langle v_i, \vec{\varepsilon}\rangle} + h(\langle v_i, \vec{\varepsilon}\rangle) \right),$$

where h is holomorphic. So the highest p-order term is $\prod_{i=1}^d \left(-\frac{1}{\langle v_i, \vec{\varepsilon}\rangle}\right)$ which is $I(C, \Lambda_C)$. □

Lemma 6.15. *For a cone C, $I(C) \neq 0$ if and only if C is strongly convex.*

Proof. We already know that $I(C, \Lambda_C) = 0$ if C is not strongly convex. So we only need to prove that if C is strongly convex, then $I(C, \Lambda_C) \neq 0$. Taking a smooth subdivision $\{C_i\}$ of C, then since C is strongly convex, $\{C_i\}$ is properly positioned. So $I(C_i, \Lambda_{C_i})$'s are linearly independent. Then their sum can not be 0. □

Lemma 6.16. *For a lattice cone (C, Λ_C), $S(C, \Lambda_C) \neq 0$ if and only if (C, Λ_C) is strongly convex.*

Proof. Again we only need to prove that if (C, Λ_C) is strongly convex, then $S(C, \Lambda_C) \neq 0$. Taking a smooth subdivision $\{(C_i, \Lambda_C)\}$ of (C, Λ_C), then since C is strongly convex, $\{C_i\}$ is properly positioned, so $I(C_i)$'s are linearly independent.

We know

$$S(C, \Lambda_C) = \sum_i S(C_i, \Lambda_C) + (\text{terms with p-order} < \dim(C)),$$

and

$$S(C_i, \Lambda_C) = I(C_i, \Lambda_C) + (\text{terms with p-order} < \dim(C)).$$

By Proposition 6.12, p-ord($S(C, \Lambda_C)$) < dim(C) implies

$$\sum_i I(C_i, \Lambda_C) = 0,$$

which is a contradiction. So p-ord($S(C, \Lambda_C)$) = dim(C), and $S(C, \Lambda_C) \neq 0$. □

At the same time, we have proved that

Lemma 6.17. *For a strongly convex lattice cone* (C, Λ_C)*,*

$$\text{p-ord}(S(C, \Lambda_C)) = \dim(C).$$

Theorem 6.18. *For a lattice cone* (C, Λ_C) *and its subdivision* $\{(C_i, \Lambda_C)\}$*, we have*

$$\text{p-res}(S(C, \Lambda_C)) = \sum_i \text{p-res}(S(C_i, \Lambda_{C_i})).$$

So the map p-res $\circ\, S$ *is compatible with subdivisions.*

Proof. For this subdivision, we have

$$S(C, \Lambda_C) = \sum_i S(C_i, \Lambda_C) + (\text{terms of p-order} < \dim(C)),$$

and

$$S(C_i, \Lambda_C) = \sum_j T_{ij} + (\text{terms of p-order} < \dim(C)),$$

where T_{ij} are polar germs, p-ord(T_{ij}) = dim(C). So

$$S(C, \Lambda_C) = \sum_{i,j} T_{ij} + (\text{terms of p-order} < \dim(C)).$$

If C is strongly convex, then p-ord($S(C, \Lambda_C)$) = dim(C), and

$$\text{p-res}(S(C, \Lambda_C)) = \sum_{i,j} T_{ij}(0) = \sum_i \text{p-res}(S(C_i, \Lambda_C))$$

by Proposition 6.12.

If C is not strongly convex, then $S(C, \Lambda_C) = 0$; on the other hand by Proposition 6.12, this means $\sum_{i,j} T_{ij} = 0$, that is $\sum_i$ p-res($S(C_i, \Lambda_C)$) = 0.

So in any case, we have the conclusion. □

As a corollary, we obtain our second main result.

Corollary 6.19. *For a lattice cone* (C, Λ_C)*, we have*

$$\text{p-res}(S(C, \Lambda_C)) = I(C, \Lambda_C).$$

Example 6.20. Take $\Lambda = \mathbb{Z}^2 \subset \mathbb{R}^2$ and $C = \langle e_1, e_1 + e_2 \rangle$ with (e_1, e_2) the canonical orthonormal basis in $\mathbb{R}^2$. Then $S^c(C, \Lambda_C) = \dfrac{1}{(1 - e^{\varepsilon_1})(1 - e^{\varepsilon_1 + \varepsilon_2})}$ has p-order 2 and p-residue $I(C, \Lambda_C) = \dfrac{1}{\varepsilon_1(\varepsilon_1 + \varepsilon_2)}$.

Acknowledgements: The authors acknowledge supports from the Natural Science Foundation of China (Grant No. 11071176, 11221101 and 11371178) and the National Science Foundation of US (Grant No. DMS 1001855). Part of the work was completed during visits of two of the authors at Sichuan University, Lanzhou University and Capital Normal University, to which they are very grateful.

References

[1] M. Beck, Counting lattice points by means of the residue theorem, *R*amanujan J. **4** (2000) 299-310.
[2] N. Berline and M. Vergne, Local Euler-Maclaurin formula for polytopes, *Mosc. Math. J.* **7** (2007) 355-386.
[3] N. Berline and M. Vergne, Local asymptotic Euler-Maclaurin expansion for Riemann sums over a semi-rational polyhedron, Xiv:1502.01671
[4] A. Bertozzi and James McKenna, Multidimensional residues, generating functions and their application on queuing networks, *S*IAM Review **35** (1993) 239-268.
[5] M. Brion and M. Vergne, Arrangement of hyperplanes. I: rational functions and Jeffrey-Kirwan residue, Annales Scientifiques de l'école Normale Supérieure, 1999, 32(5), 715-741.
[6] A. Connes and D. Kreimer, Hopf algebras, Renormalization and Noncommutative Geometry, *Comm. Math. Phys.* **199** (1988) 203-242.
[7] C. De Concini, C. Procesi, Topics in hyperplanes, arrangements, polytopes and box-splines, Universitext Springer (2011)
[8] P. Deligne et al. (eds.), Quantum Field Theory and Strings: A Course for Mathematicians, Vols. 1 and 2, AMS/IAS, 1999.
[9] N.V. Dang, Complex powers of analytic functions and meromorphic renormalization in QFT arXiv:1503.00995
[10] K. Ebrahimi-Fard, L. Guo and D. Kreimer, Spitzer's identity and the algebraic Birkhoff decomposition in pQFT, *J. Phys. A: Math. Gen.* **37** (2004) 11037-11052.
[11] S. Garoufalidis and J. Pommersheim, Sum-integral interpolators and the Euler-Maclaurin formula for polytopes, *Trans. Amer. Math. Soc.* **364** (2012) 2933-2958.
[12] L. Guo, An Introduction to Rota-Baxter Algebra, International Press, 2012.
[13] L. Guo, S. Paycha and B. Zhang, Conical zeta values and their double subdivision relations, *Adv. in Math.* **252** (2014) 343-381.
[14] L. Guo, S. Paycha and B. Zhang, Renormalization and the Euler-Maclaurin formula on cones, arXiv:1306.3420.
[15] L. Guo, S. Paycha and B. Zhang, Decompositions and residue of meromorphic functions with linear poles in the light of the geometry of cones, arXiv:1501.00426
[16] L. Guo and B. Zhang, Renormalization of multiple zeta values *J. Algebra* **319** (2008) 3770-3809.
[17] M. E. Hoffman, Multiple harmonic series, *Pacific J. Math.*, **152** (1992), no. 2, 275–290.
[18] G. Hopkins and J. Lipman, An elementary theory of Grothendieck's residue symbol, *CR Math. Rep. Academ. Sci. Can.* **1** (1978/79) 169-172.
[19] J. Lawrence, Rational-function-valued valuations on polyhedra, DIMACS Ser. Discrete Math. Theoret. Comput. Sci.,6 (1991) 199C208.
[20] D. Manchon and S. Paycha, Nested sums of symbols and renormalised multiple zeta values, Int. Math. Res. Papers 2010 issue 24, 4628-4697 (2010).
[21] K. Matsumoto, The analytic continuation and the asymptotic behaviour of certain multiple zeta-functions I, *J. Number Theory*, **101** (2003), 223–243.
[22] A. Shchuplev, Toric varieties and residues, *D*octoral dissertation, Stockholm University (2007). http://www.diva-portal.org/smash/get/diva2:197230/FULLTEXT01.pdf
[23] E. Speer, Analytic renormalization using many space-time dimensions, *Comm. Math. Phys.* **37** (1974), 83-92 (free access: http://projecteuclid.org/euclid.cmp/1103859813)
[24] E. Speer, Ultraviolet and infrared singularity structure of generic Feynman amplitudes Annales de lInst. Henri Poincaré, section A, Vol. 23, no 1 (1975), p. 1-21

(free access:http://archive.numdam.org/ARCHIVE/AIHPA
/AIHPA_1975_23_1/AIHPA_1975_23_1_1_0/AIHPA_1975_23_1_1_0.pdf)

[25] T. L. Tong, Integral representation formulae and Grothendieck residue symbol, *A*mer. J. Math. V. 4 (1973) 904-917.

[26] D. Zagier, Values of zeta functions and their applications, First European Congress of Mathematics, Vol. II (Paris, 1992), 497–512, *Progr. Math.*, **120**, Birkhäuser, Basel, 1994

[27] J. Zhao, Analytic continuation of multiple zeta functions. *Proc. Amer. Math. Soc.* **128** (2000), 1275-1283.