Wick rotation, fermion doubling

and Lorentz Symmetry for the Spectral Action

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In this seminar I will discuss some issues related to the Euclidean vs. Lorentzian symmetries, and briefly at the end connect with Clifford algebra.

The fact that the spectral action is defined in Euclidean space, while the real world had Lorentzian symmetries is usually delat with the prescription of Wick rotation, i.e. one rotates the time direction along the imaginary axis, thus changing the signature.

After showing in detail this procedure, and its nuances for the fermionic case, I will show that a proper implementation of it takes care of the proper definition of the Hilbert space of the theory, fully solving the fermion doubling problem.

At the end I will touch upon the possible physical consequences for particle physics.

A Noncommutative Geometry in the spectral approach is defined by the data given by the (extended) Spectral Triple

- An algebra $\boxed{\mathcal{A}}$ which describes the topology of spacetime. In the commutative case is the algebra of continuous functions

- A Hilbert space $\boxed{\mathcal{H}}$ on which the algebra act as operators, and which also describes the matter fields of the theory.

- A (generalized) Dirac Operator $\boxed{D_0}$ which gives the metric of the space, and other information about the fermions.

- Two operators $\boxed{\Gamma}$ (present only in the even case) and $\boxed{J}$. For physicists they are chirality and charge conjugation

The particle standard model is described by a NCG of this kind, an Almost Commutative Geometry

- The algebra is the product of functions on spacetime times a finite algebra whose unimodular elements give the gauge group: $\boxed{\mathcal{A} = C(M) \otimes \mathcal{A}_F}$ with $\boxed{\mathcal{A}_F = C(M) \otimes (\mathbb{C} \oplus \mathbb{H} \oplus \mathrm{Mat}_3(\mathbb{C}))}$

- Likewise the Hilbert space is the product of spacetime spinors times an internal space describing all 96 fermionic matter degrees of freedom: $\boxed{\mathcal{H} = \mathrm{Sp}(M) \otimes H_F}$

- The Dirac operator is the sum of the usual one and a term acting on the fermions, the latter brings the informations on the masses and mixing: $\boxed{D_0 = (\partial\!\!\!/ + \psi\!\!\!/) \otimes \mathbb{I} + \gamma_5 \otimes D_F}$

- Chirality and charge conjugations are the product of the usual ones times an internal component: $\boxed{\Gamma = \gamma_5 \otimes \gamma_f}$ and $\boxed{J = j \otimes J_F}$

The coupling with a background is done adding to $\boxed{D_0}$ a potential, i.e. a connection one-form, defined as $\boxed{D = D_0 + \not{A} = D_0 + \sum_i a_i[D_0, b_i] \quad \text{, with } a_i, b_i \in \mathcal{A}}$

The procedure gives automatically an extra field, the Higgs, as a part of the connection one-form, obtained by $\boxed{D_F}$ rater than $\boxed{\not{\partial}}$

For the bosonic part of the action, we take, the spirit of noncommutative geometry a regularized trace of the Dirac operator, the spectral action:

$$\boxed{S_B = \operatorname{Tr} \chi\left(\frac{D_A^2}{\Lambda^2}\right)}$$

where $\boxed{\Lambda}$ an energy cutoff scale, and $\boxed{\chi}$ is a cutoff function, for example the characteristic function of the unit interval. In this case the spectral action becomes the number of eigenvalues of $\boxed{D_A}$ smaller than the cutoff $\boxed{\Lambda}$

This approach is growing to the point in which it may start actually to give numbers which can be confronted with experiments. It points naturally to a generalization of the standard model based on the algebra $\boxed{\mathrm{Mat}_2(\mathbb{H}) \oplus \mathrm{Mat}_4(\mathbb{C})}$, giving the Pati-Salam gauge group $\boxed{SU(2)_L \otimes SU(2)_R \otimes SU(4)}$

This algebra makes the almost commutative geometry a Non-commutative Manifold

The model has had various refinement, until getting to its current version, which gave a prediction (Chanseddine, Connes, Marcolli) for the mass of the Higgs of $\boxed{170\ \mathrm{GeV}}$. A mathematical theory based on purely spectral reasoning, and spectral data, gives a number which is of the right order of magnitude

But the number is wrong. It should have been $\boxed{125\ \mathrm{GeV}}$

The model has to be improved. . .

Shortly after the measurement of the Higgs mass it was realised (Chamseddine Connes) that the presence of another boson, called $\boxed{\sigma}$ could save the day

This $\boxed{\sigma}$ has problems of its own, and these can in turn be solved by either considering (Devastato, FL, Martinetti) a larger "Grand" Algebra $\boxed{\mathrm{Mat}_4(\mathbb{H}) \oplus \mathrm{Mat}_8(\mathbb{C})}$, or by giving up (Chamseddine, Connes, Van Suijlekom) one of the condition for the almost commutative geometry to be a manifold

In all cases the spectral construction gives in the end an action with some fields, then one picks up his favorite quantum field theory book, and makes standard calculations to get numbers out

But, are we not forgetting something?

There are a few issues whose clarification could teach us something fundamental

A first issue is that the spectral action theory is necessarily written in a spacetime which is compact and Euclidean

Compactness is necessary because we cannot really deal with continuous spectra, and on the other side in field theory the imposition of an infrared cutoff is common. We can consider this a technical issue for the moment. At least for the remainder of this talk I will not worry about it.

The use of Euclidean actions if field theory is also common. What is usually said if that "in the end you Wick rotate to Lorentz signature".

This is not precisely true. Usually in quantum field theory one starts with a Lorentzian theory. In order to tame some infinities one then Wick rotates to Euclidean space. After this rotations some integrals become convergent, calculations can be made and, say, Green's function can be calculated. In some cases other regularizations work as well, and in principle they are just equivalent procedure which can work always, even if the technical difficulties can be very different

Then one Wick rotates back, i.e., undoes an operation. But in the spectral approach we cannot start unless we have an Euclidean theory. So we are not going back, we are going in unchartered territory

In the following I will discuss the transition form Euclidean to Lorentzian signature in detail, but before that let me mention a, seemingly unrelated, feature of the model.

Earlier I define the Hilbert space $\boxed{\mathcal{H} = \mathsf{Sp}(M) \otimes H_F}$, and I said that $\boxed{H_F}$ is a 96 dimensional space

The 96 comes form the fact that we have 8 particles: electron, neutrino, up and down quarks which come in three colours. Then there are the respective antiparticles, and the two chiralities, all of this is 3 generations: $\boxed{8 \times 2 \times 2 \times 3 = 96}$.

The total Hilbert space is overcounted since the degrees of freedom of the Dirac spinor already contains the four degrees of freedom of a particle, its antiparticle, and the two chiralities

This overcounting (FL, Mangano, Miele, Sparano) was originally called *fermion doubling* since it gives fermions with the “wrong” chirality, left on spacetime and righ in the internal space, or viceversa, but it is actually a quadruplication

Usually a Wick rotation is indicated as the transformation $\boxed{t \to \mathrm{i}t}$, even if a more correct procedure would be to rotate the viebein. Namely for each F, which depends on vierbeins

$$\boxed{\text{Wick:} \quad F\left[e^0_\mu, e^j_\mu\right] \longrightarrow \quad F\left[\mathrm{i}e^0_\mu, e^j_\mu\right], \quad j = 1,2,3.]}$$

The inverse (which is what usually people call Wick rotation) is

$$\boxed{\text{Wick}^* : \quad F\left[e^0_\mu, e^j_\mu\right] \longrightarrow \quad F\left[-\mathrm{i}e^0_\mu, e^j_\mu\right] \quad j = 1,2,3.}$$

For the bosonic part of the spectral action thing go relatively without problems, the prescription is clear and the action is rotated into a new one which makes the partition function convergent

$$\boxed{\text{Wick:} \quad S^{\mathrm{E}}_{\mathrm{bos}}\left[\text{fields}, \mathrm{g}^{\mathrm{E}}_{\mu\nu}\right] \longrightarrow S^{\mathrm{E}}_{\mathrm{bos}}\left[\text{fields}, -\mathrm{g}^{\mathrm{M}}_{\mu\nu}\right] \equiv -\mathrm{i}S^{\mathrm{M}}_{\mathrm{bos}}\left[\text{fields}, \mathrm{g}^{\mathrm{M}}_{\mu\nu}\right]}$$

The fermionic sector requires some extra considerations

The group $\boxed{\mathsf{Spin}(1,3)}$ is quite different from $\boxed{\mathsf{Spin}(4)}$, $\boxed{\gamma}$ matrices, generators, charge conjugation, change. Also the fermionic action changes, since the quadratic forms have to be invariant under the proper group transformations

$$\boxed{\bar{\psi}\,\gamma_{\mathsf{M}}^{A}\,e_{A}^{\mu}\left(\left[\nabla_{\mu}^{\mathsf{LC}}\right]^{\mathsf{M}}+\mathrm{i}A_{\mu}\right)\psi, \qquad \bar{\psi}\psi}$$

$\boxed{\bar{\psi} \equiv \psi^{\dagger}\gamma^{0}}$ and $\boxed{\nabla_{\mu}^{\mathsf{LC}}}$ the covariant derivative on the spinor bundle with the Levi-Civita spin-connection, which is different for Lorentzian and Euclidean

The corresponding terms with the required Spin(4) invariance are:

$$\boxed{\psi^{\dagger}\,\gamma_{\mathsf{E}}^{A}\,e_{A}^{\mu}\left[\nabla_{\mu}^{\mathsf{LC}}\right]^{\mathsf{E}}\psi, \qquad \psi^{\dagger}\psi}$$

The charge conjugations are:

$$\boxed{C_{\mathsf{M}}\psi = -\mathsf{i}\gamma^2_{\mathsf{M}}\psi^* \quad ; \; C_{\mathsf{E}}\psi = \mathsf{i}\gamma^0_{\mathsf{E}}\gamma^2_{\mathsf{E}} = \widehat{C}_{\mathsf{E}}\psi^*}$$

The Majorana mass term is the same in both cases:

$$\boxed{\underbrace{(C_{\mathsf{E}}\psi)^\dagger\,\psi}_{Spin(4)\text{ inv}} = (-i\gamma^0_{\mathsf{E}}\gamma^2_{\mathsf{E}}\psi^*)^\dagger\psi = \overline{(\gamma^2_{\mathsf{M}}\psi^*)}\psi = -\underbrace{i\,\overline{(C_{\mathsf{M}}\psi)}\psi}_{Spin(1,3)\text{ inv}}}$$

Also the spacetime grading is the same in the two cases

$$\boxed{\gamma^5 = \gamma^0_{\mathsf{E}}\gamma^1_{\mathsf{E}}\gamma^2_{\mathsf{E}}\gamma^3_{\mathsf{E}} = \mathsf{i}\gamma^0_{\mathsf{M}}\gamma^1_{\mathsf{M}}\gamma^2_{\mathsf{M}}\gamma^3_{\mathsf{M}}}$$

so that the definition of left and right spinor is the same

The difference between $\boxed{\psi^\dagger}$ which appers in the Euclidean, and the Lorentzian $\boxed{\bar{\psi}}$ is the presence of a $\boxed{\gamma^0}$ which must be inserted in the Lorentzian case

In NCG the fermionic spectral action is

$$\boxed{S_F = \frac{1}{2}\langle J\psi, D_A\psi\rangle}$$

Thanks to the extra degrees of freedom, the insertion of $\boxed{\gamma^0}$ by hand is not needed for this action, which therefore deals with slightly different structures.

The fermionic action is build in any case contracting the a conjugate spinor with an operator acting on a spinor. Let us look at the charge conjugation

The spacetime part of the Hilbert space splits into eigenspaces of chirality, each of which has two components, for particles and antiparticles

$$\boxed{\mathsf{Sp}(M) = H_{\mathcal{L}} \oplus H_{\mathcal{R}}}$$

with our conventions a the antiparticle of a left particle is right, and viceversa

At the same time the internal space has a similar decomposition given by the internal grading $\boxed{\gamma}$

$$\boxed{H_F = H_L \oplus H_R \oplus H_L^c \oplus H_R^c}$$

One problem with the quadruplication is the presence of "mirrors", states which have different chiralities. They have to be projected out, defining $\boxed{\mathcal{H}_+}$

$$\boxed{H_+ = (H_L)_{\mathcal{L}} \oplus (H_R)_{\mathcal{R}} \oplus (H_L^c)_{\mathcal{R}} \oplus (H_R^c)_{\mathcal{L}} = P_+ H, \quad P_+ \equiv \frac{\mathbb{I} + \Gamma}{2}}$$

This takes care of half of the extra degrees of freedom. The fermionic action is then defined as

$$\boxed{S_F = \frac{1}{2}\langle J\psi, D_A\psi\rangle \quad \psi \in H_+}$$

with $\boxed{J = C_{\mathsf{E}} \otimes J_F}$ and $\boxed{J_F = \begin{pmatrix} 0 & 0 & \mathbb{I} & 0 \\ 0 & 0 & 0 & \mathbb{I} \\ \mathbb{I} & 0 & 0 & 0 \\ 0 & \mathbb{I} & 0 & 0 \end{pmatrix} \circ cc.}$

The action reproduces correctly the Pfaffian i.e. functional integral over fermions, but this procedure only takes care of half of the extra degrees of freedom. In processes like scattering, after quatization, it is important to have the correct Hilbert space of incoming and outgoing particles.

In the bosonic spectral action the operator $\boxed{D}$ is present, not $\boxed{DP_+}$, which is not Hermitian and not a square root of the Laplacian.

One of the points of this seminar is that the remaining extra degrees of freedom are taken care by the Wick rotation. It is in fact necessary to first perform the Wick rotation in order to eliminate the charge conjugation doubling

A naive attempt to remove it from the action with the $\boxed{J}$ would break the Euclidean Spin(4) symmetry.

Only the combination of Wick rotation (and identification of states described below) and the projection renders the action viable for physical applications, and free of the fermion doubling

Let us see the procedure with some more detail

First we rotate the action as in the bosonic case:

$$\boxed{\text{Wick rotation:} \quad -S_F^{\mathsf{E}}\left[\text{spinors}, e_\mu^A\right] \longrightarrow \mathrm{i} S_F^{\mathsf{M}\,\text{doubled}}\left[\text{spinors}, e_\mu^A\right]}$$

We now have a Lorentz invariant fermionic action invariant under $\boxed{Spin(1,3)}$ but still exhibiting a doubling. The spinors are in H_+, which is not anymore a Hilbert space with respect to the $Spin(1,3)$ invariant inner product

The remaining doubling consists in presence of spinors from all four subspaces of $\boxed{H_+}$: $\boxed{\left(H_L^c\right)_{\mathcal{R}}, \left(H_R^c\right)_{\mathcal{L}}, \left(H_L\right)_{\mathcal{L}}, \left(H_R\right)_{\mathcal{R}}}$

The physical Lagrangian depends on spinors just from the last two

After the Wick rotation we should perform the following identification

$$\boxed{\begin{cases} \left(\psi_L^c\right)_{\mathcal{R}} \in \underbrace{(H_L^c)_{\mathcal{R}}}_{\in H_+} & \text{identified with} \quad C_{\mathsf{M}}\,(\psi_L)_{\mathcal{L}}\,, \quad (\psi_L)_{\mathcal{L}} \in \underbrace{(H_L)_{\mathcal{L}}}_{\in H_+} \\ \left(\psi_R^c\right)_{\mathcal{L}} \in \underbrace{(H_R^c)_{\mathcal{L}}}_{\in H_+} & \text{identified with} \quad C_{\mathsf{M}}\,(\psi_R)_{\mathcal{R}}\,, \quad (\psi_R)_{\mathcal{R}} \in \underbrace{(H_R)_{\mathcal{R}}}_{\in H_+} \end{cases} .}$$

This step leads to the same formula of Barrett, who started directly Lorentzian.

We can then apply the procedure to the spectral action:

First we restore Lorentz signature in the action

$$\boxed{\begin{aligned} -S_F^{\mathsf{E}} &\to -\int d^4x\sqrt{-g}^{\mathsf{M}} \begin{bmatrix} C_{\mathsf{E}}\left(\psi_L^c\right)_{\mathcal{R}} \\ C_{\mathsf{E}}\left(\psi_R^c\right)_{\mathcal{L}} \end{bmatrix}^{\dagger} \begin{bmatrix} \mathrm{i}\nabla\!\!\!/^{\mathsf{M}} & \mathrm{i}M_D \\ \mathrm{i}M_D^{\dagger} & \mathrm{i}\nabla\!\!\!/^{\mathsf{M}} \end{bmatrix} \begin{bmatrix} (\psi_L)_{\mathcal{L}} \\ (\psi_R)_{\mathcal{R}} \end{bmatrix} \\ &\quad -\frac{\mathrm{i}}{2}\int d^4x\sqrt{-g}^{\mathsf{M}} \left\{ \left[C_{\mathsf{E}}\left(\psi_R\right)_{\mathcal{R}}\right]^{\dagger} M_M \left(\psi_R\right)_{\mathcal{R}} + \left[C_{\mathsf{E}}\left(\psi_R^c\right)_{\mathcal{L}}\right]^{\dagger} M_M^{\dagger} \left(\psi_R^c\right)_{\mathcal{L}} \right\} \end{aligned}}$$

This action is Lorentz invariant under. No modification of the inner product, like the insertion of $\boxed{\gamma^0}$, is needed.

Since $\boxed{C_{\mathsf{E}} = \mathrm{i}\gamma^0 C_{\mathsf{M}}}$ we have the manifestly Lorentz invariant action:

$$\boxed{\begin{aligned} S_F^{\mathsf{M}} &= \int d^4x\sqrt{-g}^{\mathsf{M}} \Big\{ \overline{(\psi_{\mathcal{L}})}\, \mathrm{i}\nabla\!\!\!/^{\mathsf{M}} \psi_{\mathcal{L}} + \overline{(\psi_{\mathcal{R}})}\, \mathrm{i}\nabla\!\!\!/^{\mathsf{M}} \psi_{\mathcal{R}} \\ &\quad - \left[\overline{(\psi_{\mathcal{L}})}\, H\, \psi_{\mathcal{R}} + \tfrac{1}{2}\overline{\left[C_{\mathsf{M}}\left(\psi_{\mathcal{R}}\right)\right]}\, \omega\, \psi_{\mathcal{R}} + \mathrm{c.c.} \right] \Big\} \end{aligned}}$$

We still have extra degrees of freedom since each quantity which carries the index $\boxed{\text{“}c\text{”}}$ is independent from the one which does not.

It is remarkable that the path integral is not sensitive to the charge conjugation doubling, in particular the Pfaffian is reproduced correctly since

$$\boxed{\int [d\bar{\psi}][d\psi] e^{\mathrm{i} \int d^4x \, \bar{\psi} \, \mathrm{i} \partial\!\!\!/^{\mathrm{M}} \psi} = \int [d\bar{\xi}][d\psi] e^{\mathrm{i} \int d^4x \, \bar{\xi} \, \mathrm{i} \partial\!\!\!/^{\mathrm{M}} \psi}.}$$

The correct identification of the Hilbert space is necessary. The Lorentzian theory has to be *quantized*, and the quantum Hilbert space of asymptotic states has to be constructed. Such a space is usually referred in physical literature as a “Fock space”.

The Hamiltonian coming out of this action is not Hermitian in the Fock space. This is solved with the identification above. The rest is a straightforward excercise. In the end we obtain the correct Lorentzian signature action that you will find in textbooks.

What have we learned? I think the most intriguing element is that the Euclidean fermionic action, which uses in a crucial way the real structure of the spectral triple, and needs the fermionic quadruplication, is naturally rotated in the Lorentzian, with the elimination of the extra degrees of freedom.

There are various studies which connect spectral triples and Lorentz signatures Verch, Paschke, Sitarz, Eckstein, Franco, Besnard, Bizi, Van den Dungen, ... The considerations of this seminar suggest that a possible way to obtain Lorentzian spectral triple is a rigorous treatment of Wick rotations.

I also remind in passing that the full Hilbert space gives rise to a larger, “grand” symmetry Devastato, FL, Martinetti, which enlarges the Pati-Salam one, but breaks Lorentz symmetry. The reduction in that case also connects different spaces.

But there is more to symmetries . . .

Dabrowski and D'Andrea have given an algebraic characterization of spinors, which can be extended to the the finite (but possibly noncommutative) dimensional case

This work is another step in the construction of a dictionary translating all geometric concepts in purely algebraic terms, to generalize them to the noncommutative case

$\boxed{1}$-forms are built with the help of a Dirac operator elements of the kind $\boxed{\Omega^1_D = \text{Span}\{a, [D, b] : a, b \in \mathcal{A}}$, the Higgs field comes from the finite part $\boxed{D_F}$ of the Dirac operator

$\boxed{\mathcal{A}, \Omega^1_D}$ and in the even case $\boxed{\gamma}$ generate the Clifford algebra $\boxed{C\ell_D(\mathcal{A})}$

If the algebra is real and the first order is satisfied($\boxed{\exists J}$ such that $\boxed{[JaJ^{-1},[D,b]]=0}$), then $\boxed{J\mathcal{A}J^{-1}\subseteq C\ell_D(\mathcal{A})'}$, the commutant of the Clifford algebras.

SInce in the ordinary case the inclusion is an equality, this prompted Dabrowski and D'Andrea use an algebraic characterization of Dirac spinors, extended in an obvious way to NC spaces, and to study conditons on the spectral action

In particular they gave the definition of a spinor: "Property M":

Elements of a Hilbert space are spinors iff the above $\boxed{\subseteq}$ is an equality

Such definition has interesting consequences for physics. Which I will briefly sketch.

In order to have the description of the standard model compatible the internal grading has to change. The new grading treats differently quarks and leptons

It is known that not all Hermitean matrices are viable internal Dirac operators $\boxed{D_F}$. The number of elements whoch are different from zero are finite, and are basically the ones of the standard model (with some little extras, but see the work of Boyle and Farnsworth)

The new grading allows all the terms of the standard model, but property M requires extra terms, which correspond to new physical particles and interactions

Some of these are "good", some are "bad", i.e. make the model inconsistent

The good thing is that the "bad terms" do not appear in the Wick rotated action! there is no time details, which will appear paper due to appear shortly