The fermionic action

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Outline

1. Introduction
2. Krein spectral triples
3. Gauge theory
4. The electroweak theory
5. Conclusion
The fermionic action

Consider a real even spectral triple \((\mathcal{A}, \mathcal{H}, \mathcal{D}, J, \gamma)\) of \(KO\)-dimension 2. The fermionic action is defined as \([\text{Con06}]\)

\[
S_f := \frac{1}{2} \langle J \tilde{\zeta} | \mathcal{D} \tilde{\zeta} \rangle,
\]

where \(\tilde{\zeta}\) is a Grassmann variable corresponding to \(\zeta = \gamma \zeta \in \mathcal{H}^0\).

Two discrepancies:
- signature is Riemannian instead of Lorentzian;
- the definition involves the real structure (‘charge conjugation’).

Solution \([\text{Bar07}]\): consider an action functional of the form \(\langle \psi | \mathcal{D} \psi \rangle\), where \(\langle \cdot | \cdot \rangle\) denotes the indefinite inner product on a Krein space, and where \(\mathcal{D}\) is Krein-self-adjoint.
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Introduction

Krein spectral triples

Gauge theory

The electroweak theory

Conclusion
Krein spaces

A Krein space is a vector space $\mathcal{H}$ with a non-degenerate inner product $\langle \cdot | \cdot \rangle$ which admits a fundamental decomposition $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$ (i.e., an orthogonal direct sum decomposition into a positive-definite subspace $\mathcal{H}^+$ and a negative-definite subspace $\mathcal{H}^-$) such that the subspaces $\mathcal{H}^+$ and $\mathcal{H}^-$ are intrinsically complete (i.e., complete with respect to the norms $\| \psi \|_{\mathcal{H}^\pm} := |\langle \psi | \psi \rangle|^{1/2}$).

Given a fundamental decomposition $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$, we obtain a corresponding fundamental symmetry $J = P^+ - P^-$, where $P^\pm$ denotes the projection onto $\mathcal{H}^\pm$.

Given a fundamental symmetry $J$, we denote by $\mathcal{H}_J$ the corresponding Hilbert space for the positive-definite inner product $\langle \cdot | \cdot \rangle_J := \langle J \cdot | \cdot \rangle$.

A Krein space $\mathcal{H}$ with fundamental symmetry $J$ is called $\mathbb{Z}_2$-graded if $\mathcal{H}_J$ is $\mathbb{Z}_2$-graded and $J$ is homogeneous. This means:

- we have a decomposition $\mathcal{H}^0 \oplus \mathcal{H}^1$;
- this decomposition is respected by the positive-definite inner product $\langle \cdot | \cdot \rangle_J$. 
Krein spaces

- A **Krein space** is a vector space $\mathcal{H}$ with a non-degenerate inner product $\langle \cdot | \cdot \rangle$ which admits a *fundamental decomposition* $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$ (i.e., an orthogonal direct sum decomposition into a positive-definite subspace $\mathcal{H}^+$ and a negative-definite subspace $\mathcal{H}^-$) such that the subspaces $\mathcal{H}^+$ and $\mathcal{H}^-$ are *intrinsically complete* (i.e., complete with respect to the norms $\|\psi\|_{\mathcal{H}^\pm} := |\langle \psi | \psi \rangle|^{1/2}$).

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Lorentzian manifolds

Let \((M, g)\) be an \(n\)-dimensional space- and time-oriented Lorentzian spin manifold with an orthogonal direct sum decomposition of the tangent bundle \(TM = E_t \oplus E_s\) with \(\dim E_t = 1\) (with basis vector \(e_0\)) and \(\dim E_s = n - 1\) (with basis vectors \(e_1, \ldots, e_{n-1}\)) such that the metric \(g\) is negative-definite on \(E_t\) and positive-definite on \(E_s\).

We have a timelike projection \(T : TM \to E_t\) and a spacelike reflection \(r = 1 - 2T = (-1) \oplus 1\) on \(TM = E_t \oplus E_s\).

We can define a ‘Wick rotated’ metric \(g_r\) on \(M\) by setting

\[
gr(v, w) := g(rv, w).
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Then \((M, g_r)\) is a Riemannian manifold.
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Lorentzian spinors

- Given a decomposition $TM = E_t \oplus E_s$ there exists a positive-definite hermitian structure [Baum81]

$$ (\cdot|\cdot)_{J_M} : \Gamma_c^\infty(S) \times \Gamma_c^\infty(S) \to C_c^\infty(M). $$

which gives rise to the inner product $\langle \cdot|\cdot \rangle_{J_M} := \int_M (\cdot|\cdot)_{J_M} \, dvol_g$. The completion of $\Gamma_c^\infty(S)$ with respect to this inner product is denoted $L^2(S)$.

- The operator $J_M := \gamma(e_0)$ on $L^2(S)$ is self-adjoint and unitary, and is related to the spacelike reflection $r$ via $J_M \gamma(v) J_M = -\gamma(rv)$. Then $L^2(S)$ is a Krein space with the indefinite inner product $\langle \cdot|\cdot \rangle := \langle J_M \cdot|\cdot \rangle_{J_M}$ and with fundamental symmetry $J_M$. This indefinite inner product $\langle \cdot|\cdot \rangle$ is independent of the choice of decomposition $TM = E_t \oplus E_s$. 

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The Dirac operator

- Define the Lorentzian Dirac operator

\[ \mathcal{D} := \sum_{j=0}^{n-1} \kappa(j) \gamma(e_j) \nabla^S_{e_j}, \]

where \( \nabla^S \) is the lift of the Levi-Civita connection corresponding to \( g \), and \( \kappa(0) = -1 \) and \( \kappa(j) = 1 \) for \( j = 1, \ldots, n - 1 \).

- **Theorem [Baum81]**: Suppose there exists a decomposition \( TM = E_t \oplus E_s \) such that \( g_r \) is complete. Then \( i\mathcal{D} \) is essentially Krein-self-adjoint.

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\[ (C^\infty_c(M), L^2(S), i\mathcal{D}, \mathcal{J}_M = \gamma(e_0)) \]
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Spectral triples

**Definition:** An even spectral triple \((\mathcal{A}, \mathcal{H}, \mathcal{D})\) consists of

- a \(\mathbb{Z}_2\)-graded Hilbert space \(\mathcal{H}\);
- an even \(*\)-algebra representation \(\pi : \mathcal{A} \to B^0(\mathcal{H})\);
- a closed, odd operator \(\mathcal{D} : \text{Dom} \mathcal{D} \to \mathcal{H}\) such that:
  1. the linear subspace \(\mathcal{E} := \text{Dom} \mathcal{D}\) is dense in \(\mathcal{H}\);
  2. the operator \(\mathcal{D}\) is essentially self-adjoint on \(\mathcal{E}\);
  3. the commutator \([\mathcal{D}, \pi(a)]\) is bounded on \(\mathcal{E}\) for each \(a \in \mathcal{A}\);
  4. the map \(\pi(a) \circ \iota : \mathcal{E} \hookrightarrow \mathcal{H} \rightarrow \mathcal{H}\) is compact for each \(a \in \mathcal{A}\).

**Remark:** condition 4 is equivalent to compactness of \(\pi(a)(\mathcal{D} \pm i)^{-1}\).
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Note: \(\mathcal{E}\) is equipped with the norm \(\|\psi\|_\mathcal{E} := \|\psi\| + \|\mathcal{J}\mathcal{D}\psi\| + \|\mathcal{D}\mathcal{J}\psi\|\).

We say an even Krein spectral triple \((\mathcal{A}, \mathcal{H}, \mathcal{D}, \mathcal{J})\) is of Lorentz-type when \(\mathcal{J}\) is odd.
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- a closed, odd operator \(D: \text{Dom} D \to \mathcal{H}\) such that:
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  3. the commutator \([D, \pi(a)]\) is bounded on \(\mathcal{E}\) for each \(a \in A\);
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Almost-commutative manifolds

- Let \((M, g)\) be an even-dimensional time- and space-oriented Lorentzian spin manifold. Suppose there exists a spacelike reflection \(r\) such that \(g_r\) is complete. Then

\[
(C_c^\infty(M), L^2(S), i\mathcal{D}, J_M = \gamma(e_0))
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is a Lorentz-type spectral triple.

- A finite space \(F\) is an even Krein spectral triple \((\mathcal{A}_F, \mathcal{H}_F, \mathcal{D}_F, \mathcal{J}_F)\) such that \(\dim \mathcal{H}_F < \infty\) and \(\mathcal{J}_F\) is even.

- **Definition:** An almost-commutative Lorentzian manifold \(F \times M\) is the product of a finite space \(F\) with the manifold \(M\), given by

\[
(\mathcal{A}, \mathcal{H}, \mathcal{D}, \mathcal{J}) := \left(C_c^\infty(M, \mathcal{A}_F), \mathcal{H}_F \hat{\otimes} L^2(S), 1 \hat{\otimes} i\mathcal{D} + i\mathcal{D}_F \hat{\otimes} 1, \mathcal{J}_F \hat{\otimes} \mathcal{J}_M\right).
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The Krein action

Let \((A, \mathcal{H}, \mathcal{D}, J)\) be a Lorentz-type spectral triple. Then we have:

- \(\langle \psi | D \phi \rangle = \langle \phi | D \psi \rangle\) for any \(\psi, \phi \in \text{Dom } D\);
- \(\langle \psi_0 | D \psi_1 \rangle = 0\) for any \(\psi_0 \in \mathcal{H}^0 \cap \text{Dom } D\) and \(\psi_1 \in \mathcal{H}^1 \cap \text{Dom } D\).

We define the Krein action \(S_K : \mathcal{H}^0 \cap \text{Dom } D \to \mathbb{C}\) to be the functional

\[S_K[\psi] := \langle \psi | D \psi \rangle.\]

We note that \(S_K[\psi]\) is real-valued and (in general) non-zero.

Remark: this action is classical. In particular, there are no Grassmann variables.
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- We define the **Krein action** $S_K : \mathcal{H}^0 \cap \text{Dom} \, \mathcal{D} \to \mathbb{C}$ to be the functional
  $$S_K[\psi] := \langle \psi | \mathcal{D} \psi \rangle.$$ 

  We note that $S_K[\psi]$ is real-valued and (in general) non-zero.

- **Remark:** this action is *classical*. In particular, there are no Grassmann variables.
1 Introduction

2 Krein spectral triples

3 Gauge theory

4 The electroweak theory

5 Conclusion
The perturbation semi-group

- Let $\mathcal{A}$ be a unital $*$-algebra. Let $A = \sum_j a_j \otimes b_j^{\text{op}} \in \mathcal{A} \otimes \mathcal{A}^{\text{op}}$.

  Define $\overline{A} := \sum b_j^* \otimes a_j^{*\text{op}}$.

  - $A$ is real if $\overline{A} = A$.
  - $A$ is normalised if $\sum a_j b_j = 1 \in \mathcal{A}$.

- **Definition [CCvS13]:** The *perturbation semi-group* $\text{Pert}(\mathcal{A})$ consists of the real normalised elements in $\mathcal{A} \otimes \mathcal{A}^{\text{op}}$.

- For a Krein spectral triple $(\mathcal{B}, \mathcal{H}, \mathcal{D}, \mathcal{J})$ we consider the *generalised one-forms* given by $\Omega^1_D(\mathcal{B}) := \left\{ \sum_j a_j [\mathcal{D}, b_j] \, \middle| \, a_j, b_j \in \mathcal{B} \right\}$.

- For $\mathcal{B} = \mathcal{A} \otimes \mathcal{A}^{\text{op}}$, define the map $\eta_D : \mathcal{A} \otimes \mathcal{A}^{\text{op}} \to \Omega^1_D(\mathcal{A} \otimes \mathcal{A}^{\text{op}})$ by

  $$\eta_D \left( \sum_j a_j \otimes b_j^{\text{op}} \right) := \sum_{j,k} (a_j(a_k^*)^{\text{op}})[\mathcal{D}, b_j(b_k^*)^{\text{op}}].$$

  **Fact:** if $A \in \text{Pert}(\mathcal{A})$ is real, then $\eta_D(A)$ is Krein-self-adjoint.
The perturbation semi-group

- Let $\mathcal{A}$ be a unital $\ast$-algebra. Let $A = \sum_j a_j \otimes b_j^{\text{op}} \in \mathcal{A} \otimes \mathcal{A}^{\text{op}}$. Define $\overline{A} := \sum b_j^* \otimes a_j^{*\text{op}}$.
  - $A$ is real if $\overline{A} = A$.
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- Let $\mathcal{A}$ be a unital $\ast$-algebra. Let $A = \sum_j a_j \otimes b_j^{\text{op}} \in \mathcal{A} \otimes \mathcal{A}^{\text{op}}$. Define $\overline{A} := \sum b^*_j \otimes a^*_j$.  
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  \]

Fact: if $A \in \text{Pert}(\mathcal{A})$ is real, then $\eta_D(A)$ is Krein-self-adjoint.
The perturbation semi-group

- Let \( \mathcal{A} \) be a unital \( \ast \)-algebra. Let \( A = \sum_j a_j \otimes b_j^{\text{op}} \in \mathcal{A} \odot \mathcal{A}^{\text{op}} \).
  
  Define \( \overline{A} := \sum_j b_j^* \otimes a_j^{*\text{op}} \).

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- **Definition [CCvS13]:** The perturbation semi-group \( \text{Pert}(\mathcal{A}) \) consists of the real normalised elements in \( \mathcal{A} \odot \mathcal{A}^{\text{op}} \).

- For a Krein spectral triple \( (\mathcal{B}, \mathcal{H}, \mathcal{D}, \mathcal{J}) \) we consider the generalised one-forms given by \( \Omega^1_{\mathcal{D}}(\mathcal{B}) := \left\{ \sum_j a_j[\mathcal{D}, b_j] \mid a_j, b_j \in \mathcal{B} \right\} \).

- For \( \mathcal{B} = \mathcal{A} \odot \mathcal{A}^{\text{op}} \), define the map \( \eta_{\mathcal{D}} : \mathcal{A} \odot \mathcal{A}^{\text{op}} \to \Omega^1_{\mathcal{D}}(\mathcal{A} \odot \mathcal{A}^{\text{op}}) \) by
  
  \[
  \eta_{\mathcal{D}} \left( \sum_j a_j \otimes b_j^{\text{op}} \right) := \sum_{j,k} (a_j(a_k^*)^{\text{op}})[\mathcal{D}, b_j(b_k^*)^{\text{op}}].
  \]

  Fact: if \( A \in \text{Pert}(\mathcal{A}) \) is real, then \( \eta_{\mathcal{D}}(A) \) is Krein-self-adjoint.
Fluctuations

- If \((A \odot A^{\text{op}}, \mathcal{H}, \mathcal{D}, \mathcal{J})\) satisfies the order-one condition

\[
[a, [\mathcal{D}, b^{\text{op}}]] = 0 \quad \forall a, b \in A,
\]

then

\[
\eta_{\mathcal{D}} \left( \sum_j a_j \otimes b_j^{\text{op}} \right) = \sum_j a_j [\mathcal{D}, b_j] + \sum_j a_j^{\ast \text{op}} [\mathcal{D}, b_j^{\ast \text{op}}].
\]

- By the fluctuation of \(\mathcal{D}\) by \(A \in \text{Pert}(A)\) we mean the map

\[
\mathcal{D} \mapsto \mathcal{D}_A := \mathcal{D} + \eta_{\mathcal{D}}(A),
\]

and we refer to \(\mathcal{D}_A\) as the fluctuated Dirac operator.

- Proposition [CCvS13]: A fluctuation of a fluctuated Dirac operator is again a fluctuated Dirac operator. To be precise: \((\mathcal{D}_A)_{A'} = \mathcal{D}_{A'A}\) for all perturbations \(A, A' \in \text{Pert}(A)\).
Fluctuations

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Fluctuations

- If $(\mathcal{A} \otimes \mathcal{A}^{\text{op}}, \mathcal{H}, \mathcal{D}, \mathcal{J})$ satisfies the order-one condition

$$[a, [\mathcal{D}, b^{\text{op}}]] = 0 \quad \forall a, b \in \mathcal{A},$$

then

$$\eta_{\mathcal{D}} \left( \sum_j a_j \otimes b_j^{\text{op}} \right) = \sum_j a_j [\mathcal{D}, b_j] + \sum_j a_j^{*\text{op}} [\mathcal{D}, b_j^{*\text{op}}].$$

- By the fluctuation of $\mathcal{D}$ by $A \in \text{Pert}(\mathcal{A})$ we mean the map

$$\mathcal{D} \mapsto \mathcal{D}_A := \mathcal{D} + \eta_{\mathcal{D}}(A),$$

and we refer to $\mathcal{D}_A$ as the fluctuated Dirac operator.

- **Proposition [CCvS13]:** A fluctuation of a fluctuated Dirac operator is again a fluctuated Dirac operator. To be precise: $(\mathcal{D}_A)_{A'} = \mathcal{D}_{A'A}$ for all perturbations $A, A' \in \text{Pert}(\mathcal{A})$. 
The gauge group

- The unitary group $\mathcal{U}(\mathcal{A})$ acts on $\text{Pert}(\mathcal{A})$ via
  \[
  \Delta(u)A := (u \otimes (u^*)^{\text{op}}) \left( \sum a_j \otimes b_j^{\text{op}} \right) = \sum ua_j \otimes (b_ju^*)^{\text{op}}.
  \]

  We can compose $\Delta$ with the $\ast$-algebra representation $\pi: \mathcal{A} \otimes \mathcal{A}^{\text{op}} \to \mathcal{B}(\mathcal{H})$ to obtain a group representation
  
  \[
  \rho := \pi \circ \Delta: \mathcal{U}(\mathcal{A}) \to \mathcal{B}(\mathcal{H}).
  \]

  We define the gauge group as
  
  \[
  \mathcal{G}(\mathcal{A}) := \{ \rho(u) \mid u \in \mathcal{U}(\mathcal{A}) \} \simeq \mathcal{U}(\mathcal{A}) / \text{Ker } \rho.
  \]

- Proposition: The Krein action $S_K[\psi, A] := \langle \psi | D_A \psi \rangle$ of the fluctuated Dirac operator $D_A$ is invariant under the action of the gauge group given by $\psi \mapsto \rho(u) \psi$ and $A \mapsto \Delta(u)A$. 
The gauge group

- The unitary group $U(A)$ acts on $\text{Pert}(A)$ via

$$\Delta(u)A := (u \otimes (u^*)^{\text{op}}) \left( \sum a_j \otimes b_j^{\text{op}} \right) = \sum u a_j \otimes (b_j u^*)^{\text{op}}.$$ 

We can compose $\Delta$ with the $\ast$-algebra representation $\pi: A \otimes A^{\text{op}} \to \mathcal{B}(\mathcal{H})$ to obtain a group representation

$$\rho := \pi \circ \Delta: U(A) \to \mathcal{B}(\mathcal{H}).$$

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1. Introduction

2. Krein spectral triples

3. Gauge theory

4. The electroweak theory

5. Conclusion
The finite space (1)

- Define $\mathcal{H}_F := \mathbb{C}^4$ with the basis $\{\nu_R, e_R, \nu_L, e_L\}$ and the $\mathbb{Z}_2$-grading

$$\mathcal{H}_F^0 = \mathcal{H}_L = \text{span}\{\nu_L, e_L\}, \quad \mathcal{H}_F^1 = \mathcal{H}_R = \text{span}\{\nu_R, e_R\}$$

- Define $\mathcal{A}_F := \mathbb{C} \oplus \mathbb{H}$, with the representations

$$\pi : \mathcal{A}_F \to \mathcal{B}(\mathcal{H}_R) \oplus \mathcal{B}(\mathcal{H}_L)$$

and $\pi^\text{op} : \mathcal{A}_F^\text{op} \to \mathcal{B}(\mathcal{H}_R) \oplus \mathcal{B}(\mathcal{H}_L)$ given

for $\lambda \in \mathbb{C}$ and $q = \alpha + \beta j \in \mathbb{H}$ by

$$\pi(\lambda, q) := q\lambda \oplus q := \begin{pmatrix} \lambda & 0 \\ 0 & \overline{\lambda} \end{pmatrix} \oplus \begin{pmatrix} \alpha & \beta \\ -\beta & \overline{\alpha} \end{pmatrix}, \quad \pi^\text{op}((\lambda, q)^\text{op}) := \lambda \oplus \lambda.$$ 

- The representation $\tilde{\pi} := \pi \otimes \pi^\text{op}$ of $\mathcal{A}_F \otimes \mathcal{A}_F^\text{op}$ on $\mathcal{H}_R \oplus \mathcal{H}_L$ is then given by

$$\tilde{\pi}((\lambda, q) \otimes (\lambda', q')^\text{op}) = \lambda' q \lambda \oplus \lambda' q.$$
The finite space (1)

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- The representation $\tilde{\pi} := \pi \otimes \pi^{\text{op}}$ of $\mathcal{A}_F \otimes \mathcal{A}_F^{\text{op}}$ on $\mathcal{H}_R \oplus \mathcal{H}_L$ is then given by

$$\tilde{\pi}((\lambda, q) \otimes (\lambda', q')^{\text{op}}) = \lambda' q\lambda \oplus \lambda' q.$$
The finite space (2)

- We define the mass matrix on the basis \( \{ \nu_R, e_R, \nu_L, e_L \} \) as

\[
D_F := \begin{pmatrix}
0 & 0 & -im_\nu & 0 \\
0 & 0 & 0 & -im_e \\
-im_\nu & 0 & 0 & 0 \\
im_e & 0 & 0 & 0
\end{pmatrix}.
\]

- We then consider the even finite space \( F_{EW} := (\mathcal{A}_F, \mathcal{H}_F, D_F, J_F = 1) \).

- The gauge group of \( F_{EW} \) equals

\[
\mathcal{G}(F_{EW}) = (U(1) \times SU(2)) / \mathbb{Z}_2.
\]
The finite space (2)

- We define the mass matrix on the basis \( \{ \nu_R, e_R, \nu_L, e_L \} \) as

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\mathcal{D}_F := \begin{pmatrix}
0 & 0 & -im_\nu & 0 \\
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im_\nu & 0 & 0 & 0 \\
0 & im_e & 0 & 0 \\
\end{pmatrix}.
\]

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\[
\mathcal{G}(F_{EW}) = (U(1) \times SU(2)) / \mathbb{Z}_2.
\]
The finite space (2)

- We define the mass matrix on the basis \{ν_R, e_R, ν_L, e_L\} as

\[
D_F := \begin{pmatrix}
0 & 0 & -i m_ν & 0 \\
0 & 0 & 0 & -i m_e \\
i m_ν & 0 & 0 & 0 \\
0 & i m_e & 0 & 0
\end{pmatrix}.
\]

- We then consider the even finite space \(F_{EW} := (A_F, H_F, D_F, J_F = 1)\).

- The gauge group of \(F_{EW}\) equals

\[
G(F_{EW}) = (U(1) \times SU(2))/\mathbb{Z}_2.
\]
Fluctuations

- We consider the almost-commutative manifold

\[ F_{EW} \times M := \left( C_c^\infty (M, \mathcal{A}_F \odot \mathcal{A}_F^{op}), \mathcal{H}_F \otimes L^2(\mathfrak{s}), 1 \otimes iD + i\mathcal{D}_F \otimes 1, 1 \otimes \mathcal{J}_M \right). \]

- **Proposition:** The fluctuation of \( D := 1 \otimes iD + i\mathcal{D}_F \otimes 1 \) by \( A \in \text{Pert}(C_c^\infty (M, \mathcal{A}_F)) \) is

\[ D_A = D + \eta_D(A) = 1 \otimes iD + A_\mu \otimes i\gamma^\mu + (i\mathcal{D}_F + \phi) \otimes 1, \]

where the gauge field \( A_\mu \) and the Higgs field \( \phi \) are given by

\[
A_\mu = \begin{pmatrix}
0 & 0 \\
0 & -2\Lambda_\mu \\
Q_\mu - \Lambda_\mu
\end{pmatrix}, \quad \phi = \begin{pmatrix}
0 & 0 & m_v\overline{\phi_1} & m_v\overline{\phi_2} \\
0 & 0 & -m_e\phi_2 & m_e\phi_1 \\
-m_v\phi_1 & m_e\overline{\phi_2} & 0 & 0 \\
-m_v\overline{\phi_2} & -m_e\phi_1 & 0 & 0
\end{pmatrix},
\]

for the gauge fields \((\Lambda_\mu, Q_\mu) \in C_c^\infty (M, i\mathbb{R} \oplus \mathfrak{su}(2))\) and the Higgs field \((\phi_1, \phi_2) \in C_c^\infty (M, \mathbb{C}^2)\).
Fluctuations

- We consider the almost-commutative manifold

\[ F_{EW} \times M := \left( C_c^\infty(M, \mathcal{A}_F \otimes \mathcal{A}_F^{\text{op}}), \mathcal{H}_F \otimes L^2(S), 1 \otimes iD + iD_F \otimes 1, 1 \otimes \mathcal{J}_M \right). \]

- **Proposition:** The fluctuation of \( D := 1 \otimes iD + iD_F \otimes 1 \) by \( A \in \text{Pert}(C_c^\infty(M, \mathcal{A}_F)) \) is

\[ D_A = D + \eta_D(A) = 1 \otimes iD + A_\mu \otimes i\gamma^\mu + (iD_F + \phi) \otimes 1, \]

where the **gauge field** \( A_\mu \) and the **Higgs field** \( \phi \) are given by

\[ A_\mu = \begin{pmatrix} 0 & 0 \\ 0 & -2\Lambda_\mu \\ Q_\mu - \Lambda_\mu \end{pmatrix}, \quad \phi = \begin{pmatrix} 0 & 0 & m_v\phi_1 & m_v\phi_2 \\ 0 & 0 & -m_e\phi_2 & m_e\phi_1 \\ -m_v\phi_1 & m_e\phi_2 & 0 & 0 \\ -m_v\phi_2 & -m_e\phi_1 & 0 & 0 \end{pmatrix}, \]

for the gauge fields \((\Lambda_\mu, Q_\mu) \in C_c^\infty(M, i\mathbb{R} \oplus \mathfrak{su}(2))\) and the Higgs field \((\phi_1, \phi_2) \in C_c^\infty(M, \mathbb{C}^2)\).
The Krein action

Consider \( \xi = \nu_R \hat{\otimes} \psi^v_R + e_R \hat{\otimes} \psi^e_R + \nu_L \hat{\otimes} \psi^v_L + e_L \hat{\otimes} \psi^e_L \in \mathcal{H}^0 \), and define

\[
\Psi_L := \begin{pmatrix} \psi^v_L \\ \psi^e_L \end{pmatrix} \in L^2(S)^0 \otimes \mathbb{C}^2, \quad \Psi_R := \begin{pmatrix} \psi^v_R \\ \psi^e_R \end{pmatrix} \in L^2(S)^1 \otimes \mathbb{C}^2,
\]

\[
\Psi := \Psi_L + \Psi_R \in L^2(S) \otimes \mathbb{C}^2.
\]

**Proposition:** The Krein action for \( F_{EW} \times M \) is given by

\[
S_{EW}[\Psi, A] = \langle \Psi \mid i \slashed{D} \Psi \rangle + \langle \psi^e_R \mid -2i \gamma^\mu \Lambda_\mu \psi^e_R \rangle + \langle \Psi_L \mid i \gamma^\mu (Q_\mu - \Lambda_\mu) \Psi_L \rangle + \langle \Psi_R \mid \Phi \Psi_L \rangle + \langle \Psi_L \mid \Phi^* \Psi_R \rangle,
\]

where the Higgs field \((\phi_1, \phi_2)\) acts via

\[
\Phi := \begin{pmatrix} -m_v(\phi_1 + 1) & -m_v \phi_2 \\ m_e \phi_2 & -m_e(\phi_1 + 1) \end{pmatrix}.
\]
The Krein action

Consider \( \xi = \nu_R \otimes \psi_R^v + e_R \otimes \psi_R^e + \nu_L \otimes \psi_L^v + e_L \otimes \psi_L^e \in \mathcal{H}^0 \), and define

\[
\Psi_L := \begin{pmatrix} \psi_L^v \\ \psi_L^e \end{pmatrix} \in L^2(S)^0 \otimes \mathbb{C}^2, \quad \Psi_R := \begin{pmatrix} \psi_R^v \\ \psi_R^e \end{pmatrix} \in L^2(S)^1 \otimes \mathbb{C}^2,
\]

\[
\Psi := \Psi_L + \Psi_R \in L^2(S) \otimes \mathbb{C}^2.
\]

**Proposition:** The Krein action for \( F_{EW} \times M \) is given by

\[
S_{EW}[\Psi, A] = \langle \Psi | iD\Psi \rangle + \langle \psi_R^e | -2i \gamma^\mu \Lambda_\mu \psi_R^e \rangle + \langle \Psi_L | i \gamma^\mu (Q_\mu - \Lambda_\mu) \Psi_L \rangle
\]

\[
+ \langle \Psi_R | \Phi \Psi_L \rangle + \langle \Psi_L | \Phi^* \Psi_R \rangle,
\]

where the Higgs field \((\phi_1, \phi_2)\) acts via

\[
\Phi := \begin{pmatrix} -m_v (\phi_1 + 1) & -m_v \phi_2 \\ m_e \phi_2 & -m_e (\phi_1 + 1) \end{pmatrix}.
\]
Majorana masses

- On $\mathcal{H}_F := \mathcal{H}_F \oplus \mathcal{H}_F$, we consider the operators

$$
\hat{D}_F := \begin{pmatrix} D_F & -D_M^* \\ D_M & D_F \end{pmatrix}, \quad \hat{J}_F := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
$$

$$
\hat{\Gamma}_F := \begin{pmatrix} \Gamma_F & 0 \\ 0 & -\Gamma_F \end{pmatrix}, \quad \hat{J}_F := \begin{pmatrix} 0 & \text{c.c.} \\ \text{c.c.} & 0 \end{pmatrix},
$$

where $D_M \nu_R := im_R \overline{\nu_R}$ and $D_M \nu_R = D_M \nu_L = D_M e_L = 0$.

- Define $\hat{\pi}: A_F \to B(\mathcal{H}_F \oplus \mathcal{H}_F)$ and $\hat{\pi}^{\text{op}}: A_F^{\text{op}} \to B(\mathcal{H}_F \oplus \mathcal{H}_F)$ by

$$
\hat{\pi}(a) := \pi(a) \oplus \pi^{\text{op}}(a^t), \quad \hat{\pi}^{\text{op}}(a) := \hat{J}_F \hat{\pi}(a^*) \hat{J}_F.
$$

- We obtain a new finite space $\hat{F}_{EW} := (A_F \otimes A_F^{\text{op}}, \mathcal{H}_F, \hat{D}_F, \hat{J}_F)$ with grading operator $\hat{\Gamma}_F$ and additionally with a real structure $\hat{J}_F$.  

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Majorana masses

- On $\mathcal{H}_F := \mathcal{H}_F \oplus \mathcal{H}_{\overline{F}}$, we consider the operators

  $\hat{\mathcal{D}}_F := \begin{pmatrix} \mathcal{D}_F & -\mathcal{D}_M^* \\ \mathcal{D}_M & \mathcal{D}_F \end{pmatrix}, \quad \hat{\mathcal{J}}_F := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$

  $\hat{\Gamma}_F := \begin{pmatrix} \Gamma_F & 0 \\ 0 & -\Gamma_F \end{pmatrix}, \quad \hat{J}_F := \begin{pmatrix} 0 & \text{c.c.} \\ \text{c.c.} & 0 \end{pmatrix},$

  where $\mathcal{D}_M \nu_R := im_R \overline{\nu}_R$ and $\mathcal{D}_M e_R = \mathcal{D}_M \nu_L = \mathcal{D}_M e_L = 0$.

- Define $\hat{\pi}: \mathcal{A}_F \to \mathcal{B}(\mathcal{H}_F \oplus \mathcal{H}_{\overline{F}})$ and $\hat{\pi}^{\text{op}}: \mathcal{A}_F^{\text{op}} \to \mathcal{B}(\mathcal{H}_F \oplus \mathcal{H}_{\overline{F}})$ by

  $\hat{\pi}(a) := \pi(a) \oplus \pi^{\text{op}}(a^t), \quad \hat{\pi}^{\text{op}}(a) := \hat{J}_F \hat{\pi}(a^*) \hat{J}_F.$

- We obtain a new finite space $\hat{\mathcal{A}}_{EW} := (\mathcal{A}_F \otimes \mathcal{A}_F^{\text{op}}, \mathcal{H}_F, \hat{\mathcal{D}}_F, \hat{\mathcal{J}}_F)$ with grading operator $\hat{\Gamma}_F$ and additionally with a real structure $\hat{J}_F$. 
Majorana masses

- On $\hat{\mathcal{H}}_F := \mathcal{H}_F \oplus \mathcal{H}_F$, we consider the operators
  
  \[
  \hat{\mathcal{D}}_F := \begin{pmatrix} \mathcal{D}_F & -\mathcal{D}_M^* \\ \mathcal{D}_M & \mathcal{D}_F \end{pmatrix}, \quad \hat{\mathcal{J}}_F := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
  \]
  
  \[
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  where $\mathcal{D}_M\nu_R := \text{im} R\nu_R$ and $\mathcal{D}_M e_R = \mathcal{D}_M\nu_L = \mathcal{D}_Me_L = 0$.

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- We obtain a new finite space $\hat{\mathcal{E}}_{EW} := (\mathcal{A}_F \circ \mathcal{A}^{\text{op}}_F, \mathcal{H}_F, \hat{\mathcal{D}}_F, \hat{\mathcal{J}}_F)$ with grading operator $\hat{\Gamma}_F$ and additionally with a real structure $\hat{J}_F$. 

The Krein action + Majorana masses

- Consider $\hat{F}_{EW} \times M$ with the real structure $J := \hat{J}_F \otimes J_M$.

- Following [Bar07], consider $\eta \in H^0$ such that $J\eta = \eta$. Then $\eta = \bar{\zeta} + J\zeta$, with $\zeta \in (H_F \otimes L^2(\mathcal{S}))^0$ as before. We have

$$\langle \eta | D_A \eta \rangle = \langle \bar{\zeta} | D_A \bar{\zeta} \rangle + \langle J\zeta | D_A \bar{\zeta} \rangle + \langle \bar{\zeta} | D_A J\zeta \rangle + \langle J\zeta | D_A J\bar{\zeta} \rangle.$$

- One finds that $\langle J\zeta | D_A J\bar{\zeta} \rangle = \langle \bar{\zeta} | D_A \bar{\zeta} \rangle = S_{EW}[\Psi, A]$.

- The new contributions are

$$\langle J\zeta | D_A \bar{\zeta} \rangle = -m_R \langle J M \psi^\nu_R | \psi^\nu_R \rangle, \quad \langle \bar{\zeta} | D_A J\zeta \rangle = -m_R \langle \psi^\nu_R | J M \psi^\nu_R \rangle.$$

- Summarising, we obtain the new action $S_{EW+M}$ given by

$$S_{EW+M}[\Psi, A] = 2S_{EW}[\Psi, A] - m_R \langle \psi^\nu_R | J M \psi^\nu_R \rangle - m_R \langle J M \psi^\nu_R | \psi^\nu_R \rangle.$$
The Krein action + Majorana masses

- Consider $\hat{F}_{EW} \times \mathcal{M}$ with the real structure $J := \hat{J}_F \otimes J_M$.
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The fermionic action in Lorentzian signature (the Krein action) matches exactly with the physical Lagrangian.

- The action is purely classical; there are no anti-commuting variables.

- Majorana masses can be described by giving the finite space a Krein structure as well.
Conclusion

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References


