Morita equivalences of spectral triples

Jens Kaad

April 6, 2016

Spectral triples

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1. $a \cdot (i + D)^{-1} : H \to H$ *is compact;*
2. $\mathrm{Dom}(D) \subseteq H$ *is an invariant subspace for* $a : H \to H$ *and the commutator*
$$[D, a] : \mathrm{Dom}(D) \to H$$
is the restriction of a bounded operator $d(a) : H \to H$,

for all $a \in \mathscr{A}$.

Morita equivalence (algebraic)

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1. *$\mathscr{A}$ is $*$-isomorphic to $\mathrm{End}_{\mathscr{B}}(p\mathscr{B}^n)$;*
2. *$p\mathscr{B}^n$ is full in the sense that*

$$1_{\mathscr{B}} = \sum_{j=1}^{m} \langle \xi_j, \xi_j \rangle$$

for some $\xi_1, \ldots, \xi_m \in p\mathscr{B}^n$.

Morita equivalence (algebraic)

Theorem

Suppose that $\mathscr{A}$ and $\mathscr{B}$ are two Morita equivalent unital $*$-algebras. Then there is a bijective correspondence between the spectral triples over $\mathscr{A}$ and the spectral triples over $\mathscr{B}$ (up to bounded perturbations and unitary equivalence).

The unbounded Kasparov product (algebraic)

Proposition (Connes, Mathai, Rennie, Lord, Suijlekom, Varilly)

Let $p\mathscr{B}^n$ be a finitely generated projective module over $\mathscr{B}$ and let $(\mathscr{B}, H, D)$ be a spectral triple. Then

$$p\mathscr{B}^n \otimes_{\mathscr{B}} (\mathscr{B}, H, D) := (\mathscr{A}, pH^n, \overline{pDp})$$

is a spectral triple for $\mathscr{A} \cong \mathrm{End}_{\mathscr{B}}(p\mathscr{B}^n)$.

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2. *X is full in the sense that*

$$\langle X, X \rangle := \mathrm{span}_{\mathbb{C}} \{ \langle \xi, \eta \rangle \mid \xi, \eta \in X \}$$

is **dense** *in B.*

Morita equivalence (topological)

Theorem

Suppose that A and B are Morita equivalent C^*-algebras. Then the K-homology of A is isomorphic to the K-homology of B and the isomorphism is implemented by the **bounded** *Kasparov product by $[X] \in KK_0(A, B)$.*

Operator spaces

Definition

An **operator space** *is a vector space* $\mathcal{X}$ *equipped with a norm* $\| \cdot \|_{\mathcal{X}} : M_n(\mathcal{X}) \to [0, \infty)$ *for each* $n \in \mathbb{N}$ *such that*

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Theorem (Ruan)

Any operator space $\mathcal{X}$ is completely isometric to a closed subspace of $\mathscr{L}(H)$ for some Hilbert space H.

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3. *An injective completely contractive $*$-homomorphism*

$$i : \mathcal{A} \to A$$

with dense image in a (σ-unital) C^*-algebra A.

Operator $*$-algebras

Example

Let $\mathcal{A}$ be a $*$-algebra and let B be a (σ-unital) C^*-algebra. Suppose that we have

Then the (canonical matrix norms coming from the) algebra homomorphism

$$\mathcal{A} \to M_2(B) \qquad a \mapsto \begin{pmatrix} \pi(a) & 0 \\ \delta(a) & \pi(a) \end{pmatrix}$$

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2. *A completely contractive hermitian form*
$$\langle \cdot, \cdot \rangle : \mathcal{X} \times \mathcal{X} \to \mathcal{B}$$
3. *An injective homomorphism (also preserving the hermitian forms)*
$$i : \mathcal{X} \to X$$
with dense image in a (countably generated and non-degenerate) C^*-correspondence X from A to B.

The Haagerup tensor product

Proposition (K.)

Let $\mathcal{X}$ and $\mathcal{Y}$ be two differentiable correspondences from $\mathcal{A}$ to $\mathcal{B}$ and from $\mathcal{B}$ to $\mathcal{C}$, respectively. There exists a (balanced) tensor product

$$\mathcal{X}\widehat{\otimes}_{\mathcal{B}}\mathcal{Y}$$

which is a differentiable correspondence from $\mathcal{A}$ to $\mathcal{C}$ with C^*-completion

$$X\widehat{\otimes}_B Y$$

the interior tensor product (of the C^*-completions of $\mathcal{X}$ and $\mathcal{Y}$).

Duality

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2. $\langle U(\xi), \eta \rangle_Y \in \mathcal{B}$ *for all* $\xi \in \mathcal{X}$, $\eta \in \mathcal{Y}$;
3. *The pairing*

$$(\cdot, \cdot) : \mathcal{X} \times \mathcal{Y} \to \mathcal{B} \qquad (\xi, \eta) := \langle U(\xi), \eta \rangle_Y$$

is completely bounded.

Morita equivalence (geometric)

Definition

Two operator $*$-algebras $\mathcal{A}$ and $\mathcal{B}$ are **Morita equivalent** *when there exist a pair of* **compact** *differentiable correspondences $\mathcal{X}$ and $\mathcal{Y}$ such that*

$$\mathcal{X}\widehat{\otimes}_{\mathcal{B}}\mathcal{Y} \sim \mathcal{A} \qquad \text{and} \qquad \mathcal{Y}\widehat{\otimes}_{\mathcal{A}}\mathcal{X} \sim \mathcal{B}$$

where "$\sim$" means "in duality".

Morita equivalence (examples)

Example

Let $\mathcal{M}$ be a Riemannian manifold and let $\mathcal{M}'$ denote the same manifold but with a conformally equivalent metric. Then the operator $*$-algebras

$$C_0^1(\mathcal{M}) \qquad \text{and} \qquad C_0^1(\mathcal{M}')$$

are Morita equivalent.

Morita equivalence (examples)

Example

Let $\mathcal{M}$ be a Riemannian manifold equipped with a free and proper action ϕ of a discrete group G. Suppose that

Then the operator $*$-algebras

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Let $\mathcal{M}$ be a Riemannian manifold equipped with a free and proper action ϕ of a discrete group G. Suppose that

1. *The derivative of the action is bounded, thus*

$$\|d\phi_g\|_\infty := \sup_{x\in\mathcal{M}} \|(d\phi_g)(x)\| < \infty$$

for all $g \in G$;

Then the operator $*$-algebras

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2. $\sup_{g \in G} \|d\phi_g\|_\infty < \infty$.

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Let $\mathcal{A}$ be an operator $*$-algebra and let $\mathcal{L} \subseteq \mathcal{A}$ be a closed right ideal. Suppose that

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Let $\mathcal{A}$ be an operator $*$-algebra and let $\mathcal{L} \subseteq \mathcal{A}$ be a closed right ideal. Suppose that

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1. *The C^*-closure $L \subseteq A$ is countably generated as a Hilbert C^*-module over A;*
2. *The $*$-subalgebra*

$$L^* \cdot L = \mathrm{span}_{\mathbb{C}}\{\xi^* \cdot \eta \mid \xi, \eta \in L\} \subseteq A$$

   is dense.

Then the operator $*$-algebras

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are Morita equivalent.

Morita equivalence (geometric)

Theorem (K.)

Suppose that $\mathcal{A}$ and $\mathcal{B}$ are **Morita equivalent** *operator $*$-algebras. Then there is a* **bijective correspondence** *between the* **twisted** *spectral triples over $\mathcal{A}$ and the* **twisted** *spectral triples over $\mathcal{B}$ (up to twisted bounded perturbations and unitary equivalence).*
This bijective correspondence is implemented by the **unbounded Kasparov product** *as developed by Mesland, Lesch, K. and others.*