Morita equivalences of spectral triples

Jens Kaad

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Definition

A spectral triple \((\mathcal{A}, H, D)\) consists of

such that

for all \(a \in \mathcal{A}\).
A spectral triple \((\mathcal{A}, H, D)\) consists of

1. A \(*\)-algebra \(\mathcal{A}\) represented (non-degenerately) on a separable Hilbert space \(H\);

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A spectral triple \((\mathcal{A}, H, D)\) consists of

1. A \(*\)-algebra \(\mathcal{A}\) represented (non-degenerately) on a separable Hilbert space \(H\);
2. A selfadjoint unbounded operator \(D : \text{Dom}(D) \to H\), such that

for all \(a \in \mathcal{A}\).
A spectral triple \((\mathcal{A}, H, D)\) consists of

1. A \(*\)-algebra \(\mathcal{A}\) represented (non-degenerately) on a separable Hilbert space \(H\);
2. A selfadjoint unbounded operator \(D : \text{Dom}(D) \to H\), such that
3. \(a \cdot (i + D)^{-1} : H \to H\) is compact;

for all \(a \in \mathcal{A}\).
A **spectral triple** \((\mathcal{A}, H, D)\) consists of

1. A \(\ast\)-algebra \(\mathcal{A}\) represented (non-degenerately) on a separable Hilbert space \(H\);
2. A selfadjoint unbounded operator \(D : \text{Dom}(D) \to H\), such that
   1. \(a \cdot (i + D)^{-1} : H \to H\) is compact;
   2. \(\text{Dom}(D) \subseteq H\) is an invariant subspace for \(a : H \to H\) and the commutator \([D, a] : \text{Dom}(D) \to H\) is the restriction of a bounded operator \(d(a) : H \to H\), for all \(a \in \mathcal{A}\).
Morita equivalence (algebraic)

**Definition**

Two unital $\ast$-algebras $A$ and $B$ are **Morita equivalent** when there exists an orthogonal projection $p \in M_n(B)$ such that

1. $A$ is $\ast$-isomorphic to $\text{End}_B(pB^n)$; 
2. $pB^n$ is full in the sense that $1_B = m \sum_{j=1}^m \langle \xi_j, \xi_j \rangle$ for some $\xi_1, \ldots, \xi_m \in pB^n$. 

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$$1_\mathcal{B} = \sum_{j=1}^{m} \langle \xi_j, \xi_j \rangle$$

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Morita equivalence (algebraic)

Theorem

Suppose that $A$ and $B$ are two Morita equivalent unital ∗-algebras. Then there is a bijective correspondence between the spectral triples over $A$ and the spectral triples over $B$ (up to bounded perturbations and unitary equivalence).
The unbounded Kasparov product (algebraic)

Proposition (Connes, Mathai, Rennie, Lord, Suijlekom, Varilly)

Let $pB^n$ be a finitely generated projective module over $B$ and let $(B, H, D)$ be a spectral triple. Then

$$pB^n \otimes_B (B, H, D) := (A, pH^n, pDp)$$

is a spectral triple for $A \cong \text{End}_B(pB^n)$. 
Two (\(\sigma\)-unital) C*-algebras \(A\) and \(B\) are **Morita equivalent** when there exists a (countably generated, right) Hilbert C*-module \(X\) over \(B\) such that

\[A \cong_m K(X)\]

\(X\) is full in the sense that \(\langle X, X \rangle = \text{span} \{ \langle \xi, \eta \rangle | \xi, \eta \in X \}\) is dense in \(B\).
Definition

Two (σ-unital) C*-algebras $A$ and $B$ are Morita equivalent when there exists a (countably generated, right) Hilbert C*-module $X$ over $B$ such that

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Morita equivalence (topological)

Definition

Two (σ-unital) C*-algebras $A$ and $B$ are **Morita equivalent** when there exists a (countably generated, right) Hilbert C*-module $X$ over $B$ such that

1. $A$ is $*$-isomorphic to $\mathcal{K}(X)$;
2. $X$ is full in the sense that

$$\langle X, X \rangle := \operatorname{span}_\mathbb{C}\{\langle \xi, \eta \rangle \mid \xi, \eta \in X\}$$

is dense in $B$. 

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Morita equivalence (topological)

Theorem

Suppose that $A$ and $B$ are Morita equivalent $C^*$-algebras. Then the $K$-homology of $A$ is isomorphic to the $K$-homology of $B$ and the isomorphism is implemented by the bounded Kasparov product by $[X] \in KK_0(A, B)$. 
Definition

An operator space is a vector space $\mathcal{X}$ equipped with a norm $\| \cdot \|_{\mathcal{X}} : M_n(\mathcal{X}) \to [0, \infty)$ for each $n \in \mathbb{N}$ such that

1. $M_n(\mathcal{X})$ is complete;
2. $\|\xi \oplus \eta\|_{\mathcal{X}} = \max\{\|\xi\|_{\mathcal{X}}, \|\eta\|_{\mathcal{X}}\}$;
3. $\|v \cdot \xi \cdot w\|_{\mathcal{X}} \leq \|v\|_{C} \cdot \|\xi\|_{\mathcal{X}} \cdot \|w\|_{C}$.

Theorem (Ruan)

Any operator space $\mathcal{X}$ is completely isometric to a closed subspace of $L(H)$ for some Hilbert space $H$. 
An **operator space** is a vector space $\mathcal{X}$ equipped with a norm $\| \cdot \|_\mathcal{X} : M_n(\mathcal{X}) \to [0, \infty)$ for each $n \in \mathbb{N}$ such that

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Operator spaces

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**Theorem (Ruan)**

Any operator space $\mathcal{X}$ is completely isometric to a closed subspace of $\mathcal{L}(H)$ for some Hilbert space $H$.
An operator $\ast$-algebra is an operator space $A$ equipped with

1. A completely contractive product $m: A \times A \to A$,
2. A completely isometric involution $\ast: A \to A$,
3. An injective completely contractive $\ast$-homomorphism $i: A \to A$ with dense image in a (\(\sigma\)-unital) C$\ast$-algebra $A$. 
An operator $\ast$-algebra is an operator space $\mathcal{A}$ equipped with

1. A completely contractive product

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Definition

An **operator \( \ast \)-algebra** is an operator space \( \mathcal{A} \) equipped with

1. A *completely contractive product*

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\ast : \mathcal{A} \to \mathcal{A}
\]

3. An *injective completely contractive \( \ast \)-homomorphism*

\[
i : \mathcal{A} \to \mathcal{A}
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**with dense image in a (\( \sigma \)-unital) \( C^* \)-algebra \( \mathcal{A} \).**
Example

Let $A$ be a $\ast$-algebra and let $B$ be a ($\sigma$-unital) $C^*$-algebra. Suppose that we have

1. An injective $\ast$-homomorphism $\pi: A \to B$;
2. A closed $\ast$-derivation $\delta: A \to B$;

Then the (canonical matrix norms coming from the) algebra homomorphism

$$A \to M_2(B) \quad a \mapsto \begin{pmatrix} \pi(a) & 0 \\ \delta(a) & \pi(a) \end{pmatrix}$$

provides $A$ with an operator $\ast$-algebra structure.
Example

Let \( \mathcal{A} \) be a \(*\)-algebra and let \( \mathcal{B} \) be a (\( \sigma \)-unital) \( C^* \)-algebra. Suppose that we have

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Then the (canonical matrix norms coming from the) algebra homomorphism

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provides \( \mathcal{A} \) with an operator \(*\)-algebra structure.
Example

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**Definition**

*Let $A$ and $B$ be operator $\ast$-algebras. A **differentiable correspondence** is an operator space $\mathcal{X}$ equipped with*
Definition

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1. **Completely contractive module actions**

   $$\mathcal{A} \times \mathcal{X} \rightarrow \mathcal{X} \quad \text{and} \quad \mathcal{X} \times \mathcal{B} \rightarrow \mathcal{X}$$
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2. A **completely contractive hermitian form**

   $$\langle \cdot, \cdot \rangle : \mathcal{X} \times \mathcal{X} \to \mathcal{B}$$
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2. **A completely contractive hermitian form**

   \[ \langle \cdot, \cdot \rangle : \mathcal{X} \times \mathcal{X} \to B \]

3. **An injective homomorphism (also preserving the hermitian forms)**

   \[ i : \mathcal{X} \to X \]

with dense image in a (countably generated and non-degenerate) $C^*$-correspondence $X$ from $A$ to $B$. 
The Haagerup tensor product

Proposition (K.)

Let $\mathcal{X}$ and $\mathcal{Y}$ be two differentiable correspondences from $A$ to $B$ and from $B$ to $C$, respectively. There exists a (balanced) tensor product

$$\mathcal{X} \hat{\otimes}_B \mathcal{Y}$$

which is a differentiable correspondence from $A$ to $C$ with $C^*$-completion

$$\mathcal{X} \hat{\otimes}_B \mathcal{Y}$$

the interior tensor product (of the $C^*$-completions of $\mathcal{X}$ and $\mathcal{Y}$).
Duality

**Definition**

Two differentiable correspondences $\mathcal{X}$ and $\mathcal{Y}$ (both from $A$ to $B$) are **in duality** when there exists a unitary operator $U : \mathcal{X} \to \mathcal{Y}$ such that

1. $U \pi_{\mathcal{X}}(a) U^* = \pi_{\mathcal{Y}}(a)$ for all $a \in A$;
2. $\langle U(\xi), \eta \rangle_{\mathcal{Y}} \in B$ for all $\xi \in \mathcal{X}$, $\eta \in \mathcal{Y}$;
3. The pairing $(\cdot, \cdot) : \mathcal{X} \times \mathcal{Y} \to B(\xi, \eta) := \langle U(\xi), \eta \rangle_{\mathcal{Y}}$ is completely bounded.
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Duality

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Two differentiable correspondences $\mathcal{X}$ and $\mathcal{Y}$ (both from $A$ to $B$) are **in duality** when there exists a unitary operator $U : X \to Y$ such that

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3. The pairing
   \[
   (\cdot, \cdot) : \mathcal{X} \times \mathcal{Y} \to B \quad (\xi, \eta) := \langle U(\xi), \eta \rangle_Y
   \]
   is completely bounded.
Morita equivalence (geometric)

Definition

Two operator $\ast$-algebras $A$ and $B$ are **Morita equivalent** when there exist a pair of **compact** differentiable correspondences $\mathcal{X}$ and $\mathcal{Y}$ such that

$$\mathcal{X} \hat{\otimes}_B \mathcal{Y} \sim A \quad \text{and} \quad \mathcal{Y} \hat{\otimes}_A \mathcal{X} \sim B$$

where “$\sim$” means “in duality”.

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Example

Let $\mathcal{M}$ be a Riemannian manifold and let $\mathcal{M}'$ denote the same manifold but with a conformally equivalent metric. Then the operator $\ast$-algebras

$$C^1_0(\mathcal{M}) \quad \text{and} \quad C^1_0(\mathcal{M}')$$

are Morita equivalent.
Example

Let $\mathcal{M}$ be a Riemannian manifold equipped with a free and proper action $\phi$ of a discrete group $G$. Suppose that

$$\|d\phi_g\|_\infty := \sup_{x \in \mathcal{M}} \|(d\phi_g)(x)\| < \infty$$

for all $g \in G$;

$$\sup_{g \in G} \|d\phi_g\|_\infty < \infty.$$

Then the operator $\ast$-algebras

$$C^1_0(\mathcal{M}) \rtimes G \quad \text{and} \quad C^1_0(\mathcal{M}/G)$$

are Morita equivalent.
Morita equivalence (examples)

Example

Let $\mathcal{M}$ be a Riemannian manifold equipped with a free and proper action $\phi$ of a discrete group $G$. Suppose that

1. The derivative of the action is bounded, thus

$$\|d\phi_g\|_\infty := \sup_{x \in \mathcal{M}} \|(d\phi_g)(x)\| < \infty$$

for all $g \in G$;

Then the operator $\ast$-algebras

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Example

Let $\mathcal{M}$ be a Riemannian manifold equipped with a free and proper action $\phi$ of a discrete group $G$. Suppose that

1. The derivative of the action is bounded, thus

$$\|d\phi_g\|_{\infty} := \sup_{x \in \mathcal{M}} \|(d\phi_g)(x)\| < \infty$$

for all $g \in G$;

2. $\sup_{g \in G} \|d\phi_g\|_{\infty} < \infty$.

Then the operator $\ast$-algebras

$$C_0^1(\mathcal{M}) \rtimes G$$

and

$$C_0^1(\mathcal{M}/G)$$

are Morita equivalent.
**Example**

Let $\mathcal{A}$ be an operator $\ast$-algebra and let $\mathcal{L} \subseteq \mathcal{A}$ be a closed right ideal. Suppose that

1. The $C^\ast$-closure $\mathcal{L} \subseteq \mathcal{A}$ is countably generated as a Hilbert $C^\ast$-module over $\mathcal{A}$;
2. The $\ast$-subalgebra $\mathcal{L}^\ast \cdot \mathcal{L} = \text{span} \{ \xi^\ast \cdot \eta | \xi, \eta \in \mathcal{L} \} \subseteq \mathcal{A}$ is dense.

Then the operator $\ast$-algebras

$$\mathcal{L} \cap \mathcal{L}^*$$

and

$$\mathcal{A}$$

are Morita equivalent.

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**Morita equivalences of spectral triples**
Let $A$ be an operator $\ast$-algebra and let $\mathcal{L} \subseteq A$ be a closed right ideal. Suppose that

1. The $C^*$-closure $\mathcal{L} \subseteq A$ is countably generated as a Hilbert $C^*$-module over $A$;

Then the operator $\ast$-algebras

$$\mathcal{L} \cap \mathcal{L}^* \quad \text{and} \quad A$$

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Example

Let $A$ be an operator $\ast$-algebra and let $\mathcal{L} \subseteq A$ be a closed right ideal. Suppose that

1. The $C^*$-closure $\mathcal{L} \subseteq A$ is countably generated as a Hilbert $C^*$-module over $A$;
2. The $\ast$-subalgebra $\mathcal{L} \ast \mathcal{L} = \text{span}_{\mathbb{C}} \{ \xi^* \cdot \eta \mid \xi, \eta \in \mathcal{L} \} \subseteq A$

is dense.

Then the operator $\ast$-algebras

$\mathcal{L} \cap \mathcal{L}^* \quad \text{and} \quad A$

are Morita equivalent.
Theorem (K.)

Suppose that $A$ and $B$ are Morita equivalent operator $\ast$-algebras. Then there is a bijective correspondence between the twisted spectral triples over $A$ and the twisted spectral triples over $B$ (up to twisted bounded perturbations and unitary equivalence).

This bijective correspondence is implemented by the unbounded Kasparov product as developed by Mesland, Lesch, K. and others.