Generalized connections and Higgs fields on Lie algebroids

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Motivations

- Discovery of Higgs particle in 2012.
  ➔ need for a mathematical validation of the Higgs sector in the SM.
  ❌ No clue from “traditional” schemes and tools.

- **NCG:** Higgs field is part of a “generalized connection”.
  

  ✔ Models in NCG can reproduce the Standard Model up to the excitement connected to the diphoton resonance at 750 GeV “seen” by ATLAS and CMS!
  ❌ Mathematical structures difficult to master by particle physicists.

- **Transitive Lie algebroids:**
  ➔ generalized connections, gauge symmetries, Yang-Mills-Higgs models...
  ➔ Direct filiation from Dubois-Violette, Kerner, and Madore (1990).

  ✔ Mathematics close to “usual” mathematics of Yang-Mills theories.
  ❌ No realistic theory yet.
How to construct a gauge field theory?

The basic ingredients are:

1. A space of local symmetries (space-time dependence): ➞ a **gauge group**.
2. An implementation of the symmetry on matter fields: ➞ a **representation theory**.
3. A notion of derivation: ➞ some **differential structures**.
4. A (gauge compatible) replacement of ordinary derivations: ➞ a **covariant derivative**.
5. A way to write a gauge invariant Lagrangian density: ➞ **action functional**.

At least three mathematical schemes to construct gauge field theories:

- Ordinary differential geometry of principal fiber bundles.
- Noncommutative geometry.
- Transitive Lie algebroids (to be explained in this talk).
Ordinary differential geometry

Given a $G$-principal fiber bundle $\mathcal{P}$ over $\mathcal{M}$, the ingredients are

**gauge group:** $\mathcal{G}(\mathcal{P})$ is the group of vertical automorphisms of $\mathcal{P}$.

**representation theory:** sections of associated vector bundles.

$\quad\mapsto$ Natural action of $\mathcal{G}(\mathcal{P})$.

**differential structures:** (ordinary) de Rham differential calculus.

**covariant derivative:** connection 1-form $\omega$ on $\mathcal{P}$.

$\quad\mapsto$ covariant derivative on sections of any associated vector bundles.

**action functional:** integration on the base manifold $\mathcal{M}$, Killing form on the Lie algebra $\mathfrak{g}$ of $G$, Hodge star operator, curvature of $\omega$. 
Noncommutative geometry

Given an associative algebra $A$, the ingredients are

**representation theory**: a right module $M$ over $A$.

**gauge group**: $\text{Aut}(M)$, the group of automorphisms of the right module.

**differential structures**: any differential calculus defined on top of $A$.

$\Rightarrow$ many choices: spectral triples, derivations, twisted derivations...

**covariant derivatives**: noncommutative connections on $M$,
(need a differential calculus).

**action functional**: depends on the differential calculus.

- spectral triples: spectral action...
- derivation-based differential calculus: noncommutative integration, Hodge star operator, curvature of the connection...
Outline

1. Lie algebroids and their representations
2. Differential structures
3. Connections and covariant derivatives
4. The gauge group
5. Structures to construct an action functional
6. Gauge theories


Lie algebroids and their representations

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Generalities on Lie algebroids

\( \mathcal{M} \) a smooth manifold, \( \Gamma(T\mathcal{M}) \) the Lie algebra and \( C^\infty(\mathcal{M}) \)-module of vector fields.

Definition in terms of algebras and modules (as in NCG).

**Definition (Lie algebroids)**

A Lie algebroid \( A \) is a finite projective module over \( C^\infty(\mathcal{M}) \) equipped with a Lie bracket \([\cdot, \cdot]\) and a \( C^\infty(\mathcal{M}) \)-linear Lie morphism \( \rho : A \to \Gamma(T\mathcal{M}) \) such that

\[
[\mathcal{X}, f\mathcal{Y}] = f[\mathcal{X}, \mathcal{Y}] + (\rho(\mathcal{X}) \cdot f)\mathcal{Y}
\]

for any \( \mathcal{X}, \mathcal{Y} \in A \) and \( f \in C^\infty(\mathcal{M}) \).

\( \rho \) is the anchor of \( A \).

The usual definition uses the vector bundle \( A \) such that \( A = \Gamma(A) \).

\( A \) is viewed as a generalization of the tangent bundle.

We will never use this point of view.

Natural notion of morphisms of Lie algebroids...
Transitive Lie algebroids

A Lie algebroid $A \xrightarrow{\rho} \Gamma(TM)$ is **transitive** if $\rho$ is surjective.

**Proposition (The kernel of a transitive Lie algebroid)**

Let $A$ be a transitive Lie algebroid.

- $L = \text{Ker} \rho$ is a Lie algebroid with null anchor on $M$.
  - $L$ is called the **kernel** of $A$.
- The vector bundle $\mathcal{L}$ such that $L = \Gamma(\mathcal{L})$ is a locally trivial bundle in Lie algebras.
  - This gives the Lie structure on $L$.

One has the short exact sequence of Lie algebras and $C^\infty(M)$-modules

$$0 \rightarrow L \xrightarrow{t} A \xrightarrow{\rho} \Gamma(TM) \rightarrow 0$$

This short exact sequence is the key structure of what follows...

Very trivial example: $A = \Gamma(TM) \rightarrow L = 0$. 
Example 1: Derivations of a vector bundle

\( \mathcal{E} \) a vector bundle over \( \mathcal{M} \).
\( \text{Diff}^1(\mathcal{E}) \) the space of first order differential operators on \( \mathcal{E} \).
Symbol map:
\[
\sigma : \text{Diff}^1(\mathcal{E}) \to \text{Hom}_{C^\infty(\mathcal{M})}(T^*\mathcal{M}, \text{End}(\mathcal{E})) \cong \Gamma(TM \otimes \text{End}(\mathcal{E})) \subset \Gamma(TM)
\]
\[
\mathfrak{D}(\mathcal{E}) = \sigma^{-1}(\Gamma(TM))
\]
is the transitive Lie algebroid of derivations of \( \mathcal{E} \):
\[
0 \to \mathcal{A}(\mathcal{E}) \overset{\iota}{\longrightarrow} \mathfrak{D}(\mathcal{E}) \overset{\sigma}{\longrightarrow} \Gamma(TM) \longrightarrow 0
\]
with \( \mathcal{A}(\mathcal{E}) = \Gamma(\text{End}(\mathcal{E})) \) (0\textsuperscript{th}-order diff. op.).
\( \mathcal{A}(\mathcal{E}) \) is an associative algebra (Lie structure is the commutator).
**Representation of a Lie algebroid**

$A \overset{\rho}{\to} \Gamma(TM)$ a Lie algebroid and $\mathcal{E} \to \mathcal{M}$ a vector bundle.

**Definition (Representation of a Lie algebroid)**

A representation of $A$ on $\mathcal{E}$ is a morphism of Lie algebroids $\phi : A \to \mathfrak{D}(\mathcal{E})$.

When $A$ is transitive, one has the commutative diagram of exact rows:

\[
\begin{array}{ccccccccc}
0 & \rightarrow & L & \overset{\iota}{\rightarrow} & A & \overset{\rho}{\rightarrow} & \Gamma(TM) & \rightarrow & 0 \\
& & \downarrow{\phi_L} & & \downarrow{\phi} & & \| & & \\
0 & \rightarrow & A(\mathcal{E}) & \overset{\iota}{\rightarrow} & \mathfrak{D}(\mathcal{E}) & \overset{\sigma}{\rightarrow} & \Gamma(TM) & \rightarrow & 0
\end{array}
\]

$\phi_L : L \to A(\mathcal{E})$ is a morphism of Lie algebras.
Example 2: Atiyah Lie algebroids

\( \mathcal{P} \xrightarrow{\pi} \mathcal{M} \) a \( G \)-principal fiber bundle, \( \mathfrak{g} \) the Lie algebra of \( G \).
\( R_g: \mathcal{P} \to \mathcal{P}, \ R_g(p) = p \cdot g \), the right action of \( G \) on \( \mathcal{P} \).

\[
\Gamma_G(T\mathcal{P}) = \{ \mathfrak{X} \in \Gamma(T\mathcal{P}) / R_g * \mathfrak{X} = \mathfrak{X} \text{ for all } g \in G \}
\]

\[
\Gamma_G(\mathcal{P}, \mathfrak{g}) = \{ \nu: \mathcal{P} \to \mathfrak{g} / \nu(p \cdot g) = \text{Ad}_{g^{-1}} \nu(p) \text{ for all } g \in G \}
\]

Both are Lie algebras and \( C^\infty(\mathcal{M}) \)-modules.

\[
\Gamma_G(T\mathcal{P}) = \pi_* \text{-projectable vector fields in } \Gamma(T\mathcal{P}) \xrightarrow{\pi_*} \Gamma_G(T\mathcal{P}) \rightarrow \Gamma(T\mathcal{M}).
\]

\( \iota: \Gamma_G(\mathcal{P}, \mathfrak{g}) \to \Gamma_G(T\mathcal{P}) \) defined by \( \iota(\nu)|_p = \nu(p)|^{\mathcal{P}}_p \),

(\( \mathfrak{g} \ni \nu \mapsto \nu^{\mathcal{P}} \) fundamental vector field on \( \mathcal{P} \)).

S.E.S. of Lie algebras and \( C^\infty(\mathcal{M}) \)-modules:

\[
0 \rightarrow \Gamma_G(\mathcal{P}, \mathfrak{g}) \xrightarrow{\iota} \Gamma_G(T\mathcal{P}) \xrightarrow{\pi_*} \Gamma(T\mathcal{M}) \rightarrow 0
\]

\( \Gamma_G(T\mathcal{P}) \) is the transitive **Atiyah Lie algebroid** associated to \( \mathcal{P} \)

The representations of \( \Gamma_G(T\mathcal{P}) \) are given by the associated vector bundles to \( \mathcal{P} \).
Example 3: Trivial Lie algebroids

Trivial Lie algebroid = Atiyah Lie algebroid of a trivial principal bundle $\mathcal{M} \times \mathbb{G}$.

Concrete description in terms of the bundle $T\mathcal{M} \oplus (\mathcal{M} \times \mathfrak{g})$:

- $C^\infty(\mathcal{M})$-module: $\text{TLA}(\mathcal{M}, \mathfrak{g}) \equiv A = \Gamma(T\mathcal{M} \oplus (\mathcal{M} \times \mathfrak{g}))$.
- Bracket: $[X \oplus \gamma, Y \oplus \eta] = [X, Y] \oplus (X \cdot \eta - Y \cdot \gamma + [\gamma, \eta])$
- Anchor: $\rho(X \oplus \gamma) = X$.
- Kernel: $L = \Gamma(\mathcal{M} \times \mathfrak{g})$ (section of a trivial bundle).

Proposition

*Every transitive Lie algebroid $A$ is locally of the form $\text{TLA}(\mathcal{U}, \mathfrak{g})$ for $\mathcal{U} \subset \mathcal{M}$ open subset.*

Trivialization of an Atiyah Lie algebroid $\Gamma_G(T\mathcal{P}) \leftrightarrow$ Trivialization of $\mathcal{P}$. 
Differential structures

1 Lie algebroids and their representations

2 Differential structures

3 Connections and covariant derivatives

4 The gauge group

5 Structures to construct an action functional

6 Gauge theories
A Lie algebroid, \( \phi : A \to \mathcal{D}(\mathcal{E}) \) a representation of \( A \) on \( \mathcal{E} \).

**Definition (Differential forms)**

For \( p \in \mathbb{N} \), let \( \Omega^p(A, \mathcal{E}) \) be the linear space of \( C^\infty(\mathcal{M}) \)-multilinear antisymmetric maps \( A^p \to \Gamma(\mathcal{E}) \).

For \( p = 0 \), let \( \Omega^0(A, \mathcal{E}) = \Gamma(\mathcal{E}) \).

\( \Omega^\bullet(A, \mathcal{E}) = \bigoplus_{p \geq 0} \Omega^p(A, \mathcal{E}) \) is equipped with the natural differential

\[
(\hat{d}_\phi \hat{\omega})(\mathcal{X}_1, \ldots, \mathcal{X}_{p+1}) = \sum_{i=1}^{p+1} (-1)^{i+1} \phi(\mathcal{X}_i) \cdot \hat{\omega}(\mathcal{X}_1, \ldots, \hat{\mathcal{X}}_i \ldots, \mathcal{X}_{p+1}) \]

\[
+ \sum_{1 \leq i < j \leq p+1} (-1)^{i+j} \hat{\omega}([\mathcal{X}_i, \mathcal{X}_j], \mathcal{X}_1, \ldots, \hat{\mathcal{X}}_i \ldots, \hat{\mathcal{X}}_j \ldots, \mathcal{X}_{p+1})
\]

\( \phi(\mathcal{X}) \cdot \varphi \) is the action of the first order diff. op. \( \phi(\mathcal{X}) \) on \( \varphi \in \Gamma(\mathcal{E}) \).

One has \( \hat{d}_\phi^2 = 0 \) (since \( \phi \) is a morphism of Lie algebras).
Differential structures

Differential forms: two examples

\[ \mathcal{E} = \mathcal{M} \times \mathbb{C} \hookrightarrow \Gamma(\mathcal{E}) = C^\infty(\mathcal{M}). \]

The anchor map is a representation of \( A \) on \( C^\infty(\mathcal{M}) \) via vector fields.

**Definition (Forms with values in \( C^\infty(\mathcal{M}) \))**

\((\Omega^\bullet(A), \tilde{d}_A)\) is the graded commutative differential algebra of forms on \( A \) with values in \( C^\infty(\mathcal{M}) \) associated to the anchor as a representation.

\[ 0 \rightarrow L \xrightarrow{\iota} A \xrightarrow{\rho} \Gamma(T\mathcal{M}) \rightarrow 0 \] a transitive Lie algebroid.

\( \mathcal{E} = \mathcal{L} \) the vector bundle such that \( L = \Gamma(\mathcal{L}) \).

For \( \mathcal{X} \in A \) and \( \ell \in L \), define \( \text{ad}_\mathcal{X}(\ell) \in L \) such that \( \iota(\text{ad}_\mathcal{X}(\ell)) = [\mathcal{X}, \iota(\ell)] \) (adjoint representation of \( A \) on \( \mathcal{L} \)).

**Definition (Forms with values in the kernel)**

\((\Omega^\bullet(A, L), \tilde{d})\) is the graded differential Lie algebra of forms on \( A \) with values in the kernel \( L \) associated to the adjoint representation.

This differential space is a graded Lie algebra and a graded differential module on the graded commutative differential algebra \( \Omega^\bullet(A) \).
Differential forms on trivial Lie algebroids

\( A = \text{TLA}(\mathcal{M}, \mathfrak{g}) \) a trivial Lie algebroid.

\( \Omega^\bullet(A) \) is the total complex of the bigraded commutative algebra \( \Omega^\bullet(\mathcal{M}) \otimes \wedge^* \mathfrak{g}^* \).

\( \hat{d}_A = d + s \) with

\[ d: \Omega^\bullet(\mathcal{M}) \otimes \wedge^* \mathfrak{g}^* \to \Omega^{\bullet+1}(\mathcal{M}) \otimes \wedge^* \mathfrak{g}^* \] \text{de Rham differential}

\[ s: \Omega^\bullet(\mathcal{M}) \otimes \wedge^* \mathfrak{g}^* \to \Omega^\bullet(\mathcal{M}) \otimes \wedge^{*+1} \mathfrak{g}^* \] \text{Chevalley-Eilenberg differential}

\( \Omega^\bullet(A, L) \) is the total complex of the bigraded Lie algebra \( \Omega^\bullet(\mathcal{M}) \otimes \wedge^* \mathfrak{g}^* \otimes \mathfrak{g} \).

\( \hat{d} = d + s' \) with

\[ s' \text{ the Chevalley-Eilenberg differential on } \wedge^* \mathfrak{g}^* \otimes \mathfrak{g} \text{ (for the ad rep.)}. \]

Compact notation \((\Omega_{\text{TLA}}^\bullet(\mathcal{M}, \mathfrak{g}), \hat{d}_{\text{TLA}})\).

This is the model for trivializations of forms on any transitive Lie algebroid.

⚠️ Mathematical structure similar to the one used in BRST differential algebras.

➡️ work in progress to understand possible relations...
Differential forms on Atiyah Lie algebroids

A the Atiyah Lie algebroid of the $G$-principal fiber bundle $\mathcal{P} \xrightarrow{\pi} \mathcal{M}$.

$(\Omega^\bullet_{\text{Lie}}(\mathcal{P}, \mathfrak{g}), \tilde{d})$ the complex of forms with values in the kernel.

$$\mathfrak{g}_{\text{equ}} = \{ \xi^\mathcal{P} \oplus \xi / \xi \in \mathfrak{g} \} \subset \text{TLA}(\mathcal{P}, \mathfrak{g}) = \Gamma(T\mathcal{P} \oplus (\mathcal{P} \times \mathfrak{g}))$$

is a Lie sub-algebra, which defines a Cartan operation on $(\Omega^\bullet_{\text{TLA}}(\mathcal{P}, \mathfrak{g}), \tilde{d}_{\text{TLA}})$.

$(\Omega^\bullet_{\text{TLA}}(\mathcal{P}, \mathfrak{g})_{\text{equ}}, \tilde{d}_{\text{TLA}})$ the differential graded subcomplex of basic elements.

**Theorem (S. Lazzarini, T.M.)**

If $G$ is connected and simply connected then

$(\Omega^\bullet_{\text{Lie}}(\mathcal{P}, \mathfrak{g}), \tilde{d})$ is isomorphic to $(\Omega^\bullet_{\text{TLA}}(\mathcal{P}, \mathfrak{g})_{\text{equ}}, \tilde{d}_{\text{TLA}})$

$$\Rightarrow \Omega^\bullet_{\text{Lie}}(\mathcal{P}, \mathfrak{g}) \subset \Omega^\bullet_{\text{TLA}}(\mathcal{P}, \mathfrak{g}) \cong \Omega^\bullet(\mathcal{P}) \otimes \bigwedge^\bullet \mathfrak{g}^* \otimes \mathfrak{g}.$$
The global picture so far

- Transitive Lie algebroids $= \text{s.e.s. of Lie algebras and } C^\infty(M)\text{-modules}

$$0 \rightarrow L \overset{\iota}{\rightarrow} A \overset{\rho}{\rightarrow} \Gamma(TM) \rightarrow 0$$

- Generalized forms: $(\Omega^\bullet(A, L), \hat{d})$, graded differential Lie algebra.
  $\Rightarrow$ “Contains” ordinary de Rham calc. on $M$ (basic elements for op. of $L$).

- Local description of transitive Lie algebroids and diff. calc. using TLA.
  $\Rightarrow (\Omega^\bullet(M) \otimes \wedge^\bullet g^* \otimes g, \hat{d} = d + s')$.
  $\Rightarrow$ useful for computations and definitions of structures...

- Representation theory on derivations of a vector bundle.

$$0 \rightarrow L \overset{\iota}{\rightarrow} A \overset{\rho}{\rightarrow} \Gamma(TM) \rightarrow 0$$

$$0 \rightarrow A(E) \overset{\iota}{\rightarrow} \mathcal{O}(E) \overset{\sigma}{\rightarrow} \Gamma(TM) \rightarrow 0$$

- Principal fiber bundle $\Rightarrow$ canonical Atiyah Lie algebroid.
Connections and covariant derivatives

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Ordinary connections

0 \rightarrow L \xrightarrow{\iota} A \xrightarrow{\rho} \Gamma(TM) \rightarrow 0 \text{ a transitive Lie algebroid.}

**Definition (Connection on a transitive Lie algebroid)**

A connection on $A$ is a splitting $\nabla : \Gamma(TM) \rightarrow A$ as $C^\infty(M)$-modules of the s.e.s.

\[
\begin{array}{c}
0 \\
\xrightarrow{\iota} \\
L \\
\xrightarrow{\rho} \\
A \\
\xrightarrow{\nabla} \\
\Gamma(TM) \\
\xrightarrow{\omega^\nabla} \\
0
\end{array}
\]

The curvature of $\nabla$ is defined as the obstruction to be a morphism of Lie algebras:

\[
R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X, Y]} \in \iota(L)
\]

$\nabla$ defines $\omega^\nabla : A \rightarrow L$ (s.e.s. properties) s.t. $\mathcal{X} = \nabla_X - \iota \circ \omega^\nabla(\mathcal{X})$, $\mathcal{X} \in A$, $X = \rho(\mathcal{X})$.

**Proposition**

One has $\omega^\nabla \in \Omega^1(A, L)$ and $\omega^\nabla \circ \iota(\ell) = -\ell$ for any $\ell \in L$ (normalization on $L$).

The 2-form $R^\nabla \in \Omega^2(A, L)$ defined by $R^\nabla(\mathcal{X}, \mathcal{Y}) = (\text{d}\omega^\nabla)(\mathcal{X}, \mathcal{Y}) + [\omega^\nabla(\mathcal{X}), \omega^\nabla(\mathcal{Y})]$ vanishes when $\mathcal{X} or \mathcal{Y}$ in $\iota(L)$, and one has $\iota \circ R^\nabla(\mathcal{X}, \mathcal{Y}) = R(X, Y)$.

$\omega^\nabla$ is the connection 1-form associated to $\nabla$. 


Ordinary connections on Atiyah Lie algebroid

\[ 0 \rightarrow \Gamma_G(\mathcal{P}, \mathfrak{g}) \overset{\iota}{\rightarrow} \Gamma_G(T\mathcal{P}) \overset{\pi_*}{\rightarrow} \Gamma(TM) \rightarrow 0 \]

**Proposition (Connections)**

*Ordinary connection on the Atiyah Lie algebroid = connection on \( \mathcal{P} \).
The notions of curvature coincide.*

This example explains the terminology “ordinary connection”.

**The geometric equivalence:**
A connection on \( \mathcal{P} \) defines the horizontal lift \( \Gamma(TM) \rightarrow \Gamma_G(T\mathcal{P}), X \mapsto X^h \).

**The algebraic equivalence:**
Suppose \( G \) is connected and simply connected.

\( \omega^\mathcal{P} \in \Omega^1(\mathcal{P}) \otimes \mathfrak{g} \) a connection 1-form on \( \mathcal{P} \).
\( \theta \in \mathfrak{g}^* \otimes \mathfrak{g} \) the Maurer-Cartan 1-form on \( G \).

\( \widehat{\omega}_{\text{equ}} = \omega^\mathcal{P} - \theta \in \Omega^1_{\text{TLA}}(\mathcal{P}, \mathfrak{g}) \subset \Omega^* (\mathcal{P}) \otimes \wedge^* \mathfrak{g}^* \otimes \mathfrak{g} \) is \( \mathfrak{g}_{\text{equ}} \)-basic.

It corresponds to the connection 1-form \( \omega^\nabla \in \Omega^1_{\text{Lie}}(\mathcal{P}, \mathfrak{g}) \) associated to \( \omega^\mathcal{P} \).
Generalized connections and Higgs fields on Lie algebroids, Nijmegen, April 5, 2016

Thierry Masson, CPT-Luminy

Connections and covariant derivatives

**Generalized connections**

**Definition (Generalized connection)**

A generalized connection on a transitive Lie algebroid $A$ is a 1-form $\tilde{\omega} \in \Omega^1(A, L)$. The curvature of $\tilde{\omega}$ is the 2-form $\tilde{R} = d\tilde{\omega} + \frac{1}{2}[\tilde{\omega}, \tilde{\omega}] \in \Omega^2(A, L)$.

A generalized connection is an ordinary connection iff $\tilde{\omega} \circ \iota = -\text{Id}_L$.

Consider a representation of $A$ on $\mathcal{E}$:

$$
\begin{array}{cccc}
0 & \to & L & \xrightarrow{\omega} & A & \xrightarrow{\rho} & \Gamma(TM) & \to & 0 \\
& & \downarrow{\iota} & & \downarrow{\phi} & & \downarrow{\phi} & & \\
0 & \to & A(\mathcal{E}) & \xrightarrow{\iota} & \mathcal{D}(\mathcal{E}) & \xrightarrow{\sigma} & \Gamma(TM) & \to & 0
\end{array}
$$

$\tilde{\omega}$ defines $\tilde{\nabla} : A \to \mathcal{D}(\mathcal{E})$ by $\tilde{\nabla}_\mathcal{X} = \phi(\mathcal{X}) + \iota \circ \phi_L(\tilde{\omega}(\mathcal{X}))$.

This is the **covariant derivative** on $\mathcal{E}$ associated to $\tilde{\omega}$.

$[\tilde{\nabla}_\mathcal{X}, \tilde{\nabla}_\mathcal{Y}] - \tilde{\nabla}_{[\mathcal{X}, \mathcal{Y}]} = \iota \circ \phi_L \circ \tilde{R}(\mathcal{X}, \mathcal{Y})$ $\Rightarrow$ $\tilde{\nabla}$ is not a representation in general.

Other terminologies for $\tilde{\nabla}$: "generalized representation", "$A$-connection"...
Generalized connections and Higgs fields on Lie algebroids

Connections and covariant derivatives

Generalized connections on Atiyah Lie algebroids

\[ 0 \longrightarrow \Gamma_G(\mathcal{P}, \mathfrak{g}) \overset{\iota}{\longrightarrow} \Gamma_G(T\mathcal{P}) \overset{\pi_*}{\longrightarrow} \Gamma(TM) \longrightarrow 0 \]

To simplify the presentation: suppose \( G \) is connected and simply connected.

A generalized connection \( \tilde{\omega} \) on \( \Gamma_G(T\mathcal{P}) \) is a \( \mathfrak{g}_{\text{equ}} \)-basic 1-form \( \tilde{\omega}_{\text{equ}} \in \Omega^1_{\text{TLA}}(\mathcal{P}, \mathfrak{g}) \).

\[ \tilde{\omega}_{\text{equ}} = \omega + \varphi \in (\Omega^1(\mathcal{P}) \otimes \mathfrak{g}) \oplus (C^\infty(\mathcal{P}) \otimes \mathfrak{g}^* \otimes \mathfrak{g}). \]

Proposition (Ordinary versus generalized connections)

If \( \varphi = -\theta \), then \( \tilde{\omega} \) is an ordinary connection on \( \Gamma_G(T\mathcal{P}) \).

\[ \Rightarrow \omega \text{ is an (ordinary) connection 1-form on } \mathcal{P}. \]

Otherwise, \( \varphi + \theta \) measures the deviation of \( \tilde{\omega} \) from an ordinary connection.

\[ \Rightarrow \varphi + \theta \approx \text{Higgs scalar fields...} \]
Connections: a summary

- Ordinary connection on a transitive Lie algebroid $\equiv$ splitting:

$$0 \xrightarrow{\nabla} L \xleftarrow{\iota} A \xrightarrow{\rho} \Gamma(TM) \xrightarrow{\nabla} 0$$

$\nabla \mapsto \omega^\nabla \in \Omega^1(A, L)$ connection 1-form, curvature as a 2-form.

- Generalized connections are any 1-forms $\widehat{\omega} \in \Omega^1(A, L)$.
  - Covariant derivatives on representations.
  - Notion of curvature.

- Ordinary connection $\equiv$ normalized generalized connection:
  $\widehat{\omega} \circ \iota(\ell) = -\ell$ for any $\ell \in L$

- For Atiyah Lie algebroids:
  - space of ordinary connections on $\mathcal{P} \subset$ space of generalized connections;
  - connection 1-forms and curvatures are directly related in $\Omega^\bullet_{TLA}(\mathcal{P}, g)$. 
The gauge group

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The gauge group

**Gauge group of a representation**

Suppose given a representation of a transitive Lie algebroid $A$ on $\mathcal{E}$:

\[
\begin{array}{c}
0 \longrightarrow L \xrightarrow{l} A \xrightarrow{\rho} \Gamma(TM) \longrightarrow 0 \\
\phi_L \downarrow \quad \quad \quad \downarrow \phi \\
0 \longrightarrow A(\mathcal{E}) \xrightarrow{l} \mathcal{D}(\mathcal{E}) \xrightarrow{\sigma} \Gamma(TM) \longrightarrow 0
\end{array}
\]

**Definition (Gauge group of a representation)**

The gauge group of $\mathcal{E}$ is the group $\text{Aut}(\mathcal{E}) \subset A(\mathcal{E})$ (vertical automorphisms of $\mathcal{E}$).

No (finite) gauge transformation at the level of $A$ (similar situation in NCG).

Any $\xi \in L$ defines an infinitesimal gauge transformation on $\Gamma(\mathcal{E})$ by $\varphi \mapsto \phi_L(\xi)\varphi$.

**Definition (Infinitesimal gauge transformations)**

An infinitesimal gauge transformation on $A$ is an element $\xi \in L$. 

The gauge group

Gauge transformations

\[ \xi \in L \mapsto g = e^{\phi_L(\xi)} \approx 1 + \phi_L(\xi) + \cdots \in \text{Aut}(\mathcal{E}) \subset A(\mathcal{E}) \]

\( \widehat{\omega} \) generalized connection on \( A \), and \( \widehat{\nabla} \) its associated covariant derivative on \( \mathcal{E} \):

\[ \widehat{\nabla}_X \phi = \phi(X) \cdot \phi + \phi_L(\widehat{\omega}(X)) \phi \]

The first order diff. op. \( \widehat{\nabla}^g_{X} = g^{-1} \circ \widehat{\nabla}_X \circ g \) on \( \mathcal{E} \) can be written as

\[ \widehat{\nabla}^g_{X} \phi = \phi(X) \cdot \phi + \phi_L(\widehat{\omega}(X)) \phi + \phi_L(\text{d}\xi(X) + [\widehat{\omega}(X), \xi]) \phi + O(\xi^2) \phi \]

Definition (Infinitesimal gauge variation)

The infinitesimal gauge variation of \( \widehat{\omega} \) induced by \( \xi \) is defined to be \( \text{d}\xi + [\widehat{\omega}, \xi] \).

The infinitesimal gauge variation of the curvature \( \widehat{R} \) of \( \widehat{\omega} \) is \( [\widehat{R}, \xi] \).

The gauge principle is implemented on \( A \) at the infinitesimal level (indep. of a rep.).

Similar to ordinary differential geometry.
The gauge group

Gauge transformations on Atiyah Lie algebroids

\[ 0 \longrightarrow \Gamma_G(P, \mathfrak{g}) \overset{\iota}{\longrightarrow} \Gamma_G(TP) \overset{\pi_*}{\longrightarrow} \Gamma(TM) \longrightarrow 0 \]

\( \mathcal{G}(P) \) the gauge group of \( P \) (vertical automorphisms of \( P \)).
\( u \in \mathcal{G}(P) \) is a \( G \)-equivariant map \( u : P \rightarrow G, u(p \cdot g) = g^{-1}u(p)g \).

- Finite gauge transformations are defined.
- \( L = \Gamma_G(P, \mathfrak{g}) \) is the Lie algebra of \( \mathcal{G}(P) \).
- Infinitesimal (usual) gauge transformations are elements in \( L \).

\( \widehat{\omega} \in \Omega^1_{\text{Lie}}(P, \mathfrak{g}) \) and \( u \in \mathcal{G}(P) \).

Define \( \widehat{\omega}^u(\mathcal{X}) = u^{-1}\widehat{\omega}(\mathcal{X})u + u^{-1}(\mathcal{X} \cdot u) \) for any \( \mathcal{X} \in \Gamma_G(TP) \).

- \( \widehat{\omega}^u \in \Omega^1_{\text{Lie}}(P, \mathfrak{g}) \);
- induced by \( \widehat{\nabla} \mapsto \widehat{\nabla}^u \);
- infinitesimal gauge transformations on \( \widehat{\omega} \) are induced by \( \widehat{\omega} \mapsto \widehat{\omega}^u \);
- restricts to (ordinary) gauge transformation on ordinary connections...
  \( \mapsto \) preserves the decomposition \( \omega^P = \theta \).
Structures to construct an action functional

1. Lie algebroids and their representations
2. Differential structures
3. Connections and covariant derivatives
4. The gauge group
5. Structures to construct an action functional
6. Gauge theories
Metrics on transitive Lie algebroids

All the structures rely on a notion of metric...

**Definition (Metric on a Lie algebroid)**

A metric on $A$ is a symmetric $C^\infty(\mathcal{M})$-linear map $\widehat{\mathcal{g}} : A \otimes_{C^\infty(\mathcal{M})} A \rightarrow C^\infty(\mathcal{M})$.

$\widehat{\mathcal{g}}$ defines a metric $h = \iota^* \widehat{\mathcal{g}}$ on $L$ given by $h(\gamma, \eta) = \widehat{\mathcal{g}}(\iota(\gamma), \iota(\eta))$ for any $\gamma, \eta \in L$.

$\Rightarrow$ $\widehat{\mathcal{g}}$ is **inner non degenerate** if $h$ is non degenerate on $L$.

**Proposition (C. Fournel, S. Lazzarini, T.M.)**

An inner non degenerate metric $\widehat{\mathcal{g}}$ on $A$ is equivalent to a triple $(g, h, \nabla)$ where

- $g$ is a (possibly degenerate) metric on $\mathcal{M}$;
- $h$ is a non degenerate metric on $L$;
- $\nabla$ is an ordinary connection on $A$, with $\mathring{\omega} \in \Omega^1(A, L)$ its connection 1-form;
- $\widehat{\mathcal{g}}(\mathcal{X}, \mathcal{Y}) = g(\rho(\mathcal{X}), \rho(\mathcal{Y})) + h(\mathring{\omega}(\mathcal{X}), \mathring{\omega}(\mathcal{Y}))$.
- $\widehat{\mathcal{g}}(\nabla_X, \iota(\gamma)) = 0$ for any $X \in \Gamma(TM)$ and $\gamma \in L$.

Given $\widehat{\mathcal{g}}$, look at $\mathring{\omega}$ as a background connection...
Integration along the kernel

\[ \nabla \text{ a connection on } A, \ \hat{\omega} \in \Omega^1 (A, L) \text{ its connection 1-form.} \]
\[ h \text{ a metric on } L. \]

Suppose \( \mathcal{L} \) is orientable where \( L = \Gamma(\mathcal{L}) \) (\( \implies \) \( A \) is called inner orientable), let \( n = \text{rank}(\mathcal{L}) \).

**Proposition (Volume form along } L \text{ and inner integration)**

\( h \) and \( \hat{\omega} \) define a global form in \( \Omega^\bullet (A) \) of maximal degree in the \( L \) direction.

This volume form defines integrations
\[
\int_{\text{inner}} : \Omega^\bullet (A) \rightarrow \Omega^{\bullet-n} (\mathcal{M}) \quad \int_{\text{inner}} : \Omega^\bullet (A, L) \rightarrow \Omega^{\bullet-n} (\mathcal{M}, \mathcal{L}).
\]

They do not depend on \( \nabla \).

\( \implies \) After integration, only geometrical structures (de Rham).
Suppose also that $\mathcal{M}$ is orientable ($\mapsto A$ is called orientable) and $g$ non degenerate.

**Definition (Integration on $A$)**

Using $g$, the integration on $A$ of a form $\widehat{\omega} \in \Omega^\bullet(A)$ is defined by

$$\int_A \widehat{\omega} = \int_{\mathcal{M}} \int_{\text{inner}} \widehat{\omega} \in \mathbb{C}.$$ 

**Definition (Scalar product of forms)**

The scalar product of any 2 forms $\widehat{\omega}_1, \widehat{\omega}_2 \in \Omega^\bullet(A, L)$ is defined by

$$\langle \widehat{\omega}_1, \widehat{\omega}_2 \rangle = \int_A h(\widehat{\omega}_1, \widehat{\omega}_2) \in \mathbb{C}.$$
A non degenerate metric $\tilde{g} = (g, h, \nabla)$, with $\nabla \leftrightarrow \omega \in \Omega^1(A, L)$, gives us:

- $h \rightarrow$ scalar product on $L$;
- $h, \omega \rightarrow$ integration along $L$;
- $g \rightarrow$ integration on $\mathcal{M}$;
- $g, h, \omega \rightarrow$ integration on $A$;
- $g, h, \omega \rightarrow$ Hodge star operator (straightforward to define)
  \[ \star : \Omega^p(A, L) \rightarrow \Omega^{m+n-p}(A, L) \]
Gauge theories

1. Lie algebroids and their representations
2. Differential structures
3. Connections and covariant derivatives
4. The gauge group
5. Structures to construct an action functional
6. Gauge theories
Gauge invariant action

A orientable transitive Lie algebroid, $\widehat{g} = (g, h, \nabla)$ non degenerate metric.

Suppose $h$ is a Killing metric: $h([\xi, \gamma], \eta) + h(\gamma, [\xi, \eta]) = 0$ for any $\gamma, \eta, \xi \in L$.

$\widehat{\omega} \in \Omega^1(A, L)$ a connection on $A$ and $\widehat{R}$ its curvature 2-form.

**Proposition (C. Fournel, S. Lazzarini, T.M.)**

The action functional

$$S_{\text{Gauge}}[\widehat{\omega}] = \langle \widehat{R}, \star \widehat{R} \rangle = \int_A h(\widehat{R}, \star \widehat{R}).$$

is invariant under infinitesimal gauge transformations in $L$.

**Example (Atiyah Lie algebroid)**

$A$ the Atiyah Lie algebroid of a $G$-principal fiber bundle $\mathcal{P}$.

$h$ induced by the Killing form on $\mathfrak{g}$ (semisimple).

$\widehat{\omega}$ any generalized connection $\Rightarrow S_{\text{Gauge}}[\widehat{\omega}]$ is $\mathcal{G}(\mathcal{P})$-gauge invariant.

$\widehat{\omega}$ an ordinary connection on $A \Rightarrow S_{\text{Gauge}}[\widehat{\omega}]$ is the ordinary Yang-Mills action.

Possible to define $S_{\text{Matter}}[\varphi, \widehat{\omega}]$ for $\varphi \in \Gamma(\mathcal{E})$ where $A \to \mathcal{D}(\mathcal{E})$ is a representation of $A$. 
Decomposition of a connection

\( \hat{\omega} \in \Omega^1(A, L) \) a generalized connection on \( A \).

\( \tau \in \text{End}(L) \) defined by

\[
\tau = \hat{\omega} \circ \iota + \text{Id}_L.
\]

\( \tau \) vanishes iff \( \hat{\omega} \) is an ordinary connection on \( A \)

\[ \Rightarrow \] measures the “non Yang-Mills” part (Higgs scalar fields).

⚠️ \( \tau \) is not a Lie morphism.

\( \hat{g} = (g, h, \nabla) \) metric with \( \nabla \leftrightarrow \hat{\omega} \in \Omega^1(A, L) \).

**Proposition**

\( \omega = \hat{\omega} + \tau(\hat{\omega}) \in \Omega^1(A, L) \) is an ordinary connection on \( A \).

The induced infinitesimal gauge action of \( L \) is the one on ordinary connections.

\( \hat{g} = (g, h, \nabla) \) decomposes any connection \( \hat{\omega} \) on \( A \) as:

\[
\hat{\omega} \leftrightarrow (\omega, \tau) \quad \text{ordinary connection on } A \text{ + algebraic object on } L
\]

\( \hat{\omega} \) ordinary connection \[ \Rightarrow \tau = 0 \Rightarrow \omega = \hat{\omega} \).
The total action functional

Using the decomposition $\tilde{\omega} \leftrightarrow (\omega, \tau)$:

$$S_{\text{Gauge}}[\tilde{\omega}] + S_{\text{Matter}}[\varphi, \tilde{\omega}] = (1) \text{ Yang-Mills like term for } \omega$$

(2) covariant derivative for $\tau$ along $\omega$

(3) potential for $\tau$

(4) covariant derivative for $\varphi$ along $\omega$

(5) coupling $\varphi \leftrightarrow \tau$

The potential (3) can vanish for $\tau \neq 0$.
A development around a solution $\tau_0 \neq 0$ induces:

- A mass term for the ordinary connection $\omega$ in (2).
- A mass term for $\varphi$ in (5).
- Massive bosons ($\omega$) coupled to massive particles ($\varphi$).
- Yang-Mills-Higgs type gauge theory.
Why Higgs fields?

A pattern for Yang-Mills-Higgs gauge field theories (on “ordinary space-time”):

- **Algebraic Structure**
- **Global Structure**
- **Geometric Structure**

**Generalized connection**

\[ \hat{\omega} \simeq \omega + \tau \]

\( \tau \simeq \) scalar fields

\( \omega \simeq \) Yang-Mills fields

\[ \text{Spontaneous symmetry breaking} \]

Transitive Lie algebroids:

\[ 0 \longrightarrow L \longrightarrow A \longrightarrow \Gamma(\mathcal{TM}) \longrightarrow 0 \]

NCG:

\[ 1 \longrightarrow \text{Inn}(A) \longrightarrow \text{Aut}(A) \longrightarrow \text{Out}(A) \longrightarrow 1 \]

Conclusion

- (Geometric) Gauge field theories can be generalized in at least two directions:
  - Noncommutative Geometry
  - Transitive Lie Algebroids.

- Same pattern: add some purely algebraic directions to space-time.
  - Yang-Mills-Higgs type gauge theories.

- Gauge theories on Atiyah Lie algebroids are close to Yang-Mills gauge theories.
  - They contain ordinary gauge theories used in physics.
  - They share common mathematical structures.
  - No restriction on the gauge group.

- A lot more to investigate:
  - Relation to BRST structures...
  - Construction of realistic models...
  - Here, generalization of Ehresman’s connections:
    - we investigate generalization of Cartan’s connections (used in gravitational and conformal theories).
**Trivialization of transitive Lie algebroids**

\[0 \rightarrow L \xrightarrow{I} A \xrightarrow{\rho} \Gamma(TM) \rightarrow 0\] a transitive Lie algebroid.

A local trivialization of \(A\) is a triple \((\mathcal{U}, \Psi, \nabla^0)\) where

- \(\mathcal{U}\) is an open subset of \(\mathcal{M}\);
- \(\Psi : \Gamma(\mathcal{U} \times \mathfrak{g}) \xrightarrow{\simeq} L\mathcal{U} = \) isomorphism of Lie algebras and \(C^\infty(\mathcal{U})\)-modules;
- \(\nabla^0 : \Gamma(T\mathcal{U}) \rightarrow A\mathcal{U} = \) injective morphism of Lie algebras and \(C^\infty(\mathcal{U})\)-modules compatible \(\rho\);
- \([\nabla^0_X, \iota \circ \Psi(\gamma)] = \iota \circ \Psi(X \cdot \gamma)\) for any \(X \in \Gamma(T\mathcal{U})\) and any \(\gamma \in \Gamma(\mathcal{U} \times \mathfrak{g})\).

\(S(X \oplus \gamma) = \nabla^0_X + \iota \circ \Psi(\gamma)\) is a isomorphism of Lie algebroids \(S : TLA(\mathcal{U}, \mathfrak{g}) \xrightarrow{\simeq} A\mathcal{U}\).

Atlas for \(A =\) family of local trivializations \(\{(\mathcal{U}_i, \Psi_i, \nabla^{0,i})\}_{i \in I}\) with \(\bigcup_{i \in I} \mathcal{U}_i = \mathcal{M}\).

\(X \in A\) is decomposed as \(X^i \oplus \gamma^i \in TLA(\mathcal{U}_i, \mathfrak{g})\) such that \(S_i(X^i \oplus \gamma^i) = X|_{\mathcal{U}_i}\).

The \(X^i\)’s are the restrictions to \(\mathcal{U}_i\) of the global vector field \(X = \rho(X)\).

On \(\mathcal{U}_{ij} = \mathcal{U}_i \cap \mathcal{U}_j \neq \emptyset\) one can define \(\alpha^i_j = \Psi^{-1}_j \circ \Psi_j : \mathcal{U}_{ij} \rightarrow Aut(\mathfrak{g})\).

\(\exists \chi_{ij} \in \Omega^1(\mathcal{U}_{ij}) \otimes \mathfrak{g}\) such that \(\gamma^i = \alpha^i_j(\gamma^j) + \chi_{ij}(X)\).

Cocycle relations:

\[\alpha^i_k = \alpha^j_k \circ \alpha^i_j, \quad \alpha^i_j \circ \alpha^j_i = \text{Id}, \quad \chi_{ik} = \alpha^i_j \circ \chi_{jk} + \chi_{ij}, \quad \alpha^i_j \circ \chi_{ji} + \chi_{ij} = 0\]
Trivialization of differential forms

\[ 0 \longrightarrow L \xrightarrow{\iota} A \xrightarrow{\rho} \Gamma(TM) \longrightarrow 0 \] a transitive Lie algebroid.
\{(U_i, \Psi_i, \nabla^{0,i})\}_{i \in I} a Lie algebroid atlas for \( A \).

\( \omega \in \Omega^q(A, L) \) \( \mapsto \) family of local \( q \)-forms \( \omega^i_{\text{loc}} \in \Omega^q_{\text{TLA}}(U_i, g) \)

\[ \omega^i_{\text{loc}} = \Psi_i^{-1} \circ \omega \circ S_i \]

\( s^j_i = S^{-1}_j \circ S_i : \text{TLA}(U_{ij}, g) \xrightarrow{\sim} \text{TLA}(U_{ij}, g) = \text{isomorphism of TLA.} \)

\( \tilde{\alpha}^i_j : \Omega^q_{\text{TLA}}(U_{ij}, g) \to \Omega^q_{\text{TLA}}(U_{ij}, g) \) defined by \( \tilde{\alpha}^i_j(\omega^j_{\text{loc}}) = \alpha^i_j \circ \omega^j_{\text{loc}} \circ s^j_i \).

### Proposition

- A family of local forms \( \omega^i_{\text{loc}} \in \Omega^*_{\text{TLA}}(U_i, g) \) is a system of trivializations of a global form \( \omega \in \Omega^\bullet(A, L) \) if and only if \( \tilde{\alpha}^i_j(\omega^j_{\text{loc}}) = \omega^i_{\text{loc}} \) on any \( U_{ij} \neq \emptyset \).
- For any \( \omega \in \Omega^\bullet(A, L) \) trivialized on \( U \) as \( \omega_{\text{loc}} \), one has \( \tilde{d}_{\text{TLA}}\omega_{\text{loc}} = \Psi^{-1} \circ (d\omega) \circ S \).
- \( \tilde{\alpha}^i_j : \Omega^\bullet_{\text{TLA}}(U_{ij}, g) \to \Omega^\bullet_{\text{TLA}}(U_{ij}, g) = \text{isomorphism of grad. diff. Lie algebras.} \)
Local mixed basis

\( \mathfrak{g} \) the Lie algebra fiber of \( \mathcal{L} \) where \( L = \Gamma(\mathcal{L}), \mathcal{U} \subset \mathcal{M} \) open subset which trivializes \( A \).

\( \{E_a\}_{1 \leq a \leq n} \) basis of \( \mathfrak{g} \), \( \{\theta^a\}_{1 \leq a \leq n} \) dual basis of \( \mathfrak{g}^* \).

\( \widehat{\omega} \in \Omega^p(A, L) \) and \( \widehat{\omega}_{\text{loc}} \in \Omega^p_{\text{TLA}}(U, \mathfrak{g}) \) its local description:

\[
\widehat{\omega}_{\text{loc}} = \sum_{r+s=p} \widehat{\omega}^{\theta}_{\mu_1 \ldots \mu_r a_1 \ldots a_s} \, dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_r} \wedge \theta^{a_1} \wedge \cdots \wedge \theta^{a_s}, \quad \widehat{\omega}^{\theta}_{\mu_1 \ldots \mu_r a_1 \ldots a_s} : U \to \mathfrak{g}
\]

⚠️ \( \theta^a \) is not convenient \( \Rightarrow \) inhomogeneous transformations!

\( \nabla \) ordinary connection on \( A \), \( \widehat{\omega} \) its connection 1-form

\( \Rightarrow \) \( \widehat{\omega}_{\text{loc}} = (A^a - \theta^a)E_a \) with \( A^a \in \Omega^1(U) \otimes \mathfrak{g} \),

\( \Rightarrow \) the \( \omega^a = A^a - \theta^a \in \Omega^1_{\text{TLA}}(U) \) define the \textit{mixed basis} in \( \Omega^1_{\text{TLA}}(U) \).

Then one can write

\[
\widehat{\omega}_{\text{loc}} = \sum_{r+s=p} \widehat{\omega}_{\mu_1 \ldots \mu_r a_1 \ldots a_s} \, dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_r} \wedge \omega^{a_1} \wedge \cdots \wedge \omega^{a_s}, \quad \widehat{\omega}_{\mu_1 \ldots \mu_r a_1 \ldots a_s} : U \to \mathfrak{g}
\]

**Proposition (Homogeneous transformations)**

The \( \widehat{\omega}_{\mu_1 \ldots \mu_r a_1 \ldots a_s} \)'s have homog. transf. in a change of local trivializations.
Integration along the kernel

$\mathcal{U} \subset \mathcal{M}$ an open subset which trivializes $A$. \( \{E_a\} \) basis of $\mathfrak{g}$, \( \{\theta^a\} \) dual basis of $\mathfrak{g}^*$.  

$\nabla$ a connection on $A$, $\hat{\omega} \in \Omega^1(A, L)$ its connection 1-form.  

$\hat{\omega}^a = A^a - \theta^a \in \Omega^1_{\text{TLA}}(\mathcal{U})$ mixed basis in $\Omega^1_{\text{TLA}}(\mathcal{U})$.

$h$ a metric on $L$.  

$h_{\text{loc}} = \text{trivialization of } h \text{ over } \mathcal{U}$, $h_{ab} = h_{\text{loc}}(E_a, E_b) \in C^\infty(\mathcal{U})$, $|h_{\text{loc}}| = |\det(h_{ab})|$.  

Suppose $\mathcal{L}$ is orientable where $L = \Gamma(\mathcal{L})$ ($A$ is called inner orientable),

**Proposition (Volume form along $L$ and inner integration)**

$$\hat{\omega}_{h, \text{loc}} = (-1)^n \sqrt{|h_{\text{loc}}|} \hat{\omega}^1_{\text{loc}} \wedge \cdots \wedge \hat{\omega}^n_{\text{loc}}$$

defines a global form $\hat{\omega}_{h, \hat{\omega}} \in \Omega^\bullet(A)$ of maximal degree $n = \dim \mathfrak{g}$ in the $L$ direction.  

This volume form defines integrations

$$\int_{\text{inner}} : \Omega^\bullet(A, L) \to \Omega^{\bullet-n}(\mathcal{M}, \mathcal{L})$$

$$\int_{\text{inner}} : \Omega^\bullet(A) \to \Omega^{\bullet-n}(\mathcal{M})$$

They do not depend on $\nabla$. 
Hodge star operator

Let $\mathcal{A}$ be an orientable transitive Lie algebroid. A metric $\hat{g} = (g, h, \nabla)$ on $\mathcal{A}$, and $\hat{\omega}$ the connection 1-form of $\nabla$.

\[ \hat{\omega} \in \Omega^p(A, L), \] written locally as
\[ \hat{\omega}_{\text{loc}} = \sum_{r+s=p} \hat{\omega}_{\mu_1 \ldots \mu_r a_1 \ldots a_s} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_r} \wedge \hat{\omega}^{a_1} \wedge \cdots \wedge \hat{\omega}^{a_s} \]

The map $\star: \Omega^p(A, L) \to \Omega^{m+n-p}(A, L)$ is well defined globally.

This is the Hodge star operator associated to $\hat{g}$ on $A$.

Proposition (Hodge star operator)

The map $\star: \Omega^p(A, L) \to \Omega^{m+n-p}(A, L)$ is well defined globally.
Gauge transformations on Atiyah Lie algebroids

Suppose $G$ is connected and simply connected.

$$\hat{\omega} \in \Omega^1_{\text{Lie}}(\mathcal{P}, \mathfrak{g}) \mapsto \hat{\omega}_{\text{equ}} \in \Omega^1_{\Lambda^1}(\mathcal{P}, \mathfrak{g}), \text{equ}-\text{basic}.$$ 

The gauge action $\hat{\omega} \mapsto \hat{\omega}^u$ induces

$$\hat{\omega}_{\text{equ}} \mapsto u\hat{\omega}_{\text{equ}} u^{-1} + u\tilde{\text{d}}_{\Lambda^1} u^{-1} \in \Omega^1_{\Lambda^1}(\mathcal{P}, \mathfrak{g})$$

where

$$u\tilde{\text{d}}_{\Lambda^1} u^{-1} = u\text{d} u^{-1} + u\theta u^{-1} - \theta$$

($\theta = \text{Cartan 1-form}$)

Notice that $u\theta u^{-1} - \theta = u[\theta, u^{-1}]$ is more or less “s” applied to $u$.

**Proposition (Ordinary gauge transformations)**

If $\hat{\omega}_{\text{equ}} = \omega^\mathcal{P} - \theta$ is an ordinary connection on $\Gamma_G(T\mathcal{P})$, this action reduces to the usual gauge transformation $\omega^\mathcal{P} \mapsto u\omega^\mathcal{P} u^{-1} + u\text{d} u^{-1}$ on the (ordinay) connection 1-form $\omega^\mathcal{P}$.
**Decomposition of a connection**

\( \tilde{\omega} \in \Omega^1(A, L) \) a generalized connection on \( A \).

**Definition (Reduced kernel endomorphism)**

The reduced kernel endomorphism \( \tau \in \text{End}(\mathcal{L}) \) associated to \( \tilde{\omega} \) is defined by

\[
\tau = \tilde{\omega} \circ \iota + \text{Id}_L.
\]

\( \tau \) vanishes iff \( \tilde{\omega} \) is an ordinary connection on \( A \)

\( \implies \) measures the “non Yang-Mills” part.

\( \tau \) is not a Lie morphism. Define \( R_{\tau}(\gamma, \eta) = [\tau(\gamma), \tau(\eta)] - \tau([\gamma, \eta]) \) for any \( \gamma, \eta \in L \).

Let \( \tilde{\omega} \in \Omega^1(A, L) \) be a fixed ordinary connection on \( A \) (“background connection”).

**Theorem**

\( \tilde{\omega} \in \Omega^1(A, L) \) a connection and \( \tau \) its reduced kernel endomorphism.

\[
\omega = \tilde{\omega} + \tau(\tilde{\omega})
\]

is an ordinary connection on \( A \).

The induced infinitesimal gauge action of \( L \) is the one on ordinary connections.

\( \tilde{\omega} \) ordinary connection \( \implies \tau = 0 \implies \omega = \tilde{\omega} \).

\( \implies \) \( \tilde{\omega} \) only relevant for connections which are not ordinary connections.
Decomposition of curvature and covariant derivative

\[ \hat{\omega} = \omega - \tau(\hat{\omega}) \] connection on \( A \).

\( \hat{\nabla}, \nabla : \Gamma(TM) \to A \) the splittings associated to the ordinary connections \( \hat{\omega}, \omega \).

\( \hat{R}, R \in \Omega^2(\mathcal{M}, \mathcal{L}) \) the curvature 2-forms of \( \hat{\omega}, \omega \).

\[ \hat{F} = R - \tau \circ \hat{R} \in \Omega^2(\mathcal{M}, \mathcal{L}) \implies \rho^* \hat{F} \in \Omega^2(A, L). \]

For \( X \in \Gamma(TM) \), define \( D_X \tau \in \text{End}(\mathcal{L}) \) by, for any \( \gamma \in L \),
\[ (D_X \tau)(\gamma) = [\nabla_X, \tau(\gamma)] - \tau([\hat{\nabla}_X, \gamma]) \implies (\rho^* D\tau) \circ \hat{\omega} \in \Omega^2(A, L). \]

\( \nabla^E \) the (ordinary) covariant derivative induced on \( E \) by the (ordinary) connection \( \omega \).

For any \( \phi \in \Gamma(E) \), one has \( \rho^* \phi(\nabla) \cdot \phi = \rho^* \nabla^E \phi \).

**Proposition (Decomposition of the curvature and the covariant derivative)**

The curvature \( \hat{R} \in \Omega^2(A, L) \) of \( \hat{\omega} \) can be decomposed as

\[ \hat{R} = \rho^* \hat{F} - (\rho^* D\tau) \circ \hat{\omega} + R_\tau \circ \hat{\omega} \]

The covariant derivative \( \hat{\nabla}^E \phi \in \Omega^1(A, E) \) can be decomposed as

\[ \hat{\nabla}^E \phi = \rho^* \phi(\nabla) \cdot \phi - (\phi_L(\tau) \phi) \circ \hat{\omega} \]

Under infinitesimal gauge transformations, each term has homog. transf.

“\( \circ \hat{\omega} \)” = along the mixed basis and “\( \rho^* \)” = along \( \Gamma(TM) \).
Coupling to matter fields

Matter fields are sections $\varphi \in \Gamma(\mathcal{E})$ of a representation $\phi : A \to \mathcal{D}(\mathcal{E})$ of $A$.

Definition ($\phi_L$-compatible metric)

A metric $h^\mathcal{E}$ on $\mathcal{E}$ is $\phi_L$-compatible if

$$h^\mathcal{E}(\phi_L(\xi)\varphi_1, \varphi_2) + h^\mathcal{E}(\varphi_1, \phi_L(\xi)\varphi_2) = 0$$

for any $\varphi_1, \varphi_2 \in \Gamma(\mathcal{E})$ and any $\xi \in L$.

Generalization of “Killing metric”.
One can define a Hodge star operator on $\Omega^\bullet(A, \mathcal{E})$.

$\hat{\omega}$ connection on $A$ and $\hat{\nabla}^\mathcal{E}$ the induced covariant derivative on $\Gamma(\mathcal{E})$.

Proposition

The action functional

$$S_{\text{Matter}}[\varphi, \hat{\omega}] = \int_A h^\mathcal{E}(\hat{\nabla}^\mathcal{E}\varphi, \star \hat{\nabla}^\mathcal{E}\varphi)$$

is invariant under infinitesimal gauge transformations in $L$. 
Decomposition of the action functional

\( \hat{\omega} \) connection on \( A, \varphi \in \Gamma(\mathcal{E}) \) matter field.

\[
S[\varphi, \hat{\omega}] = S_{\text{Gauge}}[\hat{\omega}] + S_{\text{Matter}}[\varphi, \hat{\omega}] \quad \text{total action functional}
\]

\( \hat{g} = (g, h, \nabla) \), with \( \nabla \leftrightarrow \hat{\omega} \in \Omega^1(A, L) \), metric on \( A \).

The decomposition \( \hat{\omega} = \omega - \tau(\hat{\omega}) \) induces the decomposition:

\[
S[\varphi, \hat{\omega}] = \langle \rho^* \hat{F}, \star \rho^* \hat{F} \rangle \quad (1) \text{ spatial term: Yang-Mills like}
\]

\[
+ \langle (\rho^* \mathcal{D} \tau) \circ \hat{\omega}, \star (\rho^* \mathcal{D} \tau) \circ \hat{\omega} \rangle \quad (2) \text{ mixed term: covariant derivative of } \tau
\]

\[
+ \langle R_\tau \circ \hat{\omega}, \star R_\tau \circ \hat{\omega} \rangle \quad (3) \text{ algebraic term: potential for } \tau
\]

\[
+ \langle \rho^* \phi(\nabla) \cdot \varphi, \star \rho^* \phi(\nabla) \cdot \varphi \rangle \quad (4) \text{ spatial term: covariant derivative of } \varphi
\]

\[
+ \langle (\phi_L(\tau) \varphi) \circ \hat{\omega}, \star (\phi_L(\tau) \varphi) \circ \hat{\omega} \rangle \quad (5) \text{ algebraic term: coupling } \varphi \leftrightarrow \tau
\]