Noncommutative geometry of generalized Weyl algebras

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References:

TB, N Ciccoli, L Dąbrowski & A Sitarz, *Twisted reality condition for Dirac operators,* arXiv:1601.07404
TB, *Noncommutative differential geometry of generalized Weyl algebras*, arXiv:1602.07456.

Outline

- In 2008 Connes and Moscovici introduced twisted Dirac operators (commutators replaced by twisted commutators).
- In 2016 Landi and Martinetti studied twisted real spectral triples.
- TB, N Ciccoli, L Dąbrowski and A Sitarz observed that even for an untwisted Dirac operator, the real structure could (or even should) be twisted.
- Quantum generalized Weyl algebras (introduced by Bavula in 1996) serve as examples of algebras which enforce the reality to be twisted.

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Reality twisted by an automorphism

Let B be a complex $*$-algebra, (H, π) a representation of B, D a linear operator on H, let ν be a linear automorphism of H. Set

$$\bar{\nu} : \mathrm{End}(H) \to \mathrm{End}(H), \qquad \phi \mapsto \nu \circ \phi \circ \nu^{-1}.$$

(B, H, D) admits a ν-*twisted real structure* if there exists an anti-linear map $J : H \to H$ such that $J^2 = \epsilon\,\mathrm{id}$, and, for all $a, b \in B$,

$$[\pi(a), J\pi(b)J^{-1}] = 0,$$

$$[D, \pi(a)]J\bar{\nu}^2(\pi(b))J^{-1} = J\pi(b)J^{-1}[D, \pi(a)],$$

$$DJ\nu = \epsilon' \nu JD,$$

$$\nu J \nu = J,$$

where $\epsilon, \epsilon' \in \{+, -\}$.

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Reality twisted by an automorphism

If (B, H, D) admits a grading operator $\gamma : H \to H$,

$$\gamma^2 = \mathrm{id}, \qquad [\gamma, \pi(a)] = 0, \qquad \gamma D = -D\gamma,$$

such that

$$\nu^2\gamma = \gamma\nu^2,$$

then the twisted real structure J is also required to satisfy

$$\gamma J = \epsilon'' J\gamma,$$

where ϵ'' is another sign.

Construction from graded algebras

- Let G be a group and $A = \oplus_{g \in G} A_g$ be a G-graded $*$-algebra, and set $B := A_e$, where e is the neutral element of G.
- The grading is assumed to be compatible with the $*$-structure in the sense that, for all $g \in G$,

$$A_g^* \subseteq A_{g^{-1}},$$

so, in particular, B is a $*$-subalgebra of A.
- Let $G_+ \subset G$ and set

$$G_- := \{g^{-1} \mid g \in G_+\}, \quad H_\pm = \bigoplus_{g \in G_\pm} A_g, \quad H = H_+ \oplus H_-.$$

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Construction from graded algebras

- Let ν be a graded (i.e. degree preserving) algebra automorphism of A that satisfies

$$\nu \circ * \circ \nu = *.$$

- Let $\partial_\pm : A \to A$ be ν^2-twisted $q^{\pm 2}$-skew derivations of A, i.e.

$$\partial_\pm(ab) = \partial_\pm(a)\nu^2(b) + a\partial_\pm(b),$$

and

$$\nu \circ \partial_\pm \circ \nu^{-1} = q^{\pm 2}\, \partial_\pm,$$

for a real number q.

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Construction from graded algebras

- Assume that $\partial_\pm$ are compatible with the $*$-structure, automorphism ν and the grading, i.e. that

$$\nu\left(\partial_\pm(a)^*\right) = \nu^{-1}\left(\partial_\mp(a^*)\right).$$

and

$$\partial_\pm(H_\mp) \subseteq H_\pm,$$

- View H as a left B-module by

$$\pi(a)(h_\pm) = \nu^2(a)h_\pm, \qquad \text{for all } a \in B,\ h_\pm \in H_\pm.$$

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Construction from graded algebras

Define:

- the ("q-Dolbeault") Dirac operator:

$$D : H \to H, \qquad (h_+, h_-) \mapsto \Big(-q^{-1}\partial_+(h_-),\ q\partial_-(h_+) \Big) \, ,$$

- the grading operator:

$$\gamma : H \to H, \qquad (h_+, h_-) \mapsto (h_+, -h_-) \, .$$

Then

$$J : H \to H, \qquad (h_+, h_-) \mapsto (-h_-^*, h_+^*) \, ,$$

equips (B, H, D) with a ν-twisted real structure.

Differenital calculi from skew derivations: An interlude

- A *first-order differential calculus* on A is an A-bimodule Ω with a $\mathbb{K}$-linear map $d : A \to \Omega$ such that
  (a) d satisfies the Leibniz rule: for all $a, b \in A$,
  $$d(ab) = d(a)b + ad(b);$$
  (b) Ω satisfies the *density condition*: $\Omega = Ad(A)$.
- If A is a complex $*$-algebra, then the calculus (Ω, d) is said to be a $*$-*calculus* provided Ω is equipped with an anti-linear operation $*$ such that, for all $a, b \in A$, $\omega \in \Omega$,
  $$(a\omega b)^* = b^*\omega^* a^* \quad \text{and} \quad d(a^*) = d(a)^*.$$

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- Fix a finite indexing set I, and let (∂_i, ν_i), $i \in I$, be a collection of skew derivations on an algebra A.
- Let Ω be a free left A-module with a free basis ω_i, $i \in I$.
- Define the (free) right A-module structure on Ω by setting

$$\omega_i a := \nu_i(a)\omega_i.$$

- Then the map

$$d : \mathcal{A} \to \Omega, \qquad a \mapsto \sum_{i \in I} \partial_i(a)\omega_i, \tag{1}$$

satisfies the Leibniz rule.

- There is no guarantee in general that the density condition be satisfied.

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Generalized Weyl algebras

- [Bavula] Let A be an algebra, σ an automorphism of A and p an element of the centre of A. A *degree-one generalized Weyl algebra over A* is an algebraic extension $\mathcal{A}(p, \sigma)$ of A obtained by supplementing A with additional generators x, y subject to the following relations

$$yx = \sigma(p), \quad xy = p, \quad ya = \sigma(a)y, \quad xa = \sigma^{-1}(a)x.$$

- The algebras $\mathcal{A}(p, \sigma)$ share many properties with A, in particular, if A is a Noetherian algebra, so is $\mathcal{A}(p, \sigma)$, and if A is a domain and $p \neq 0$, so is $\mathcal{A}(p, \sigma)$.

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Generalized Weyl $*$-algebras over $\mathbb{C}[z]$

- If $A = \mathbb{C}[z]$, then every automorphism σ of A necessarily takes the form $\sigma(z) = qz + r$, for $q, r \in \mathbb{C}$, $q \neq 0$.
- Any generalized Weyl algebra over $\mathbb{C}[z]$ coincides with an algebra $\mathcal{B}(p; q, r)$ generated by x, y, z subject to the relations

$$yx = p(qz + r), \qquad xy = p(z),$$
$$yz = (qz + r)y, \qquad xz = q^{-1}(z - r)x,$$

  where $p(z) \in \mathbb{C}[z]$ and $q, r \in \mathbb{K}$, $q \neq 0$.
- If p has real coefficients and $q, r \in \mathbb{R}$, then $\mathcal{B}(p; q, r)$ can be made into $*$-algebra by setting

$$x^* = y, \qquad z^* = z.$$

- $\mathcal{B}(p; q, r)$ can be understood as coordinate algebras of noncommutative surfaces, for example quantum spheres or quantum weighted projective lines (spindles).

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Regular gen. Weyl algebras

Definition

Let $\mathbb{K}$ be a field of characteristic 0, $q \in \mathbb{K}$, and let $p(z)$ be a polynomial in one variable with coefficients from $\mathbb{K}$. Let

$$p_q(z) := \frac{p(qz) - p(z)}{(q-1)z},$$

denote the q-derivative of p. We say that p is a *q-separable polynomial* if $p(z)$ is coprime with $p_q(z)$.

Definition

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Quantum cones and teardrop-like gen. Weyl algebras

- In a regular gen. Weyl algebra $\mathcal{B}(p; q, 0)$, either
  (a) $p(0) \neq 0$ or
  (b) p has a simple root at $z = 0$.
- Quantum cone algebras

$$\mathcal{O}(C_q^N) = \mathcal{B}(\prod_{l=0}^{N-1}\left(1 - q^{-2l}z\right); q^{2N}, 0),$$

  are examples of generalized Weyl algebras in class (a).
- Algebras in class (b) include quantum teardrop algebras

$$\mathcal{B}(z\prod_{l=0}^{N-1}\left(1 - q^{-2l}z\right); q^{2N}, 0).$$

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Twisted-real Dirac operators for quantum cones

- $\mathcal{O}(C_q^1) = \mathcal{O}(D_q)$, the quantum disc algebra, generated by v, v^*,
$$v^* v - q^2 v v^* = 1 - q^2.$$

- $\mathcal{O}(D_q)$ is $\mathbb{Z}_N$-graded, with $|v| = 1$.
- $\mathcal{O}(C_q^N)$ can be identified with the degree-0 subalgebra via the embedding
$$\iota : \mathcal{O}(C_q^N) \hookrightarrow \mathcal{O}(D_q), \qquad x \mapsto v^N, \quad z \mapsto 1 - vv^*.$$

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Define the automorphism of $\mathbb{Z}_N$-graded algebras

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Theorem

(1) The automorphism ν is $*$-regular, i.e. $\nu \circ * \circ \nu = *$.

(2) The maps $\partial_\pm : \mathcal{O}(D_q) \to \mathcal{O}(D_q)$, given by

$$\partial_-(v) = v^*, \quad \partial_-(v^*) = 0, \qquad \partial_+(v) = 0, \quad \partial_+(v^*) = q^2 v,$$

are degree ± 2, ν^2-twisted $q^{\pm 2}$-skew derivations.

(3) For all $a \in \mathcal{O}(D_q)$,

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Theorem

Let:

$$H_+ = \mathcal{O}(D_q)_1, \quad H_- = \mathcal{O}(D_q)_{N-1}, \qquad H = H_+ \oplus H_-,$$

and represent $\mathcal{O}(C_q^N)$ *in* H *as*

$$\pi(a)(h_\pm) = \nu^2(\iota(a))h_\pm, \qquad \text{for all } a \in \mathcal{O}(C_q^N).$$

Then $(\mathcal{O}(C_q^N), H, D)$*, where*

$$D : H \to H, \qquad (h_+, h_-) \mapsto \left(-q^{-1}\partial_+(h_-),\ q\partial_-(h_+)\right),$$

is a KO-dim 2 spectral triple with grading $(h_+, h_-) \mapsto (h_+, -h_-)$*,*
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Differential structure for quantum cones

- The system (∂_-, ν), (∂_+, ν) defines differential structure on $\mathcal{O}(D_q)$.
- The module of one-forms restricted to $\mathcal{O}(C_q^N)$ is ismorphic to

$$\Omega^1 C_q^N = \mathcal{O}(D_q)_2 \oplus \mathcal{O}(D_q)_{N-2}.$$

- The differential on $\mathcal{O}(C_q^N)$ can be represented by the commutator with D.
- The Dirac operator is obtained by the Clifford action on a connection on the spinor bundle $\mathcal{O}(D_q)_1 \oplus \mathcal{O}(D_q)_{N-1}$,

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Real Dirac operators for regular gen. Weyl algebras

- Consider regular gen. Weyl algebra $\mathcal{B}(p; q^4, 0)$ with $p(0) = 0$ and set
$$\tilde{p}(z) = \frac{p(z)}{z}.$$
- Let $\mathcal{A}(\tilde{p}; q^2)$ be an affine algebra generated by $z_\pm$ and $x_\pm$ subject to the relations
$$z_+z_- = z_-z_+, \quad x_+x_- = \tilde{p}(z), \quad x_-x_+ = \tilde{p}(q^4 z),$$
$$x_+z_\pm = q^{-2}z_\pm x_+, \qquad x_-z_\pm = q^2 z_\pm x_-,$$
where $z = z_-z_+$.
- $\mathcal{A}(\tilde{p}; q^2)$ is a $*$-algebra with $z_+^* = z_-$, $x_+^* = x_-$.

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Real Dirac operators for regular gen. Weyl algebras

- The algebra $\mathcal{A}(\tilde{p}; q^2)$ is a $\mathbb{Z}$-graded algebra with the grading given on the generators by $|z_\pm| = |x_\pm| = \pm 1$.
- $\mathcal{B}(p; q^4, 0)$ embeds into $\mathcal{A}(\tilde{p}; q^2)$ as the degree-zero subalgebra, by the map

$$\iota : \mathcal{B}(p; q^4, 0) \hookrightarrow \mathcal{A}(\tilde{p}; q^2), \qquad x \mapsto z_- x_+, \quad z \mapsto z_- z_+.$$

- Define the degree-preserving automorphism of $\mathcal{A}(\tilde{p}; q^2)$,

$$\nu : \mathcal{A}(\tilde{p}; q^2) \to \mathcal{A}(\tilde{p}; q^2), \quad z_\pm \mapsto q^{\pm 1} z_\pm, \quad x_\pm \mapsto q^{\pm 1} x_\pm.$$

- Note that ν restricted to $\iota\left(\mathcal{B}(p; q^4, 0)\right)$ is the identity map.

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Real Dirac operators for regular gen. Weyl algebras

Theorem

(1) The automorphism ν is $*$-regular, i.e. $\nu \circ * \circ \nu = *$.

(2) The maps $\partial_\pm : \mathcal{A}(\tilde{p}; q^2) \to \mathcal{A}(\tilde{p}; q^2)$, given by

$$\partial_\pm(x_\pm) = \partial_\pm(z_\mp) = 0, \quad \partial_\pm(x_\pm) = c(z)z_\mp, \quad \partial_\pm(z_\pm) = x_\mp,$$

where

$$c(z) = q^2 \frac{\tilde{p}(q^4 z) - \tilde{p}(z)}{(q^4 - 1)z},$$

are degree ∓ 2, ν^2-twisted $q^{\pm 2}$-skew derivations.

(3) For all $a \in \mathcal{A}(\tilde{p}; q^2)$,

$$\nu\left(\partial_\pm(a)^*\right) = \nu^{-1}\left(\partial_\mp(a^*)\right).$$

Real Dirac operators for regular gen. Weyl algebras

Theorem

Let:

$$H_+ = \mathcal{A}(\tilde{p}; q^2)_{-1}, \quad H_- = \mathcal{A}(\tilde{p}; q^2)_1, \qquad H = H_+ \oplus H_-,$$

and represent $\mathcal{B}(p; q^4, 0)$ *in* H *as*

$$\pi(a)(h_\pm) = \iota(a)h_\pm, \qquad \text{for all } a \in \mathcal{B}(p; q^4, 0).$$

Then $(\mathcal{B}(p; q^4, 0), H, D)$*, where*

$$D : H \to H, \qquad (h_+, h_-) \mapsto \left(-q^{-1}\partial_+(h_-),\ q\partial_-(h_+)\right),$$

is a KO-dim 2 spectral triple with grading $(h_+, h_-) \mapsto (h_+, -h_-)$*,*
and ν*-twisted real structure* $(h_+, h_-) \mapsto (-h_-^*, h_+^*)$*.*

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Differential structure for regular gen. Weyl algebras

- The system (∂_-, ν), (∂_+, ν) supplemented by the (vertical) skew derivation (∂_0, ν_0), defines differential structure on $\mathcal{A}(\tilde{p}; q^2)$.
- The module of one-forms restricted to $\mathcal{B}(p; q^4, 0)$ is ismorphic to

$$\Omega^1\mathcal{B}(p; q^4, 0) = \mathcal{A}(\tilde{p}; q^2)_2 \oplus \mathcal{A}(\tilde{p}; q^2)_{-2}.$$

- The differential on $\mathcal{B}(p; q^4, 0)$ can be represented by the commutator with D.
- The Dirac operator is obtained by the Clifford action on a connection on the spinor bundle $\mathcal{A}(\tilde{p}; q^2)_1 \oplus \mathcal{A}(\tilde{p}; q^2)_{-1}$,

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