Cuntz-Pimsner Algebras and Mapping Cone Exact Sequences

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Gysin sequences naturally appear in the study of

- $\mathrm{U}(1)$-gauge theory;
- T-duality;
- Chern Simons field theories.

Both the classical Gysin sequence and the six term exact sequences by Pimsner show some connection to the mapping cone construction.

We have seen an instance of the Gysin sequence in K-theory for circle bundles. It has the form of a cyclic six term exact sequence:

$$\begin{array}{ccccc} K^0(X) & \xrightarrow{\alpha} & K^0(X) & \xrightarrow{\pi^*} & K^0(P) \\ \delta_{1,0}\uparrow & & & & \downarrow\delta_{0,1} \\ K^1(P) & \xleftarrow[\pi^*]{} & K^1(X) & \xleftarrow[\alpha]{} & K^1(X) \end{array} , \tag{1}$$

where α is the mutiliplication by the Euler class

$$\chi(L) = 1 - [L] \tag{2}$$

of the line bundle $L \to X$ with associated circle bundle $\pi : P \to X$.

More generally, let V be a Hermitian vector bundle over X.
Denote by $B(V)$ and $S(V)$ the ball bundle and the sphere bundle of V, resp.
$S(V)$ is closed in $B(V)$.
Then $B(V) - S(V)$ denotes the open ball bundle.

Let $K^*(B(V), S(V))$ denote the relative K-theory group.
Then we have a six term exact sequence in topological K-theory:

$$\begin{tikzcd} K^0(B(V), S(V)) \arrow[r] & K^0(B(V)) \arrow[r] & K^0(S(V)) \arrow[d, "\delta_{\mathbf{0},\mathbf{1}}"] \\ K^1(S(V)) \arrow[u, "\delta_{\mathbf{1},\mathbf{0}}"] & K^1(B(V)) \arrow[l] & K^1(B(V), S(V)) \arrow[l] \end{tikzcd} , \qquad (3)$$

- Since $B(V)$ is compact and $S(V)$ is closed in $B(V)$ $K^*(B(V), S(V)) \simeq K^*(B(V) - S(V))$.
- The total space of the open ball bundle is homeomorphic to X. This yields $K^*(B(V), S(V)) \simeq K^*(X)$.
- Finally, the total space of $B(V)$ is homotopic to X, hence we have isomorphisms $K^*(B(V)) \simeq K^*(X)$.

We can use these facts to simplify the exact sequence.

Then one gets

$$\begin{tikzcd} K^0(X) \arrow[r, "\alpha"] & K^0(X) \arrow[r, "\pi^*"] & K^0(S(V)) \arrow[d, "\delta_{\mathbf{0},\mathbf{1}}"] \\ K^1(S(V)) \arrow[u, "\delta_{\mathbf{1},\mathbf{0}}"] & K^1(X) \arrow[l, "\pi^*"] & K^1(X) \arrow[l, "\alpha"] \end{tikzcd} , \tag{4}$$

which is refered to as the *Gysin exact sequence*.
Here π^* is the map induced by the projection $\pi : S(V) \to X$ and α is the cup-product with $1 - \chi(V)$, where $\chi(V)$ is the Euler class of the bundle.

Open problem: how to construct an analogue of this sequence for noncommutative spaces.

For the line bundle case: SMEBs and their corresponding Pimsner algebras. What about vector bundles? A (A, B) bi-Hilbertian bimodule is an

(A, B)-bimodule ${}_A E_B$ endowed with a right Hilbert module structure $\langle \cdot, \cdot \rangle_B$ and a left Hilbert module structure ${}_A\langle \cdot, \cdot \rangle$ such that there exist $\lambda, \lambda' > 0$ satisfying

$$\lambda' \| \langle x, x \rangle_B \| \leq \| {}_A\langle x, x \rangle \| \leq \lambda \| \langle x, x \rangle_B \|.$$

When $B = A$: bi-Hilbertian bimodule over A.
A bi-Hilbertian bimodule is a special case of a C*-correspondence (E, ϕ), which is a right Hilbert A-module E_A together with a left action $\phi : A \to \mathrm{End}^*_A(E)$.

We assume E to be countably generated. Then there exist a frame for E, i.e. vectors $\{e_j\}_{j\geq 1} \subset E$ such that

$$\sum_{j\geq 1} |e_j\rangle\langle e_j| \to \mathrm{I}_E \quad \text{strictly.}$$

The quantity

$$e^{\beta} := \sum_{j\geq 1} {}_A\langle e_j | e_j\rangle \in M(A) \tag{5}$$

if and only if the $\phi(A) \subseteq \mathcal{K}_A(E)$.

If this happens, E is said to have *finite right Jones-Watatani index*.

For A unital, a module of finite right index is finitely generated projective (hence a noncommutative vector bundle).

We forget the left inner product and look at E as a correspondence over A, that we assume to be injective.

We can take interior tensor product of correspondences.

For every $n \in \mathbb{N}$ we define

$$E^{(n)} := \begin{cases} E^{\widehat{\otimes}_\phi n} & n > 0 \\ A & n = 0 \end{cases}.$$

Out of these we construct the Fock module

$$\mathcal{F}_E := \bigoplus_{n \in \mathbb{N}} E^{(n)}.$$

We have natural creation and annihilation operators $S_\xi, S_\xi^* : \mathcal{F}_E \to \mathcal{F}_E$:

$$S_\xi(\eta_1 \cdots \otimes \eta_n) = \xi \otimes \eta_1 \cdots \otimes \eta_n \qquad S_\xi(a) = \xi a$$

$$S_\xi^*(\eta_1 \cdots \otimes \eta_n) = \langle \xi, \eta_1 \rangle \otimes \eta_2 \cdots \otimes \eta_n \qquad S_\xi^*(a) = 0$$

Definition

The *Toeplitz algebra* of E, denoted $\mathcal{T}_E$, is the smallest C^*-subalgebra of $\mathrm{End}_A^*(\mathcal{F}_E)$ which contains the operators $S_\xi : \mathcal{F}_E \to \mathcal{F}_E$ for all $\xi \in E$.

Definition

The *Pimsner algebra* of E, denoted $\mathcal{O}_E$, is the quotient algebra in the extension

$$0 \longrightarrow \mathcal{K}(\mathcal{F}_E) \longrightarrow \mathcal{T}_E \xrightarrow{\pi} \mathcal{O}_E \longrightarrow 0. \tag{6}$$

Whenever A is separable and nuclear, so are $\mathcal{T}_E$ and $\mathcal{O}_E$.

Then the above sequence admits a completely positive splitting.

The representation of $\mathrm{U}(1)$ on $\mathcal{F}_E$ given by

$$t \circ x = t^n x \quad \forall t \in S^1,\ x \in E^{(n)}$$

induces an circle action γ on $\mathcal{O}_E$ called the *gauge action*.
We denote by $\mathcal{O}_E^\gamma$ the fixed point for this action.

$$\rho(x) = \int_{S^1} \gamma_z(x) dz. \tag{7}$$

Proposition

E is a self-Morita equivalence bimodule if and only if $\mathcal{O}_E^\gamma \simeq A$.

Since the defining extension . is semi-split, it induces six term exact sequences in KK-theory.

These simplify by using:

- The class of the correspondence $E \in KK(A, A)$;
- The class of the Morita equivalence $[\mathcal{F}_E] \in KK(\mathcal{K}(\mathcal{F}_E), A)$;
- The class of the KK-equivalence $[\alpha]^{-1} \in KK(\mathcal{T}_E, A)$, which is the inverse to the class of the inclusion $\alpha : A \hookrightarrow \mathcal{T}_E$.

These satisfy:

$$[\mathcal{F}_E] \otimes_A (1 - [E]) = [j] \otimes_{\mathcal{T}_E} [\alpha]^{-1}$$

We denote by $[\partial] \in KK_1(\mathcal{O}_E, A)$ the class of the product $[\mathrm{ext}] \otimes_{\mathcal{K}(\mathcal{F}_E)} [\mathcal{F}_E]$.
For $C = \mathbb{C}$ we get exact sequences in K-theory

$$\begin{tikzcd}
K_0(A) \arrow[r, "1-[E]"] & K_0(A) \arrow[r, "j_*"] & K_0(\mathcal{O}_E) \arrow[d, "{[\partial]}"] \\
K_1(\mathcal{O}_E) \arrow[u, "{[\partial]}"] & K_1(A) \arrow[l, "j_*"] & K_1(A) \arrow[l, "1-[E]"]
\end{tikzcd} ,$$

and in K-homology

$$\begin{tikzcd}
K^0(A) \arrow[d, "{[\partial]}"] & K^0(A) \arrow[l, "1-[E]"] & K^0(\mathcal{O}_E) \arrow[l, "j^*"] \\
K^1(\mathcal{O}_E) \arrow[r, "j^*"] & K^1(A) \arrow[r, "1-[E]"] & K^1(A) \arrow[u, "{[\partial]}"]
\end{tikzcd} . \quad (8)$$

In the case of a self-Morita equivalence bimodule, the conditional expectation ρ defines an A-valued inner product on $\mathcal{O}_E$.
We denote the completion with Ξ_A.

Then the generator of the circle action, i.e the the number operator, defines an unbounded self-adjoint regular operator D on Ξ_A.
We obtain ll defined unbounded Kasparov module $(\mathcal{O}_E, \Xi_A, D)$
The connecting homomorphism is realised as a Kasparov product with the class $[(\mathcal{O}_E, \Xi_A, D)] \in KK^1(\mathcal{O}_E, A)$.

- In the case of SMEBs, the Pimsner algebra can be thought of as a noncommutative associated circle bundle construction.
- The corresponding six-term exact sequence generalises the Gysin sequence for circle bundles.
- Unfortunately, this analogy doesn't seem to work anymore for higher rank vector bundles.

Motivated by explicit computations (cf. A. Brain & Landi) we would like to compare the Gysin exact sequences with the exact sequences associated to the mapping cone of the inclusion $A \to \mathcal{O}_E$.

The *cone* over a C*-algebra A is the C*-algebra

$$CA := \{f \in C([0,1), A) \mid f(0) = 0\},$$

with point-wise operation and norm the supremum norm.
The *mapping cone* for a morphism $\alpha : A \to B$ is the C*-algebra

$$M(A,B) := \{a \oplus f \in A \oplus CB \mid f(1) = \alpha(a)\}.$$

The mapping cone $M(A, B)$ is related to A and B via the exact sequence

$$0 \longrightarrow SB \xrightarrow{\iota} M(A, B) \xrightarrow{\pi} A \longrightarrow 0 \ ,$$

where $\iota(f \otimes a)(t) := f(t)a$ and π is the projections $\pi(a \oplus f) = a$. The exact sequence admits a completely positive cross section given by $\phi(a) = (a, (1 - t)\alpha(a))$. Hence it induces six term exact sequences in KK-theory.

For the inclusion $\iota : A \to \mathcal{O}_E$:

$$0 \longrightarrow S\mathcal{O}_E \xrightarrow{j_*} M(A, O_E) \xrightarrow{\mathrm{ev}} A \longrightarrow 0, \tag{9}$$

where $\mathrm{ev}(f) = f(0)$ and $j(g \otimes a)(t) = g(t)a$.
We focus on the case of K-theory.

$$\begin{tikzcd} K_0(A) \arrow[r, "\partial'"] & K_0(S\mathcal{O}_E) \arrow[r, "j_*"] & K_1(M) \arrow[d, "\mathrm{ev}_*"] \\ K_0(M) \arrow[u, "\mathrm{ev}_*"] & K_1(S\mathcal{O}_E) \arrow[l, "j_*"] & K_1(A) \arrow[l, "\partial'"] \end{tikzcd} \tag{10}$$

We use the identification $\mathrm{Bott} : K_j(\mathcal{O}_E) \to K_{j+1}(S\mathcal{O}_E)$ to define a map $j_*^B : K_i(\mathcal{O}_E) \to K_{i+1}(M)$ given by $j_* \circ \mathrm{Bott}$.

We now compare the six term exact sequences induced by the mapping cone of the inclusion with the Gysin six term exact sequences.

$$\begin{tikzcd} \cdots \arrow[r, "\iota_*"] & K_0(\mathcal{O}_E) \arrow[r, "j_*^B"] \arrow[d, "="] & K_1(M) \arrow[r, "\mathrm{ev}_*"] \arrow[d, "?"] & K_1(A) \arrow[r, "\iota_*"] \arrow[d, "="] & K_1(\mathcal{O}_E) \arrow[r, "j_*^B"] \arrow[d, "="] & \cdots \\ \cdots \arrow[r, "\iota_*"] & K_0(\mathcal{O}_E) \arrow[r, "\partial"] & K_1(A) \arrow[r, "1-[E]"] & K_1(A) \arrow[r, "\iota_*"] & K_1(\mathcal{O}_E) \arrow[r, "\partial"] & \cdots \end{tikzcd} \tag{11}$$

Cuntz & Skandalis ([CK86]): for any semi-split extension of C*-algebras

$$0 \longrightarrow J \longrightarrow E \xrightarrow{\pi} Q \longrightarrow 0,$$

the ideal J is KK-equivalent to the mapping cone $M(E, Q)$ of the morphism π. In the induced six term exact sequence in KK-theory, the connecting homomorphism is given up to Bott periodicity, by the Kasparov product with

$$\iota^*([u]) \in KK(SQ, J)$$

where $u \in KK(M(E,Q), J))$ is the KK-equivalence and $\iota : SQ \to M(E, Q)$ is the natural inclusion.

Meyer & Nest ([MR06]): the KK category is a *triangulated category*, whose exact triangles are mapping cone triangles with isomorphisms given by KK-equivalence (cf. [MR06]). Indeed, for every semisplit extension with quotient map π, one has an isomorphism of triangles where all vertical arrows are KK-equivalences.

For the defining extension

$$\begin{tikzcd} S\mathcal{O}_E \arrow[r] \arrow[d, "="] & M(\mathcal{T}_E, \mathcal{O}_E) \arrow[r] \arrow[d] & \mathcal{T}_E \arrow[r] \arrow[d, "\alpha"] & \mathcal{O}_E \arrow[d, "="] \\ S\mathcal{O}_E \arrow[r] & \mathcal{K}(\mathcal{F}_E) \arrow[r] & \mathcal{T}_E \arrow[r] & \mathcal{O}_E \end{tikzcd} \tag{12}$$

Using the KK-equivalence between A and $\mathcal{T}_E$ and the natural Morita equivalence between A and $\mathcal{K}(\mathcal{F}_E)$, one can show that one has an isomorphism of mapping cone triangles

$$\begin{tikzcd} S\mathcal{O}_E \arrow[r] \arrow[d, "="] & M(\mathcal{T}_E, \mathcal{O}_E) \arrow[r] \arrow[d] & \mathcal{T}_E \arrow[r] \arrow[d] & \mathcal{O}_E \arrow[d, "="] \\ SO_E \arrow[r] & M(A, \mathcal{O}_E) \arrow[r] & A \arrow[r] & \mathcal{O}_E \end{tikzcd} \tag{13}$$

This follows from the axioms of a triangulated category which imply that the mapping cone of $A \to O_E$ is unique up to a (non-canonical) isomorphism in KK.

Combining the two isomorphisms of triangles, one obtains the isomorphism of exact triangles

$$\begin{tikzcd}
S\mathcal{O}_E \arrow[r] \arrow[d, "="] & M(A,\mathcal{O}_E) \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, "\alpha"] & \mathcal{O}_E \arrow[d, "="] \\
S\mathcal{O}_E \arrow[r] & \mathcal{K}(\mathcal{F}_E) \arrow[r] & \mathcal{T}_E \arrow[r] & \mathcal{O}_E
\end{tikzcd}$$

which induces an isomorphism of the corresponding KK-exact sequences.

The missing map can be realised as the Kasparov product with the class

$$[\tilde{\alpha}] \otimes_{M(\mathcal{T}_E, \mathcal{O}_E)} [u] \otimes_{\mathcal{K}(\mathcal{F}_E)} [\mathcal{F}_E] \in KK(M, A),$$

- $\tilde{\alpha} : M(A, \mathcal{O}_E) \to M(\mathcal{T}_E, \mathcal{O}_E)$ is the inclusion of mapping cones induced by $\alpha : A \to \mathcal{T}_E$,
- $[\mathcal{F}_E] \in KK(\mathrm{K}(\mathcal{F}_E), A)$ is the class of the Morita equivalence,
- $[u] \in KK(M(\mathcal{T}_E, \mathcal{O}_E), \mathrm{K}(\mathcal{F}_E))$ is the KK-equivalence.

We will provide the isomorphism between the associated six-term exact sequences explicitly at the level of unbounded KK-cycles

Goffeng, Mesland & Rennie ([GMR15]): an unbounded representative for the class of $[\partial]$, under some technical assumptions on the bimodule E.
Construct a positive A-bilinear expectation

$$\phi_\infty : \mathcal{O}_E \to A.$$

This will furnish $\mathcal{O}_E$ with an A-valued inner product

$$\langle S_1, S_2 \rangle_A := \phi_\infty(S_1^* S_2).$$

Denote by Ξ_A the completion of $\mathcal{O}_E$ in the norm induced by this inner product.

The module Ξ_A admits a decomposition

$$\Xi_A = \bigoplus_{n\in\mathbb{Z},\ r\geq \max\{0,n\}} \Xi_{n,r} \tag{14}$$

into bimodules of finite Jones-Watatani index.

We denote the projections onto these sub-modules by $P_{n,r}$ and consider the operator

$$Q = \sum_{n=0}^{\infty} P_{n,n},$$

with respect to the above decomposition (14).

The tuple $(O_E, \Xi_A, 2Q - 1)$ is an odd Kasparov module representing the class of the extension (6).

The projection Q has range isometrically isomorphic to the Fock module $\mathcal{F}_E$.

To construct an unbounded representative, one defines $D = \sum_{n,r} \psi(n,r) P_{n,r}$ and ψ is a suitable function.

Theorem ([GMR15],A.-Rennie for the nonunital case)

If the bi-Hilbertian A-bimodule E satisfies the Assumptions of [GMR15], then the tuple $(\mathcal{O}_E, \Xi_A, D)$ is an odd unbounded Kasparov module representing the class of the extension (6). The spectrum of D can be chosen to consist of integers with bi-Hilbertian A-bimodule eigenspaces of finite right Watatani index, and non-negative spectral projection Q.

When E is a self-Morita equivalence bimodule, $\Phi_\infty : O_E \to A$ coincides with the expectation $\rho : \mathcal{O}_E \to \mathcal{O}_E^\gamma \simeq A$ discussed in .

Therefore

$$\Xi_A = \bigoplus_{n \in \mathbb{Z}} E^{\otimes n}$$

with the convention that $E^{\otimes(-|n|)} = \overline{E}^{\otimes |n|}$, where $\overline{E}$ is the conjugate module, which agrees with the C*-algebraic dual of E.

One gets back the class $[(\mathcal{O}_E, \Xi_A, D)] \in KK(\mathcal{O}_E, A)$ described before.

As noted earlier, D has discrete spectrum and commutes with the left action of A, hence we have $\iota^*_{A,O_E}[(\mathcal{O}_E, \Xi_A, D)] = 0$.
There is a class $[\widehat{D}] \in KK(M(A, O_E), A)$ such that $j^{B*}[\widehat{D}] = [(\mathcal{O}_E, \Xi_A, D)]$.
An explicit unbounded representative for the class $[\widehat{D}]$, provided by the main result of [CPR10]. One obtains commutativity of

$$\begin{tikzcd}
\cdots \arrow[r, "\iota_*"] & K_0(\mathcal{O}_E) \arrow[r, "j^B_*"] \arrow[d, "="] & K_1(M) \arrow[r, "\mathrm{ev}_*"] \arrow[d, "\widehat{D}"] & \cdots \\
\cdots \arrow[r, "\iota_*"] & K_0(\mathcal{O}_E) \arrow[r, "\partial"] & K_1(A) \arrow[r, "1-[E]"] & \cdots
\end{tikzcd} \tag{15}$$

$$\begin{tikzcd} \cdots \arrow[r, "j_*^B"] & K_i(M) \arrow[r, "\mathrm{ev}_*"] \arrow[d, "\widehat{D}"] & K_i(A) \arrow[r, "\iota_*"] \arrow[d, "="] & \cdots \\ \cdots \arrow[r, "\partial"] & K_i(A) \arrow[r, "1-[E]"] & K_i(A) \arrow[r, "\iota_*"] & \cdots \end{tikzcd} \tag{16}$$

- We use the characterisation of the K-theory group $K_0(M)$ due to Putnam.
- For any $v \in K_*(M)$, we need to evaluate the product $[v] \otimes_{\mathcal{O}_E} [\widehat{D}] \otimes_A ([\mathrm{Id}_{KK(A,A)}] - [E])$. Our strategy is to use [CPR10], to find that the latter product is given by an index.
- This works for $i = 0$. For $i = 1$ we have to adapt the argument to suspended algebras.

The argument works readily thanks to the following result

Proposition

Let E be a bi-Hilbertian A-bimodule with finite right Watatani index. We can define a suspended bi-Hilbertian SA-bimodule SE over the suspension SA. Then SE has finite right Watatani index given by $1 \otimes e^\beta$ where $e^\beta \in M(A)$ is the right Watatani index of E. If E_A is full so too is SE_{SA} and if the left action of A on E is injective, so too is the left action of SA on SE.

Theorem (A.-Rennie 16)

Let E be a bi-Hilbertian A-bimodule of finite right Watatani index, full as a right module with injective left action, and satisfying the assumptions of [GMR15]. $(\mathcal{O}_E, \Xi_A, D)$ be the unbounded representative of the defining extension and $(M(A, \mathcal{O}_E), \widehat{\Xi}_A, \widehat{D})$ the lift to the mapping cone.
Then

$$\cdot \otimes_{M(A,\mathcal{O}_E)} [(M(A, \mathcal{O}_E), \widehat{\Xi}_A, \widehat{D})] : \; K_*(M(A, \mathcal{O}_E)) \to K_*(A)$$

is an isomorphism that makes diagrams in K-theory commute.
If furthermore the algebra A belongs to the bootstrap class, the Kasparov product with the class $[(M(A, \mathcal{O}_E), \widehat{\Xi}_A, \widehat{D})] \in KK(M(A, \mathcal{O}_E), \mathcal{O}_E)$ is a KK-equivalence.

- Pimsner algebras for SMEBs are the analogue of associated circle bundles.
- We made the relationship between Pimsner's exact sequences and mapping cone exact sequences explicit.
- In order to deal with suspensions, we generalised the results of [GMR15] to non-unital C^*-algebras.
- Open problems:
  - Commutativity of diagrams in KK-theory.
  - Uniqueness of such a class.
  - Noncommutative sphere bundles.

Thank you very much for your attention!

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