From Equivariant Quantization to Locally Compact Quantum Groups

Victor Gayral
Laboratoire de mathématique
Université de Reims Champagne-Ardenne
Locally compact quantum groups

A) von Neumann algebraic setting (Kustermans-Vaes 2000)

**Definition:** A locally compact quantum group in the von Neumann algebraic setting is $G = (\mathcal{M}, \Delta, \Phi_\lambda, \Phi_\rho)$ where

- $\mathcal{M}$ is a von Neumann algebra
- $\Delta : \mathcal{M} \rightarrow \mathcal{M} \bar{\otimes} \mathcal{M}$ is a co-associative morphism (coproduct)
- $\Phi_\lambda, \Phi_\rho$ are left- and right-invariant NSF weights on $\mathcal{M}$, i.e.

$$\Phi_\lambda(\omega \otimes \text{Id}(\Delta(a))) = \omega(1)\Phi_\lambda(a), \quad \forall a \in \mathcal{N}^+, \forall \omega \in \mathcal{M}_*^+$$
Well established theory

→ Antipode

→ Pontryagin duality (denote by \( \hat{G} = (\hat{M}, \hat{\Delta}, \hat{\Phi}_\lambda, \hat{\Phi}_\rho) \) the dual LCQG)

But possesses pathologies

→ Measurable and not topological theory as it should be

→ Very few examples
Cocycle deformation of a LCQG

Definition: A dual unitary 2-cocycle for a locally compact quantum group $G$ is an element $F \in \mathcal{U}(\hat{\mathcal{M}} \bar{\otimes} \hat{\mathcal{M}})$ satisfying

$$(F \otimes 1)(\hat{\Delta} \otimes \text{Id})(F) = (1 \otimes F)(\text{Id} \otimes \hat{\Delta})(F)$$

→ Deformation of the coproduct:

$$\hat{\Delta}_F : \hat{\mathcal{M}} \to \hat{\mathcal{M}} \bar{\otimes} \hat{\mathcal{M}}, \quad a \mapsto F^* \hat{\Delta}(a)F$$

Theorem [De Commer 2010]: There exist NSF weights on $\hat{\mathcal{M}}$ invariant for $\hat{\Delta}_F$
Hence: Using duality, one can deform a LCQG $\mathcal{G}$ from a dual unitary 2-cocycle $F: \mathcal{G}_F$

Open problem: Construct a dual unitary 2-cocycle already on a genuine group!

Open solution: Non-formal equivariant quantization on a group!
Part I: What we (I?) want to do
Non-formal $G$-equivariant quantization on a group $G$:

**Data:**
- $\tilde{G}$ locally compact group (covariance group)
- $(\pi, \mathcal{H}_\pi)$ projective representation (square integrable) of $\tilde{G}$
- $X = \tilde{G}/H$ homogeneous space (substitute for the phases space)

**Definition:** $\tilde{G}$-equivariant quantization on $X$:

$$\Omega : \mathcal{D}(X) \to \mathcal{B}(\mathcal{H}_\pi), \quad \pi(g) \Omega(f) \pi(g)^* = \Omega(f^g), \quad [f^g(x) := f(g^{-1}.x)]$$

**Examples:** Weyl quantization, Berezin quantization, coherent states quantization, Fuchs calculus (Unterberger), $p$-adic Weyl calculus (Haran-Unterberger), BCH quantization of coadjoint orbits of exponential Lie groups.....
Geometrical assumptions:

(H1) \( \tilde{G} \) possesses a subgroup acting simply transitively on \( X \) (and the restriction of \( \pi \) is still irreducible and square-integrable)

(H1’) \( H \) is a normal subgroup of \( \tilde{G} \) and \([\pi(H), \Omega(D(X))] = 0\)

In both cases: The phases space \( X \) is endowed with a group structure (\( X \) is then denoted \( G \))

In the first case: we have a \( G \)-equivariant quantization on \( G \) (\( G \) acts on \( \mathcal{H}_\pi \))

In the second case: we have a \( G \)-quasi-equivariant quantization on \( G \) (\( G \) does not act on \( \mathcal{H}_\pi \))
Analytical assumptions:

(H2) The map $\Omega$ extends to a unitary from $L^2(G)$ to $L^2(\mathcal{H}_\pi)$
$\rightarrow$ If not finite, then $G$ non-compact and $\mathcal{H}_\pi$ infinite dimensional

- $L^2(G)$ becomes an algebra equivariant under the left action of $G$

$$f_1 \ast f_2 := \Omega^{-1}(\Omega(f_1)\Omega(f_2))$$

(H3) The distribution $K \in \mathcal{D}'(G^3)$ is a regular function (Bruhat)

$$\langle K, \varphi_1 \otimes \varphi_2 \otimes \varphi_3 \rangle := \text{Tr}(\Omega(f_1)\Omega(f_2)\Omega(f_3))$$

- Tri-point kernel $K$ is invariant under left diagonal action of $G$

- On $\mathcal{D}(G)$ the non-formal star-product reads

$$f_1 \ast f_2 = \int_{G \times G} K(e, g_1, g_2) \rho_{g_1}(f_1) \rho_{g_2}(f_2) \, dg_1 \, dg_2$$
Paradigmatic example:
Fix a densely defined self-adjoint operator $\Sigma$ on $\mathcal{H}_\pi$ (which commutes with $\pi(H)$ in the $G$-quasi-equivariant case) and set

$$\Omega(f) = \int_G f(g) \pi(g) \Sigma \pi(g)^* \, dg$$

• In bad exemple $\Sigma$ is compact and positive (Berezin $\Sigma = \langle \varphi \rangle \langle \varphi |$)
• In good examples (Weyl, Unterberger...), $\Sigma$ is bounded and self-adjoint
• In very good examples (BCH, what follows) $\Sigma$ is unbounded and self-adjoint
• Typically, $\mathcal{H}_\pi = L^2(Q)$, $\sigma$ involution on $Q$, where $Q$ (the configuration space) is a sub-group of $G$

$$\Sigma \varphi(q) = \text{Jac}^{1/2}_\Psi(q) \varphi(\sigma(q))$$
In conclusion: Non-formal $G$-equivariant (or $G$-quasi-equivariant) quantizations on a group $G$ are associated with

- $(\pi, \mathcal{H}_\pi)$ projective representation of $G$ (or of a group $\tilde{G}$ for which $G$ is a quotient)

- $\Sigma$ non necessarily bounded self-adjoint operator on $\mathcal{H}_\pi$ (which commutes with $\pi(H)$ in the $G$-quasi-equivariant case)
Natural candidate for a unitary dual 2-cocycle on $G$:

$$F = \int_{G \times G} K(e, g_1, g_2) \lambda g_1^{-1} \otimes \lambda g_2^{-1} \, dg_1 dg_2$$

- 2-cocyclicity for $F$ equivalent to associativity of $\star$

- Unitarity has to be checked

Slogan: What seems to be deep is not and what seems to be trivial is not....
Part II: What we have done
Negatively curved Kählerian Lie groups

Pyatetskii-Shapiro: Every Kählerian Lie group with negative sectional curvature as the form

\[ \left( \left( \cdots \left( \left( G_d \ltimes G_{d-1} \right) \ltimes \cdots \right) \ltimes G_2 \right) \ltimes G_1 \right) \]

• \( G_j \simeq AN_j, \ AN_jK = SU(1, n_j), \ N_j = \text{Heisenberg group} \)

• Solvable, non-unimodular, exponential

• Extension homomorphisms land in \( \text{Sp}(V_j) \)

• 2 classes of (infinite dimensional) irreducible unitary representations \( U_\pm \) on \( L^2(Q) \), where \( G = Q \ltimes P \)

• \( \text{En dim} \ 2: \ G = ax + b \)
Extra geometric structures

• Elementary block $G = AN$ symmetric symplectic space $G$-covariant

• Mid-point map $\text{mid} : G \times G \to G$, $s_{\text{mid}(x,y)}(x) = y$

• $\Phi : G^3 \to G^3$, $(g, g', g'') \mapsto (\text{mid}(g, g'), \text{mid}(g', g''), \text{mid}(g'', g))$
  global diffeomorphism invariant under the diagonal left action of $G$

• The symmetric structure of $G$ restrict to $Q$ ($G = Q \ltimes P$)
Elementary case:

- $\Sigma \varphi(q) = \text{Jac}_{\psi}^{1/2}(q) \varphi(s_e(q))$ ($\psi$ is an intrinsic diffeomorphism of $Q$)
- Representations $U_{\pm}$ on $L^2(Q)$
- Tri-point kernel
  
  $$K^G_{\pm}(g, g', g'') = |\text{Jac}_{\Phi^{-1}}|^{1/2}(g, g', g'') \exp \left\{ \pm 2i \text{Area} \left( \Phi^{-1}(g, g', g'') \right) \right\}$$

General case: (stupid) gluing $g = g_1 g_2 \in G = G_2 \times G_1$, $g_j \in G_j$

$$K^G_{\pm, \pm}(g, g', g'') = K^G_{\pm}(g_1, g'_1, g''_1)K^G_{\pm}(g_2, g'_2, g''_2)$$
Theorem [G-Bieliavsky 2013]:
• Quantization unitary from \( L^2(G) \) to \( L^2(\mathcal{H}_\pm) \)
• It extends as a continuous map from a larger set of functions to \( B(\mathcal{H}_\pm) \)
  (Calderon-Vaillancourt type estimate)
• It allows to deform any \( C^* \)-algebra endowed with a continuous action of \( G \) (i.e.
  we generalize Rieffel construction for a non-Abelian group)

Theorem [Neshveyev-Tuset 2014]:
The quadratic form on \( \mathcal{D}(G \times G) \) defined by
\[
F^*_\pm[\varphi_1, \varphi_2] := \int K^G_{\pm}(e, g_1, g_2) \bar{\varphi}_1 * \varphi_2(g_1, g_2) \, dg_1dg_2
\]
extends to a unitary operator on \( L^2(G \times G) \) and its adjoint
defines a dual 2-cocycle on \( G \)

→ First example of a unitary dual 2-cocycle on a non-Abelian group!
Variation 1: geometric structures

- If \( \text{Dim}(G) \geq 4 \), there is two classes of non equivalent symplectic symmetric space structures on an elementary Kahlerian Lie group \( G \) (Voglaire)

→ There is no (yet) quantization behind but a direct star-product approach (what does change is the covariance group \( \tilde{G} \))

Theorem [G-Jondreville]:
The associated dual 2-cocycle is unitary on \( L^2(G \times G) \)
• Projective representations

$\rightarrow$ Group cohomology in degree 2 is nontrivial if $G$ is not elementary (i.e. if there is at least two factors in the Pyatetskii-Shapiro decomposition of $G$)

$\rightarrow$ There exists non trivial projective representations if $G$ is not elementary

**Theorem [Bieliaevsky-G-De Goursac]:**
The associated dual 2-cocycle is unitary on $L^2(G \times G)$
**Clopen Question:** Are the different negatively curved quantum Kählerian Lie groups constructed so far isomorphic?

- **Representation theory:** NO (conjecture supported by my feeling)
- **Symplectic symmetric space structures:** NO (conjecture supported by the fact that covariance groups are different)
- **Projective representations:** NO (proof from Duflo-Moore theory)
Variation 2: semi-simple covariance group

- Up to now, all the variations lead to solvable covariance groups ($\tilde{G}$)

For elementary Kahlerians Lie groups, there is ways to modify the star-product (not yet the quantization map) to get for covariance group $\tilde{G} = SU(1,n)$

- But the associated dual 2-cocycle is no longer unitary, probably (?) invertible
Variation 3: no more geometry

Motivation: The BCH quantization of an exponential Lie group

- Let $G$ be an exponential Lie group with $\mathfrak{g}$ its Lie algebra
- Assume $G$ possesses a coadjoint orbit $\mathcal{O} \subset \mathfrak{g}^*$ on which $G$ acts simply transitively
- Let $U_\mathcal{O}$ the KKS representation of $G$
- For $f \in C_c^\infty(\mathcal{O}) \subset C_c^\infty(\mathfrak{g}^*)$ define
  \[
  \Omega_\mathcal{O}(f) := \int_{\mathfrak{g}} \mathcal{F}(f)(X) U_\mathcal{O}(\exp\{X\}) \, dX
  \]
- After identification $G \simeq \mathcal{O}$, get a $G$-covariant quantization on $G$
• These assumptions are satisfied for $G = \mathbb{R} \times \mathbb{R}$. In this case, two possible orbits $\Pi_{\pm} := \{(x, y) \in g^* : \pm y > 0\}$

• The quantization map reads

$$\Omega_{\pm}(f) = \int_G f(g) U_{\pm}(g) \Sigma U_{\pm}(g) dg,$$

• where, realizing $\mathcal{H}_{\pm}$ as $L^2(\mathbb{R})$, we have

$$\Sigma \varphi(t) = |\gamma'(t)| \varphi(\sigma(t)),$$

• $\sigma$ is the involutive diffeomorphism of $\mathbb{R}$ given by

$$\sigma = \text{Id} - \gamma : \mathbb{R} \to \mathbb{R}$$

and $\gamma$ is the inverse diffeomorphism of $\log \circ \lambda : \mathbb{R} \to \mathbb{R}$ where

$$\lambda : \mathbb{R} \to \mathbb{R}^*_+, \quad t \mapsto t(1 - e^{-t})^{-1}$$
Generalization: no more geometry

- $G = \mathbb{R} \times \mathbb{R}, U_{\pm}$

- Let $\sigma \in \text{Diff}(\mathbb{R})$ be an involution such that

  (1) $\gamma := \sigma - \text{Id} \in \text{Diff}(\mathbb{R})$

  (2) $\phi := [a \mapsto e^a - e^{\sigma(a)}] \in \text{Diff}(\mathbb{R})$

Set $\Sigma \varphi(a) := |\gamma'(a)|^{1/2} |\phi'(a)|^{1/2} \varphi(\sigma(a))$
Theorem [G-Jondreville]:

(i) The \( \mathbb{R} \times \mathbb{R} \)-covariant quantization map on \( \mathbb{R} \times \mathbb{R} \) associated with \((U_\pm, \Sigma)\) defines a unitary operator from \( L^2(G) \) to \( \text{HS}(L^2(\mathbb{R})) \)

(ii) Let \( \kappa : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) be the unique solution of

\[
\sigma(\sigma(\kappa(a_1, a_2, a_3) - a_1) + a_1 - a_2) + a_2 - a_3) + a_3 = \kappa(a_1, a_2, a_3)
\]

Then, the associated 2-points kernel reads (\( \kappa := \kappa(0, a_1, a_2) \))

\[
K_\sigma(a_1, t_1; a_2, t_2) = \sqrt{\gamma' \phi' \left( \kappa \right) \gamma' \phi' \left( \sigma(\kappa) - a_1 \right) \gamma' \phi' \left( \sigma(\kappa - a_2) \right)} \left| 1 - \sigma'(\kappa) \sigma'(\sigma(\kappa) - a_1) \sigma'(\sigma(\kappa - a_2)) \right| \\
\times \exp\{i\phi(a_1 - \sigma(\kappa))t_1 + i\phi(a_2 - \kappa)t_2 \}
\]
(iii) The 2-cocycle

\[
F_\sigma := \int_{G^2} K_\sigma(e; g, g') \lambda_{g^{-1}} \otimes \lambda_{g'^{-1}} \, dg \, dg'
\]

is a unitary element of \( W^*((G \times G) \) if and only if \( \sigma = -\text{Id} \)

- **Unitarity** for the 2-cocycle selects the (unique) symplectic symmetric space structure of the affine group of the real line

- **Need framework without unitarity** for the dual 2-cocycle, instead invertibility (on suitable domains)
Variation 4: base field

* $k$ Non Archimedean local field (of characteristic $\neq 2$ and which is not an extension of $\mathbb{Q}_2$)

* $\mathcal{O}_k$ ring of integers, $\varpi$ a generator of its unique maximal ideal, $\Psi$ non-trivial additive character of $k$ constant on $\mathcal{O}_k$

* $\tilde{G}_n = (1 + \varpi^n \mathcal{O}_k) \rtimes k$, $n = 1, 2, \ldots$

* $G_n = \tilde{G}_n/\varpi^{-n} \mathcal{O}_k \cong \mathcal{O}_k \ltimes \alpha_n \mathcal{O}_k$ (countable family of non-isomorphic groups which are not discrete, nor compact but unimodular)

* Representations of $\tilde{G}_n$ on $L^2(1 + \varpi^n \mathcal{O}_k)$

$$U(a, t)\varphi(a_0) = \Psi(aa_0^{-1}t)\varphi(a^{-1}a_0)$$

(Mackey: all the infinite dimensional reps are of this form)

* $\Sigma \varphi(a_0) = \varphi(a_0^{-1})$ (bounded and self-adjoint)
→ Trikernel given by

\[ K(a_1, [t_1]; a_2, [t_2]; a_3, [t_3]) = \psi \left( \left( \frac{a_1}{a_2} - \frac{a_2}{a_1} \right) t_2 + \left( \frac{a_2}{a_3} - \frac{a_3}{a_2} \right) t_1 + \left( \frac{a_3}{a_1} - \frac{a_1}{a_3} \right) t_2 \right) \]

**Theorem [G-Jondreville]:**

- The quantization is unitary and extends as a continuous map from a larger space of functions to \( B(L^2(1 + \varpi^n O_k)) \) (non-Abelain and p-adic Calderon-Vaillancourt) and allows to deform any \( C^* \)-algebra endowed with a continuous action of \( G \) and ....

- The associated dual 2-cocycle is unitary on \( L^2(G_n \times G_n) \)
Locally compact quantum groups

B) Manageable multiplicative unitary (Woronowicz 1995)

Definition: A multiplicative unitary on $\mathcal{H}$ is a unitary $W$ on $\mathcal{H}\bar{\otimes}\mathcal{H}$ satisfying the pentagonale equation

$$W_{23}W_{12} = W_{12}W_{13}W_{23}$$

$W$ is manageable is there exists a densely defined positive and self-adjoint operator $Q$ with densely defined inverse and a unitary operator $\tilde{W}$ on $\mathcal{H}\bar{\otimes}\mathcal{H}$ such that

$$W^* Q \otimes Q W = Q \otimes Q$$

and such that for all $\varphi_1, \varphi_3 \in \mathcal{H}$ and all $\varphi_2 \in \text{Dom}(Q)$, $\varphi_4 \in \text{Dom}(Q^{-1})$

$$\langle \varphi_1 \otimes \varphi_2, W\varphi_3 \otimes \varphi_4 \rangle = \langle \bar{\varphi}_3 \otimes Q\varphi_2, \tilde{W}\bar{\varphi}_4 \otimes Q^{-1}\varphi_4 \rangle$$
Less well established theory

→ Generalize Baaj-Skandalis theory

→ Contains all the locally compact quantum group in the von Neumann algebraic setting

→ Topological theory as it should be

Still possesses a pathology

→ Very few examples
Multiplicative unitary from an invertible dual 2-cocycle
(Joint work with Bieliavsky, Bonneau and D’Andrea)

• Starts with a densely defined inverse dual 2-cocycle on a group $G$ (coming from quantization or not)

$$F = \int_{G \times G} K(g_1, g_2) \lambda_{g_1} \otimes \lambda_{g_2} dg_1 dg_2 \quad \text{affiliated with} \quad W_\lambda^*(G \times G)$$

• Assume

$$F^{-1} = \int_{G \times G} \tilde{K}(g_1, g_2) \lambda_{g_1} \otimes \lambda_{g_2} dg_1 dg_2 \quad \text{affiliated with} \quad W_\lambda^*(G \times G)$$

and that $\tilde{F}$ is still a dual 2-cocycle, where

$$\tilde{F} := \int_{G \times G} \tilde{K}(g_1^{-1}, g_2^{-1}) \lambda_{g_1} \otimes \lambda_{g_2} dg_1 dg_2 \quad \text{affiliated with} \quad W_\lambda^*(G \times G)$$

• Set

$$\tilde{F}_\rho := \int_{G \times G} \tilde{K}(g_1, g_2) \rho_{g_1} \otimes \rho_{g_2} dg_1 dg_2 \quad \text{affiliated with} \quad W_\rho^*(G \times G)$$
• Assume the associative doubly deformed product $\varphi \star_{\lambda, \rho}$ well defined

$$\varphi_1 \star_{\lambda, \rho} \varphi_2 := \mu \circ F \circ \tilde{F}_\rho(\varphi_1 \otimes \varphi_2)$$

• Natural candidate for a manageable multiplicative unitary

$$W_\star(\varphi_1 \otimes \varphi_2) = \Delta(\varphi_1) \star_{\lambda, \rho} (1 \otimes \varphi_2)$$

• Pentagonale equation is automatic!

• Unitarity on the completion of $\mathcal{D}(G)$ of

$$\langle \varphi_1, \varphi_2 \rangle := \int_G \overline{\varphi_1} \star_{\lambda, \rho} \varphi_2(g) \, d^\rho(g)$$

Provided it is a scalar product!

True when $\delta^\alpha_G \star_{\lambda, \rho} \delta^\beta_G = \delta^\alpha + \beta^\beta$

• Manageability has to be checked......