Unification of Gravity and Gauge Interactions

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Unification of gravity with electromagnetism is an idea that occupied Einstein for more than 20 years.

First attempt: Kaluza-Klein approach of a 5-dimensional constrained space.

Second attempt: Hermitian metric on a real four-dimensional manifold with Hermitian connection

$$g_{\mu\nu} = G_{\mu\nu} + i B_{\mu\nu} , \qquad g^{*}_{\mu\nu} = g_{\nu\mu}$$

$$G_{\mu\nu} = G_{\nu\mu} \qquad B_{\mu\nu} = - B_{\nu\mu}$$

$$0 = \nabla_{\mu} g_{\nu\rho} = \partial_{\mu} g_{\nu\rho} - \Gamma_{\nu\mu}^{\sigma} g_{\sigma\rho} - \Gamma_{\rho\mu}^{\sigma} g_{\nu\sigma}$$

$$\Gamma_{\mu\nu}^{\rho *} = \Gamma_{\nu\mu}^{\rho}$$

Einstein-Strauss

String theory, Supergravity are all based on Kaluza-Klein approach and have to deal with infinite number of massive modes.

Noncommutative Geometry: Unifies gravity, gauge and Higgs fields by tensoring a discrete space with a 4-d continuous manifold. There is distinction between gauge fields and spin-connection.

Coupling to Spinors Dirac action in Minkowski space must be made invariant under General Coordinate transformations and local Lorentz symmetry.

$$\langle \psi, D\psi \rangle = \int d^4x \, \bar{\psi} \gamma^\mu \partial_\mu \psi$$

Start with

$$\{\gamma^\mu, \gamma^\nu\} = -2\eta^{\mu\nu} \quad ; \quad \gamma^{\mu\dagger} = -\gamma^0 \gamma^\mu \gamma^0$$

$$\psi_\alpha \rightarrow \left(e^{\frac{1}{4}\Lambda^{ab}\gamma_{ab}}\right)_\alpha{}^\beta \psi_\beta \quad ; \quad x^\mu \rightarrow \Lambda^\mu{}_\nu x^\nu$$

$$\gamma_{ab} = \frac{1}{2}(\gamma_a\gamma_b - \gamma_b\gamma_a)$$

In curved space

$$x^\mu \rightarrow x'^\mu(x) \quad , \quad \Lambda^{ab} \rightarrow \Lambda^{ab}(x)$$

$$\partial_\mu \longrightarrow \partial_\mu + \frac{1}{4}\omega_\mu{}^{ab}(e)\gamma_{ab} \equiv \mathcal{D}_\mu$$

$$\langle \psi, D\psi \rangle = \int d^4x \, e \, \bar{\psi} \gamma^\mu D_\mu \psi$$

$$e = \det e_\mu{}^a \quad ; \quad \gamma^\mu = e^\mu{}_a \gamma^a$$

Spin connection is determined from torsion free condition. In Cartan formulation

$$T^a = de^a + \omega^a{}_b \wedge e^b \quad ; \quad e^a = e^a_\mu \, dx^\mu$$

$$T^a = T^a_{\mu\nu}\, dx^\mu \wedge dx^\nu \quad ; \quad \omega^a{}_b = dx^\mu \omega_\mu{}^a{}_b$$

Set torsion to zero: $\partial_\mu e_\nu^a - \partial_\nu e_\mu^a + \omega_\mu{}^a{}_b e_\nu^b - \omega_\nu{}^a{}_b e_\mu^b = 0$

- 24 equations for 24 variables $\omega_\mu^{ab} = -\omega_\mu^{ba}$

The metric satisfy metricity condition with symmetric affine connection

$$g_{\mu\nu} = e_\mu^a e_\nu^b \eta_{ab} \,, \qquad \Gamma_{\mu\nu}^\rho = \Gamma_{\nu\mu}^\rho$$

$$0 = \nabla_\rho g_{\mu\nu} = \partial_\rho g_{\mu\nu} - \Gamma_{\mu\rho}^\sigma g_{\sigma\nu} - \Gamma_{\rho\nu}^\sigma g_{\mu\sigma}$$

Combine both conditions in one on soldering form

$$0 = \nabla_\mu e_\nu^a = \partial_\mu e_\nu^a + \omega_\mu{}^a{}_b e_\nu^b - \Gamma_{\mu\nu}^\rho e_\rho^a$$

64 conditions for 24 $\omega_\mu{}^a{}_b$ and 40 $\Gamma_{\mu\nu}^\rho$

The action $\langle \Psi, D\Psi \rangle = \langle D\Psi, \Psi \rangle$ with D Hermitian and $\omega_\mu{}^{ab}$ are dependent functions completely determined in terms of e_μ^a.

The spin-connection has curvature : $R^a{}_b = d\,\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b$

$$\text{or } R_{\mu\nu}{}^{ab} = \partial_\mu \omega_\nu{}^{ab} + \omega_\mu{}^{ac}\,\omega_\nu{}^{cb} - \mu \leftrightarrow \nu$$

is related to the affine connection curvature

$$[D_\mu, D_\nu]\, e_\rho^a = R_{\mu\nu}{}^a{}_b(\omega)\, e_\rho^b = R^\sigma{}_{\rho\mu\nu}(\Gamma)\, e_\sigma^a$$

Proof: $D_\nu e_\rho^a = \Gamma_{\nu\rho}^\sigma\, e_\sigma^a$ (metricity)

Thus $e_a^\mu\, e_b^\nu\, R_{\mu\nu}{}^{ab}(\omega) = g^{\rho\nu} R^\sigma{}_{\rho\sigma\nu}(\Gamma)$

$$\text{or} \qquad R(\omega) = R(\Gamma)$$

$$\boxed{\int d^4x\, e\, R(\omega) = \int d^4x \sqrt{g}\, R(\Gamma)}$$

$e_\mu^{\ a}$ has 16 components, 6 of which are gauged away by the local Lorentz transformation

$$\delta e_\mu^{\ a} = \Lambda^a_{\ b}\, e_\mu^{\ b}$$

The spinors transform like scalars under general coordinate transformations, but are acted on by the local Lorentz transformations formed from Clifford algebra

$$\psi_\alpha(x) \to \left(e^{\frac{1}{4}\Lambda^{ab}(x)\gamma_{ab}}\right)_\alpha^{\ \beta}\psi_\beta$$

Taking a hint from NCG where the number of components of the spinors are larger than the four component spinors of the 4d tangent bundle, we allow for the dimensions of the tangent space to be N>d. Weinberg investigated the case of N<d, but found nothing interesting.

Looking at metricity condition

$$0 = \nabla_\mu e_\nu^{\ A} = \partial_\mu e_\nu^{\ A} + \omega_\mu^{\ A}{}_B\, e_\nu^{\ B} - \Gamma_{\mu\nu}^{\ \rho}\, e_\rho^{\ A}$$

where =0,1,...,d-1, and A=0, 1, .., N-1. We have d^2N conditions for d(N)(N-1)/2 $\omega_\mu^{\ A}{}_B$ and $d^2(d+1)/2$ $\Gamma_{\mu\nu}^{\ \rho}$. This will have a solution provided

$$d^2 N = dN(N-1)/2 + d^2(d+1)/2$$

There are two possibilities: 1- N=d which we have already studied and

2- N=d+1.

The remarkable thing in this case is that the curvature identities keep holding.

Let $D = \Gamma^A e_A^\mu \left(\partial_\mu + \frac{1}{4}\omega_\mu^{BC}\Gamma_{BC}\right)$; $\{\Gamma^A, \Gamma^B\} = -2\eta^{AB}$

Define soldering form e_μ^A and its inverse $e_\mu^A e_A^\nu = \delta_\mu^\nu$ but $e_\mu^A e_B^\mu \neq \delta_B^A$

instead $e^A{}_\mu e^\mu_B = \delta^A_B - n^A n_B$; $n^A e^\mu_A = 0$

The metric is given by $g_{\mu\nu} = e_\mu^A \eta_{AB} e_\nu^B$; $e^\mu_A = g^{\mu\nu} e_\nu^A$

$$\int d^4x\, e\, R(\omega) = \int d^4x \sqrt{g}\, R(\Gamma)$$

The identities hold even for all second order invariants in curvature.

$$R_{\mu\nu}{}^{AB}(\omega)\, e^\kappa_A\, e^\lambda_B\, R_{\kappa\lambda}{}^{CD}(\omega)\, e^\mu_C\, e^\nu_D = R^2_{\mu\nu\kappa\lambda}(\Gamma), \ldots$$

Vectors and tensors are not sensitive to the tangent space, but spinors are. These are also sensitive to the signature of the Minkowski space η_{AB}. In particular when d=4 and N=5 we have two possibilities for the signature of η_{AB}: $\eta_{AB} = (-++++)$ or $(--+++)$ corresponding to the two groups SO(1,4) and SO(2,3).

In case of SO(1,4) there could only be Dirac spinors. No Majorana or Weyl conditions are possible.

There are 4 independent components for the spinor.

In case of SO(2,3) a Majorana condition is possible and there are only two independent components for the spinor. Gravity and gauge interactions are equivalent in this case to the one with SO(1,3) tangent group. Projecting vectors in tangent space will give ghosts: $V_A = e_A^{\mu} V_{\mu} + n_A \phi$

Only SO(1,4) will avoid ghost states for phi.

One also gets a consistent system by allowing the soldering from $e_\mu^{\ A}$ *to be complex with the spin connection, anti-Hermitian and the affine connection Hermitian*

$$0 = \nabla_\mu e_\nu^{\ A} = \partial_\mu e_\nu^{\ A} + \omega_\mu{}^A{}_B \, e_\nu^{\ B} - \Gamma_{\nu\mu}{}^\rho \, e_\rho^{\ A}$$

$$e_\mu^{\ A *} = e_{\mu A} \;, \qquad g_{\mu\nu} = e_\mu^{\ A} e_{\nu A}$$

$$g_{\mu\nu}^* = g_{\nu\mu} \,, \qquad (\omega_\mu{}^A{}_B)^* = -\omega_{\mu A}{}^B \,, \quad \Gamma_{\mu\nu}{}^{\rho *} = \Gamma_{\nu\mu}{}^\rho$$

The symmetry of the tangent space is U(1,d-1) with N=d. This give Einstein-Strauss gravity and reproduces

all the work of Einstein in a simple way explaining all the symmetries and definitions that he had to

struggle with. We have d^3 *complex conditions with* d^3 *anti-Hermitian spin-connections and* d^3 *Hermitian .*

The question now is whether it is possible to have a sensible system where the number of constraints is

less than the number of variables ?

We require $\quad 0 = \nabla_\mu e_\nu^{\ A} = \partial_\mu e_\nu^{\ A} + \omega_\mu^{\ A}{}_B\, e_\nu^{\ B} - \Gamma_{\mu\nu}^{\ \rho}\, e_\rho^{\ A}$

where $\omega_\mu^{\ AB} = -\omega_\mu^{\ BA}$ and $\Gamma_{\mu\nu}^{\ \ \gamma} = \Gamma_{\nu\mu}^{\ \ \rho}$, A=0,1,..., N-1 and =0,1,...,d.

We have d^2N constraints for dN(N-1)/2 $\omega_\mu^{\ AB}$ and $d^2(d+1)/2$ $\Gamma_{\mu\nu}^{\ \ \rho}$

$d^2N< dN(N-1)/2 + d^2(d+1)$

We analyze the constraints for this system, by writing A=a, I where a=0,1,..., d-1 and I=d,....., N-1.

$$0 = \partial_\mu e_\nu^{\ a} + \omega_\mu^{\ a}{}_b\, e_\nu^{\ b} - \Gamma_{\mu\nu}^{\ \rho}\, e_\rho^{\ a} + \omega_\mu^{\ a}{}_I\, e_\nu^{\ I} \qquad (1)$$

$$0 = \partial_\mu e_\nu^{\ I} + \omega_\mu^{\ I}{}_J\, e_\nu^{\ J} - \Gamma_{\mu\nu}^{\ \rho}\, e_\rho^{\ I} + \omega_\mu^{\ I}{}_a\, e^a{}_\nu \qquad (2)$$

In equation 1, the number of constraints and number of variables $\omega_\mu{}^a{}_b$ and $\Gamma_{\mu\nu}^\rho$ match. They could be determined. In equation 2, the number of constraints match the variables $\omega_\mu{}^a{}_I$. This leaves the fields $\omega_\mu{}^I{}_J$ undetermined.

To see this, the soldering form transforms under the SO(1,N-1) Lorentz symmetry as

$$\delta e_\mu{}^A = \Lambda^A{}_B\, e_\mu{}^B \quad , \quad \Lambda^{AB} = -\Lambda^{BA}$$

In particular

$$\delta e_\mu{}^I = \Lambda^I{}_a\, e_\mu{}^a + \Lambda^I{}_J\, e_\mu{}^J$$

and we can chose the gauge $\boxed{e_\mu{}^I = 0}$ by fixing $\Lambda^I{}_a$. In this gauge equation 2 can be solved to give

$$\boxed{\omega_\mu{}^I{}_a = 0}$$

The gauge symmetry is broken from SO(1,N-1) to SO(1,d-1)x SO(N-d) as the gauge fixed symmetry. The gauge fields for SO(N-d) are $\omega_\mu{}^{IJ} = -\omega_\mu{}^{JI} \equiv A_\mu{}^{IJ}$.

Curvatures for the spin-connection are in this gauge:

$$R_{\mu\nu}{}^{ab} = \partial_\mu \omega_\nu{}^{ab} + \omega_\mu{}^{ac}\,\omega_{\nu c}{}^{b} - \mu \leftrightarrow \nu$$

$$R_{\mu\nu}{}^{aI} = 0$$

$$R_{\mu\nu}{}^{IJ} = \partial_\mu A_\nu{}^{IJ} + A_\mu{}^{IK} A_{\nu K}{}^{J} - \mu \leftrightarrow \nu \equiv F_{\mu\nu}{}^{IJ}$$

the last being the SO(N-d) curvature. We prove the following identities which hold without gauge fixing:

$$R_{\mu\nu}{}^{AB}(\omega)\, e_A^\mu\, e_B^\nu = R(\Gamma)$$

$$R_{\mu\nu}{}^{AB}(\omega)\, R_{\rho\sigma}{}^{CD}(\omega)\, e_A^\mu\, e_B^\sigma\, e_C^\nu\, e_D^\rho = -R_{\mu\nu}^2(\Gamma)$$

$$R_{\mu\nu}{}^{AB}(\omega)\, R_{\rho\sigma}{}^{CD}(\omega)\, e_A^\rho\, e_B^\sigma\, e_C^\mu\, e_D^\nu = R_{\mu\nu\rho\sigma}^2(\Gamma)$$

$$\boxed{g^{\mu\rho}\, g^{\nu\sigma}\, R_{\mu\nu}{}^{AB}\, R_{\rho\sigma AB} = R_{\mu\nu\rho\sigma}^2(\Gamma) + (F_{\mu\nu}{}^{IJ})^2}$$

The last relation shows the unification of gauge and gravity.

After using the above identities, the most general action, up to quadratic terms in spin-curvature is

$$I = \int d^4x \sqrt{g} \left[\frac{1}{16\pi G} R(\Gamma) + a R^2(\Gamma) - b R_{\mu\nu}^2(\Gamma) + \left(c - \frac{1}{4}\right) R_{\mu\nu\rho\sigma}^2(\Gamma) - \frac{1}{4} \left(F_{\mu\nu}^{IJ}\right)^2 \right]$$

Compare this action with the spectral action associated with the Dirac operator

$$D = \Gamma^A e_A^\mu \left(\partial_\mu + \frac{1}{4} \omega_\mu^{BC} \Gamma_{BC} \right)$$

in four dimensions, where ω_μ^{AB} is subject to metricity condition $\nabla_\mu e_A^\nu = 0$

$$\mathrm{Tr}\, f(D^2) = \frac{2^{N/2}}{16\pi^2} \left[f_4 \int d^4x \sqrt{g} + \frac{1}{12} f_2 \int d^4x \sqrt{g}\, R \right.$$

$$\left. + \frac{1}{360} f_0 \int d^4x \sqrt{g} \left(3 R_{;\mu}{}^{\mu} + \frac{5}{4} R^2 - 2 R_{\mu\nu}^2 - \frac{7}{4} R_{\mu\nu\rho\sigma}^2 + \frac{15}{4} \left(F_{\mu\nu}^{IJ}\right)^2 \right) \right]$$

All curvatures are functions of the metric $g_{\mu\nu}$ and the a_4 is conformally invariant.

The dimensions of the spinors ψ_α is $2^{[N/2]}r$ where r is a reduction factor, r=1/2 when Weyl or Majorana condition is imposed, and r=1/4 when both conditions could be imposed simultaneously.

To obtain gauge unification for a realistic model in four dimensions, the first realistic gauge group is SO(10): N-4=10 and thus N=14. The symmetry is SO(1,13). Dimension of the spinor is $2^7\left(\frac{1}{2}\right)=2^6$ because Weyl condition could be imposed. This is a 4-spinor with respect to SO(1,3) and 16 with respect to SO(10). Thus it is $16_s + \overline{16}_s$ space-time spinors with respect to SO(1,3).

The Weyl condition is $\left(\Gamma^{15}\right)_\alpha^{\hat\beta}\,\psi_{\hat\beta} = \psi_{\hat\alpha}$, $\quad \Gamma^{15} = \gamma_5 \otimes \Gamma$

$\gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3$, $\quad \Gamma = \Gamma_4 \cdots \Gamma_{13}$

The spinors feel the full SO(1,13) symmetry, unlike gravity and vectors where it is hidden.

The problem now is to obtain chiral spinors in 4d. The symmetry SO(1,13) must be broken spontaneously

to SO(1,3)x SO(10). For this we use Higgs like mechanism.

Use the Higgs fields H_{ABCD} with vev $\langle H_{abcd} \rangle = \frac{v}{4!} \epsilon_{abcd}$ $\quad a = 0,1,2,3$

$$\langle H_{ijkl} \rangle = \frac{v'}{4!} \epsilon_{ijkl} \qquad i = 4,5,6,7$$

This breaks SO(1,13) to SO(1,3) x SO(4) x SO(6). We recognize this as the Pati-Salam group

SO(4) isomorphic to SU(2)x SU(2) and SO(6) isomorphic to SU(4).

Add Majorana mass $\quad \Psi_{\hat{\alpha}} \left(C^{\hat{\alpha}\hat{\beta}} \phi - (C\,\Gamma^{ABCD})^{\hat{\alpha}\hat{\beta}} \right) \Psi_{\hat{\beta}}$

$$\langle \phi \rangle = v \longrightarrow \quad v\, \Psi_{\hat{\alpha}} \left(C (1-\gamma_5) \right)^{\hat{\alpha}\hat{\beta}} \Psi_{\hat{\beta}}$$

$(1-\gamma_5)\,\psi_{\tilde{\alpha}}$ acquire Majorana mass

$(1+\gamma_5)\,\psi_{\dot{\alpha}}$ remain massless.

SU(4) can be further broken to SU(3) x SU(2) x U(1) with Higgs $H_{ABCDE}\,\Gamma^{ABCDE}$

and to SU(3) x U(1) with Higgs $H_A\,\Gamma^A$ and $H_{ABC}\,\Gamma^{ABC}$.

Need for NCG

From the above it is clear that we have succeeded in combining space-time symmetry with the gauge symmetry in one group. There is need to unify further by including the Higgs fields, which requires NCG and discrete symmetries to extend the Dirac operator. This must take the Clifford algebra that was used above into consideration.

There is an ambiguity connected with the Euclidean signature in NCG as the number of independent components of the spinors depends on it. For SM this was solved by using a Dirac action of the form $\langle J\Psi, D\psi\rangle$. For the SO(1,13) example the Euclidean versionn requires SO(14) instead which has KO dimensions 6.

Conclusions

1- The role of the tangent group in gravity has been underestimated. Although vectors are not dependent on the tangent group, the spinors are very sensitive to it.

2- The metricity condition on the soldering form (vielbein) insures the solvability of the system and determines both the spin-connnection and the symmetric affine connection.

3- When dimensions of the tangent group N=d+1, gravity for SO(1,d-1) and SO(1,d) are identical but spinors know the difference.

4- For complex vielbeins the tangent group is the Unitary group U(1,d-1) instead of SO(1,d-1) and gives Einstein-Strauss Hermitian gravity. The new formulation is extremely simple in contrast to Einstein's work.

5- Enlarging tangent space so that N>d and taking associated Lorentz group as the symmetry implies a generalized curvature. The soldering form admit the same metricity condition, which now only determines the d- dimensional spin-connection and d-dimensional symmetric affinee connection. The SO(N-d) gauge connection remains arbitrary.

6- The curvature of the N-dimensional spin-connection includes both the gravity and gauge sectors. The difference is that the dynamics of the gravity sector is governed by linear terms in curvature and the gauge fields by quadratic terms. The spin-connection is not an independent field while the gauge connection is. This explains renormalizability of gauge fields.

7- This is the first step towards geometric unification and realizes Einstein's dream of unifying gravity and electromagnetism.

8- The spinors in the unified theory of SO(1,13) have unwanted partners which must get masses through spontaneous symmetry breaking. The idea of SO(1,13) breaking to SO(3,1)x SO(10) is not a good idea.

9- Spectral action for the new Dirac operator works, and there is need to extend it to include Higgs fields.