From Equivariant Quantization to Locally Compact Quantum Groups

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Locally compact quantum groups

A) von Neumann algebraic setting (Kustermans-Vaes 2000)

Definition: A locally compact quantum group in the von Neumann algebraic setting is $\mathbb{G} = (\mathcal{M}, \Delta, \Phi_\lambda, \Phi_\rho)$ where

- $\mathcal{M}$ is a von Neumann algebra
- $\Delta : \mathcal{M} \to \mathcal{M} \bar{\otimes} \mathcal{M}$ is a co-associative morphism (coproduct)
- Φ_λ, Φ_ρ are left- and right-invariant NSF weights on $\mathcal{M}$, i.e.

$$\Phi_\lambda\Big(\omega \otimes \mathrm{Id}(\Delta(a)\Big) = \omega(1)\Phi_\lambda(a)\,, \quad \forall a \in \mathcal{N}^+,\ \forall \omega \in \mathcal{M}_*^+$$

Well established theory

→ Antipode

→ Pontryagin duality (denote by $\widehat{\mathbb{G}} = (\widehat{\mathcal{M}}, \widehat{\Delta}, \widehat{\Phi}_\lambda, \widehat{\Phi}_\rho)$ the dual LCQG)

But possesses pathologies

→ Measurable and not topological theory as it should be

→ Very few examples

Cocycle deformation of a LCQG

Definition: A dual unitary 2-cocycle for a locally compact quantum group $\mathbb{G}$ is an element $F \in \mathcal{U}(\hat{\mathcal{M}} \bar{\otimes} \hat{\mathcal{M}})$ satisfying

$$(F \otimes 1)(\hat{\Delta} \otimes \mathrm{Id})(F) = (1 \otimes F)(\mathrm{Id} \otimes \hat{\Delta})(F)$$

$\rightarrow$ Deformation of the coproduct:

$$\hat{\Delta}_F : \hat{\mathcal{M}} \rightarrow \hat{\mathcal{M}} \bar{\otimes} \hat{\mathcal{M}}, \quad a \mapsto F^* \hat{\Delta}(a) F$$

Theorem [De Commer 2010]: There exist NSF weights on $\hat{\mathcal{M}}$ invariant for $\hat{\Delta}_F$

Hence: Using duality, one can deform a LCQG $\mathbb{G}$ from a dual unitary 2-cocycle F: $\mathbb{G}_F$

Open problem: Construct a dual unitary 2-cocycle already on a genuine group!

Open solution: Non-formal equivariant quantization on a group!

Part I: What we (I?) want to do

Non-formal G-equivariant quantization on a group G :

Data:

- $\widetilde{G}$ locally compact group (covariance group)
- $(\pi, \mathcal{H}_\pi)$ projective representation (square integrable) of $\widetilde{G}$
- $X = \widetilde{G}/H$ homogeneous space (substitute for the phases space)

Definition: $\widetilde{G}$-equivariant quantization on X :

$$\mathbf{\Omega} : \mathcal{D}(X) \to \mathcal{B}(\mathcal{H}_\pi), \quad \pi(g)\,\mathbf{\Omega}(f)\,\pi(g)^* = \mathbf{\Omega}(f^g)\,, \quad \left[f^g(x) := f(g^{-1}.x)\right]$$

Examples: Weyl quantization, Berezin quantization, coherent states quantization, Fuchs calculus (Unterberger), p-adic Weyl calculus (Haran-Unterberger), BCH quantization of coadjoint orbits of exponential Lie groups.....

Geometrical assumptions:

(H1) $\tilde{G}$ possesses a subgroup acting simply transitively on X (and the restriction of π is still irreducible and square-integrable)

(H1') H is a normal subgroup of $\tilde{G}$ and $[\pi(H), \mathbf{\Omega}(\mathcal{D}(X))] = 0$

In both cases: The phases space X is endowed with a group structure (X is then denoted G)

In the first case: we have a G-equivariant quantization on G (G acts on $\mathcal{H}_\pi$)

In the second case: we have a G-quasi-equivariant quantization on G (G does not act on $\mathcal{H}_\pi$)

Analytical assumptions :

(H2) The map $\mathbf{\Omega}$ extends to a unitary from $L^2(G)$ to $\mathcal{L}^2(\mathcal{H}_\pi)$
$\rightarrow$ If not finite, then G non-compact and $\mathcal{H}_\pi$ infinite dimensional

- $L^2(G)$ becomes an algebra equivariant under the left action of G

$$f_1 \star f_2 := \mathbf{\Omega}^{-1}\Big(\mathbf{\Omega}(f_1)\mathbf{\Omega}(f_2)\Big)$$

(H3) The distribution $\mathbf{K} \in \mathcal{D}'(G^3)$ is a regular function (Bruhat)

$$\langle \mathbf{K}, \varphi_1 \otimes \varphi_2 \otimes \varphi_3 \rangle := \mathrm{Tr}\Big(\mathbf{\Omega}(f_1)\mathbf{\Omega}(f_2)\mathbf{\Omega}(f_3)\Big)$$

- Tri-point kernel $\mathbf{K}$ is invariant under left diagonal action of G

- On $\mathcal{D}(G)$ the non-formal star-product reads

$$f_1 \star f_2 = \int_{G\times G} \mathbf{K}(e, g_1, g_2)\, \rho_{g_1}(f_1)\, \rho_{g_2}(f_2)\, dg_1 dg_2$$

Paradigmatic example:
Fix a densely defined self-adjoint operator Σ on $\mathcal{H}_\pi$ (which commutes with $\pi(H)$ in the G-quasi-equivariant case) and set

$$\Omega(f) = \int_G f(g)\, \pi(g)\Sigma\pi(g)^*\, dg$$

• In bad exemple Σ is compact and positive (Berezin $\Sigma = |\varphi\rangle\langle\varphi|$)

• In good examples (Weyl, Unterberger...), Σ is bounded and self-adjoint

• In very good examples (BCH, what follows) Σ is unbounded and self-adjoint

• Typically, $\mathcal{H}_\pi = L^2(Q)$, σ involution on Q, where Q (the configuration space) is a sub-group of G

$$\Sigma\varphi(q) = \mathrm{Jac}_\Psi^{1/2}(q)\, \varphi(\sigma(q))$$

In conclusion : Non-formal G-equivariant (or G-quasi-equivariant) quantizations on a group G are associated with

• $(\pi, \mathcal{H}_\pi)$ projective representation of G (or of a group $\widetilde{G}$ for which G is a quotient)

• Σ non necessarily bounded self-adjoint operator on $\mathcal{H}_\pi$ (which commutes with $\pi(H)$ in the G-quasi-equivariant case)

Natural candidate for a unitary dual 2-cocycle on G:

$$F = \int_{G\times G} \overline{\mathbf{K}(e, g_1, g_2)}\, \lambda_{g_1^{-1}} \otimes \lambda_{g_2^{-1}}\, dg_1 dg_2$$

- 2-cocyclicity for F equivalent to associativity of $\star$

- Unitarity has to be checked

Slogan: What seems to be deep is not and what seems to be trivial is not....

Part II: What we have done

Negatively curved Kählerian Lie groups

Pyatetskii-Shapiro : Every Kählerian Lie group with negative sectional curvature as the form

$$\Big(\Big(\dots\Big(G_d \ltimes G_{d-1}\Big) \ltimes \dots\Big) \ltimes G_2\Big) \ltimes G_1$$

- $G_j \simeq AN_j$, $AN_jK = SU(1, n_j)$, $N_j =$ Heisenberg group

- Solvable, non-unimodular, exponential

- Extension homomorphisms land in $\mathrm{Sp}(V_j)$

- 2 classes of (infinite dimensional) irreducible unitary representations $U_\pm$ on $L^2(Q)$, where $G = Q \ltimes P$

- En dim 2 : $G = ax + b$

Extra geometric structures

- Elementary block $G = AN$ symmetric symplectic space G-covariant

- Mid-point map mid : $G \times G \to G$, $s_{\text{mid}(x,y)}(x) = y$

- $\Phi : G^3 \to G^3$, $(g, g', g'') \mapsto (\text{mid}(g, g'), \text{mid}(g', g''), \text{mid}(g'', g))$
global diffeomorphism invariant under the diagonal left action of G

- The symmetric structure of G restrict to Q ($G = Q \ltimes P$)

Elementary case:

- $\Sigma\varphi(q) = \mathsf{Jac}_{\psi}^{1/2}(q)\varphi(s_e(q))$ (ψ is an intrinsic diffeomorphism of Q)

- Representations $U_{\pm}$ on $L^2(Q)$

- Tri-point kernel

$$\mathbf{K}_{\pm}^{G}(g,g',g'') = |\mathsf{Jac}_{\Phi^{-1}}|^{1/2}(g,g',g'')\ \exp\Big\{\pm 2i\mathsf{Area}\Big(\Phi^{-1}(g,g',g'')\Big)\Big\}$$

General case: (stupid) gluing $g = g_1 g_2 \in G = G_2 \ltimes G_1$, $g_j \in G_j$

$$\mathbf{K}_{\pm,\pm}^{G}(g,g',g'') = \mathbf{K}_{\pm}^{G_1}(g_1,g_1',g_1'')\mathbf{K}_{\pm}^{G_2}(g_2,g_2',g_2'')$$

Theorem [G-Bieliavsky 2013]:

- Quantization unitary from $L^2(G)$ to $\mathcal{L}^2(\mathcal{H}_\pm)$
- It extends as a continuous map from a larger set of functions to $\mathcal{B}(\mathcal{H}_\pm)$ (Calderon-Vaillancourt type estimate)
- It allows to deform any C^*-algebra endowed with a continuous action of G (i.e. we generalize Rieffel construction for a non-Abelian group)

Theorem [Neshveyev-Tuset 2014]:

The quadratic form on $\mathcal{D}(G \times G)$ defined by

$$F^*_\pm[\varphi_1, \varphi_2] := \int \mathbf{K}^G_\pm(e, g_1, g_2)\, \bar{\varphi}_1 * \varphi_2(g_1, g_2)\, dg_1 dg_2$$

extends to a unitary operator on $L^2(G \times G)$ and its adjoint defines a dual 2-cocycle on G

$\rightarrow$ First example of a unitary dual 2-cocycle on a non-Abelian group!

Variation 1: geometric structures

• If $\mathrm{Dim}(G) \geq 4$, there is two classes of non equivalent symplectic symmetric space structures on an elementary Kahlerian Lie group G (Voglaire)

$\rightarrow$ There is no (yet) quantization behind but a direct star-product approach (what does change is the covariance group $\widetilde{G}$)

Theorem [G-Jondreville]:
The associated dual 2-cocycle is unitary on $L^2(G \times G)$

• Projectives representations

$\rightarrow$ Group cohomology in degree 2 is nontrivial if G is not elementary (i.e. if there is at least two factors in the Pyatetskii-Shapiro decomposition of G)

$\rightarrow$ There exists non trivial projective representations if G is not elementary

Theorem [Bieliavsky-G-De Goursac]:
The associated dual 2-cocycle is unitary on $L^2(G \times G)$

Clopen Question: Are the different negatively curved quantum Kählerian Lie groups constructed so far isomorphic?

• Representation theory: NO (conjecture supported by my feeling)

• Symplectic symmetric space structures: NO (conjecture supported by the fact that covariance groups are differents)

• Projective representations: NO (proof from Duflo-Moore theory)

Variation 2: semi-simple covariance group

• Up to now, all the variations lead to solvable covariance groups ($\widetilde{G}$)

For elementary Kahlerians Lie groups, there is ways to modify the star-product (not yet the quantization map) to get for covariance group $\tilde{G} = SU(1,n)$

• But the associated dual 2-cocycle is no longer unitary, probably (?) invertible

Variation 3: no more geometry

Motivation: The BCH quantization of an exponential Lie group

- Let G be an exponential Lie group with $\mathfrak{g}$ its Lie algebra

- Assume G possesses a coadjoint orbit $\mathcal{O} \subset \mathfrak{g}^\star$ on which G acts simply transitively

- Let $U_{\mathcal{O}}$ the KKS representation of G

- For $f \in C_c^\infty(\mathcal{O}) \subset C_c^\infty(\mathfrak{g}^\star)$ define

$$\mathbf{\Omega}_{\mathcal{O}}(f) := \int_{\mathfrak{g}} \mathcal{F}(f)(X)\, U_{\mathcal{O}}\Big(\exp\{X\}\Big)\, dX$$

- After identification $G \simeq \mathcal{O}$, get a G-covariant quantization on G

• These assumptions are satisfied for $G = \mathbb{R} \ltimes \mathbb{R}$. In this case, two possible orbits $\Pi_\pm := \{(x, y) \in \mathfrak{g}^\star : \pm y > 0\}$

• The quantization map reads

$$\mathbf{\Omega}_\pm(f) = \int_G f(g)\, U_\pm(g)\, \mathbf{\Sigma}\, U_\pm(g) dg,$$

• where, realizing $\mathcal{H}_\pm$ as $L^2(\mathbb{R})$, we have

$$\Sigma\varphi(t) = |\gamma'(t)|\, \varphi\Big(\sigma(t)\Big),$$

• σ is the involutive diffeomorphism of $\mathbb{R}$ given by

$$\sigma = \mathrm{Id} - \gamma : \mathbb{R} \to \mathbb{R}$$

and γ is the inverse diffeomorphism of $\log \circ \lambda : \mathbb{R} \to \mathbb{R}$ where

$$\lambda : \mathbb{R} \to \mathbb{R}^*_+ , \quad t \mapsto t(1 - e^{-t})^{-1}$$

Generalization: no more geometry

- $G = \mathbb{R} \ltimes \mathbb{R}$, $U_\pm$

- Let $\sigma \in \mathsf{Diff}(\mathbb{R})$ be an involution such that

(1) $\gamma := \sigma - \mathsf{Id} \in \mathsf{Diff}(\mathbb{R})$

(2) $\phi := [a \mapsto e^a - e^{\sigma(a)}] \in \mathsf{Diff}(\mathbb{R})$

Set $\Sigma\varphi(a) := |\gamma'(a)|^{1/2}\,|\phi'(a)|^{1/2}\,\varphi\Big(\sigma(a)\Big)$

Theorem [G-Jondreville]:

(i) The $\mathbb{R} \ltimes \mathbb{R}$-covariant quantization map on $\mathbb{R} \ltimes \mathbb{R}$ associated with $(U_\pm, \Sigma)$ defines a unitary operator from $L^2(G)$ to HS$(L^2(\mathbb{R}))$

(ii) Let $\kappa : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be the unique solution of

$$\sigma(\sigma(\sigma(\kappa(a_1, a_2, a_3) - a_1) + a_1 - a_2) + a_2 - a_3) + a_3 = \kappa(a_1, a_2, a_3)$$

Then, the associated 2-points kernel reads ($\kappa := \kappa(0, a_1, a_2)$)

$$\mathbf{K}_\sigma(a_1, t_1; a_2, t_2) = \frac{\sqrt{\left|\gamma'\phi'\right|\Big(\kappa\Big)\left|\gamma'\phi'\right|\Big(\sigma(\kappa) - a_1\Big)\left|\gamma'\phi'\right|\Big(\sigma(\kappa - a_2)\Big)}}{\Big|1 - \sigma'(\kappa)\sigma'(\sigma(\kappa) - a_1)\sigma'(\sigma(\kappa - a_2))\Big|}$$
$$\times \exp\{i\phi(a_1 - \sigma(\kappa))t_1 + i\phi(a_2 - \kappa)t_2\}$$

(iii) The 2-cocycle

$$F_\sigma := \int_{G^2} \overline{\mathbf{K}_\sigma(e; g, g')}\, \lambda_{g^{-1}} \otimes \lambda_{g'^{-1}}\, dg dg'$$

is a unitary element of $W^*((G \times G)$ if and only if $\sigma = -\mathrm{Id}$

• Unitarity for the 2-cocycle selects the (unique) symplectic symmetric space structure of the affine group of the real line

• Need framework without unitarity for the dual 2-cocycle, instead invertibility (on suitable domains)

Variation 4: base field

- $\mathbf{k}$ Non Archimedean local field (of characteristic $\neq 2$ and which is not an extension of $\mathbb{Q}_2$)

- $\mathcal{O}_{\mathbf{k}}$ ring of integers, ϖ a generator of its unique maximal ideal, Ψ non-trivial additive character of $\mathbf{k}$ constant on $\mathcal{O}_{\mathbf{k}}$

- $\widetilde{G}_n = (1 + \varpi^n \mathcal{O}_{\mathbf{k}}) \ltimes \mathbf{k}$, $n = 1, 2, \ldots$

- $G_n = \widetilde{G}_n / \varpi^{-n} \mathcal{O}_{\mathbf{k}} \simeq \mathcal{O}_{\mathbf{k}} \ltimes_{\alpha_n} \widehat{\mathcal{O}_{\mathbf{k}}}$ (countable family of non-isomorphic groups which are not discrete, nor compact but unimodular)

- Representations of $\widetilde{G}_n$ on $L^2(1 + \varpi^n \mathcal{O}_{\mathbf{k}})$

$$U(a,t)\varphi(a_0) = \Psi(a a_0^{-1} t)\varphi(a^{-1} a_0)$$

(Mackey: all the infinite dimensional reps are of this form)

- $\Sigma\varphi(a_0) = \varphi(a_0^{-1})$ (bounded and self-adjoint)

$\rightarrow$ Trikernel given by

$$\mathbf{K}(a_1,[t_1];a_2,[t_2];a_3,[t_3]) = \Psi\Big((\frac{a_1}{a_2}-\frac{a_2}{a_1})t_2+(\frac{a_2}{a_3}-\frac{a_3}{a_2})t_1+(\frac{a_3}{a_1}-\frac{a_1}{a_3})t_2\Big)$$

Theorem [G-Jondreville]:

• The quantization is unitary and extends as a continuous map from a larger space of functions to $\mathcal{B}(L^2(1+\varpi^n\mathcal{O}_{\mathrm{k}}))$ (non-Abelain and p-adic Calderon-Vaillancourt) and allows to deform any C^*-algebra endowed with a continuous action of G and ....

• The associated dual 2-cocycle is unitary on $L^2(G_n\times G_n)$

Locally compact quantum groups

B) Manageable multiplicative unitary (Woronowicz 1995)

Definition : A multiplicative unitary on $\mathcal{H}$ is a unitary W on $\mathcal{H}\bar{\otimes}\mathcal{H}$ satisfying the pentagonale equation

$$W_{23}W_{12} = W_{12}W_{13}W_{23}$$

W is manageable is there exists a densely defined positive and self-adjoint operator Q with densely defined inverse and a unitary operator $\widetilde{W}$ on $\bar{\mathcal{H}}\bar{\otimes}\mathcal{H}$ such that

$$W^* Q \otimes Q W = Q \otimes Q$$

and such that for all $\varphi_1, \varphi_3 \in \mathcal{H}$ and all $\varphi_2 \in \mathrm{Dom}(Q)$, $\varphi_4 \in \mathrm{Dom}(Q^{-1})$

$$\langle \varphi_1 \otimes \varphi_2, W \varphi_3 \otimes \varphi_4 \rangle = \langle \bar{\varphi}_3 \otimes Q\varphi_2, \widetilde{W} \bar{\varphi}_4 \otimes Q^{-1}\varphi_4 \rangle$$

Less well established theory

$\rightarrow$ Generalize Baaj-Skandalis theory

$\rightarrow$ Contains all the locally compact quantum group in the von Neumann algebraic setting

$\rightarrow$ Topological theory as it should be

Still possesses a pathology

$\rightarrow$ Very few examples

Multiplicative unitary from an invertible dual 2-cocycle
(Joint work with Bieliavsky, Bonneau and D'Andrea)

- Starts with a densely defined with densely defined inverse dual 2-cocycle on a group G (coming from quantization or not)

$$F = \int_{G\times G} K(g_1,g_2)\lambda_{g_1}\otimes\lambda_{g_2}dg_1dg_2 \quad \text{affiliated with} \quad W^*_\lambda(G\times G)$$

- Assume

$$F^{-1} = \int_{G\times G} \widetilde{K}(g_1,g_2)\lambda_{g_1}\otimes\lambda_{g_2}dg_1dg_2 \quad \text{affiliated with} \quad W^*_\lambda(G\times G)$$

and that $\widetilde{F}$ is still a dual 2-cocycle, where

$$\widetilde{F} := \int_{G\times G} \widetilde{K}(g_1^{-1},g_2^{-1})\lambda_{g_1}\otimes\lambda_{g_2}dg_1dg_2 \quad \text{affiliated with} \quad W^*_\lambda(G\times G)$$

- Set

$$\widetilde{F}_\rho := \int_{G\times G} \widetilde{K}(g_1,g_2)\rho_{g_1}\otimes\rho_{g_2}dg_1dg_2 \quad \text{affiliated with} \quad W^*_\rho(G\times G)$$

- Assume the associative doubly deformed product $\star_{\lambda,\rho}$ well defined

$$\varphi_1 \star_{\lambda,\rho} \varphi_2 := \mu \circ F \circ \widetilde{F}_\rho(\varphi_1 \otimes \varphi_2)$$

- Natural candidate for a manageable multiplicative unitary

$$W_\star(\varphi_1 \otimes \varphi_2) = \Delta(\varphi_1) \star_{\lambda,\rho} (1 \otimes \varphi_2)$$

- Pentagonale equation is automatic!

- Unitarity on the completion of $\mathcal{D}(G)$ of

$$\langle \varphi_1, \varphi_2 \rangle := \int_G \overline{\varphi_1} \star_{\lambda,\rho} \varphi_2(g)\, d^\rho(g)$$

Provided it is a scalar product !

True when $\delta_G^\alpha \star_{\lambda,\rho} \delta_G^\beta = \delta_G^{\alpha+\beta}$

- Manageability has to be checked......