# Homology and K-theory of torsion free ample groupoids and Smale spaces

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# invariants for singular spaces

**Question**

*how do we understand singular spaces?*

- how to make sense of (co)homology for such spaces?
- how do we compute such invariants?

# Deligne's approach

to understand cohomology of $X$ (like non-smooth algebraic variety):

- take a covering map from smooth one $Y \to X$
- $Y_n =$ '$Y \times_X \cdots \times_X Y$' ($n+1$ times), simplicial object $(Y_n)_{n=0}^{\infty}$
- collect complexes $C^{n,k}$ computing cohomology of $Y_n$

$\rightsquigarrow$ cohomology of $X$ from total complex of $(C^{n,k})_{n,k}$

$$E_2^{p,q} = H^p(Y_q) \Rightarrow H^{p+q}(X)$$

also applicable for $BG$, simplicial spaces, …

# groupoid homology after Crainic–Moerdijk

given

- locally compact (locally) Hausdorff étale groupoid $G$
- coefficient in $G$-sheaf $A$ (could be $\underline{\mathbb{Z}}$)

simplicial space: nerve

$$G^{(k)} = \{(g_1, \dots, g_k) \mid sg_j = rg_{j+1}\}$$

$\rightsquigarrow$ simplicial space $(G^{(n)})_{n=0}^{\infty}$

- concatenate (or drop first / last) $G^{(n+1)} \to G^{(n)}$
- insert unit $G^{(n-1)} \to G^{(n)}$

for cohomology of space:

- (c-)soft resolution $A \to C^0 \to C^1 \to$

$\rightsquigarrow$ bicomplex $\Gamma_c(G^{(n)}, C^m)$, homology $H_k(G, A)$

# Connes's approach

given locally compact (locally) Hausdorff groupoid $G$, take

- convolution product on $C_c(G)$ (or $C_c^\infty(G)$, ...)
- C*-algebraic completion $C^*_r G$

and look at (co)homological invariants like

- operator K-groups $K_*(C^*_r G)$
- cyclic (co)homologies of $C_c^\infty(G)$

# groupoid homology to K-theory

**Theorem (P.–Y.)**

*$G$: locally compact Hausdorff groupoid;*

- *ample; totally disconnected, $s, r$ local homeomorphisms*
- *stabilizers $G_x^x = \{g \mid sg = rg = x\}$ are torsion free*
- *satisfies the strong Baum–Connes conjecture (e.g., amenable)*

*then $\exists$ spectral sequence*

$$E^2_{p,q} = H_p(G, \mathbb{Z}) \otimes K_q(\mathbb{C}) \Rightarrow K_{p+q}(C^*_r G)$$

# examples

Cantor system $\Gamma \curvearrowright X$ with torsion free stabilizers

- $\Gamma$: discrete group with the Haagerup property
- $X$: Cantor space

$$E^2_{p,q} = H_p(\Gamma, C(X, \mathbb{Z})) \otimes K_q(\mathbb{C}) \Rightarrow K_{p+q}(C^*_r\Gamma \ltimes X)$$

Smale space $(X, \phi)$ with totally disconnected stable sets

- $X$: compact metric space
- $\phi$: hyperbolic homeomorphism of $X$
- $H^s(X, \phi)$: Putnam's homology from symbolic presentation
- $R^u(X, \phi)$: unstable equivalence relation; $d(\phi^{-N}x, \phi^{-N}x') \to 0$

$$E^2_{p,q} = H^s_p(X, \phi) \otimes K_q(\mathbb{C}) \Rightarrow K_{p+q}(C^*_r R^u(X, \phi))$$

# why such correspondence?

simplicial $G$-space $(G^{(n+1)})_{n=0}^{\infty}$ (not $(G^{(n)})_{n=0}^{\infty}$!)

- concatenate (or drop last) $G^{(n+2)} \to G^{(n+1)}$
- insert unit $G^{(n)} \to G^{(n+1)}$ into other than the 0-th leg

cosimplicial $G$-C*-algebra $(C_0(G^{(n+1)}))_{n=0}^{\infty}$

- resolution of $C_0(G^{(0)})$ in $KK^G$
- $K_*(G \ltimes C_0(G^{(n+1)})) \simeq K^*(G^{(n)})$

ample $G$: $K^*(G^{(n)}) \simeq C_c(G^{(n)}, \mathbb{Z})$; defining complex of $H(G, \mathbb{Z})$

# prototype of spectral sequence

Kasparov

$\Gamma$: torsion free discrete group with 'nice' Riemannian manifold model of $E\Gamma$ (e.g., sectional curvature $\leq 0$)

$$E^2_{p,q} = H_p(\Gamma, K_q(A)) \Rightarrow K_{p+q}(\Gamma \ltimes A)$$

for $\Gamma$-$C^*$-algebra $A$

**Example (Pismner–Voiculescu)**

from $\alpha \in \mathrm{Aut}(A)$: extension

$H_0(\mathbb{Z}, K_*(A)) = K_*(A)_{\alpha\text{-coinv}} \to K_*(\mathbb{Z} \ltimes_\alpha A) \to K_{*+1}(A)^{\alpha\text{-inv}} = H_1(\mathbb{Z}, K_{*+1}(A))$,

or

$$\begin{array}{ccccc} K_0(A) & \xrightarrow{1-\alpha_*} & K_0(A) & \longrightarrow & K_0(\mathbb{Z} \ltimes_\alpha A) \\ \uparrow & & & & \downarrow \\ K_1(\mathbb{Z} \ltimes_\alpha A) & \longleftarrow & K_1(A) & \xleftarrow[1-\alpha_*]{} & K_1(A) \end{array}$$

# triangulated category and Adams spectral sequence

Christensen, Meyer–Nest, Meyer

- $\mathscr{C}$: triangulated category, suspension $\Sigma$
- $\mathscr{I}(A,B) \lhd \mathscr{C}(A,B)$: homological ideal
- $(P_n)_{n=0}^{\infty}$: $\mathscr{I}$-projective resolution of $A$
- $\mathscr{D}$: Abelian category with suspension
- $F\colon \mathscr{C} \to \mathscr{D}$: stable homological functor

approximation $P_A \in \langle \mathscr{I}\text{-proj}\rangle$; ghost $N_A \in \mathscr{I}$

$$P_A \to A \to N_A \to \Sigma P_A \quad \text{exact triangle}$$

spectral sequence

$$E^1_{p,q} = F(\Sigma^{-q} P_p) \Rightarrow F(\Sigma^{-(p+q)} P_A)$$

# triangulated category structure of $KK^G$

Le Gall; for (Hausdorff) locally compact groupoid $G$ on $X$

- continuous action of $G$: $C_0(G) \otimes_{s C_0(X)} A \to C_0(G) \otimes_{r C_0(X)} A$
- $KK^G(A, B)$ for separable $G$-C*-algebras, composition
- descent $KK^G(A, B) \to KK(G \ltimes_r A, G \ltimes_r B)$
- $KK^{\Gamma \ltimes X}(A, B) \simeq RKK^\Gamma(X; A, B)$ for $\Gamma$-$C_0(X)$-algebras

from Oyono-Oyono's result

- up to stabilization: any cycle $(E_B, T)$ for $KK^G(A, B)$ is representable by an equivariant one; $gTg^{-1} = T$
- $C^*_G \to KK^G$: universal functor with stability, homotopy invariance, split exactness
- $KK^G$ triangulated with mapping cone triangles

$$(A \to B \to C \to \Sigma A) \simeq (\Sigma B' \to Con(f) \to A' \to B')$$

# projective resolution from adjoint pairs

- $\mathscr{S}, \mathscr{T}$: triangulated categories
- $E\colon \mathscr{S} \to \mathscr{T}$, $F\colon \mathscr{T} \to \mathscr{S}$: triangulated functors, with adjunction

$$\mathscr{S}(B, FA) \simeq \mathscr{T}(EB, A)$$

- $\mathscr{I} = \ker F$: homological ideal
- $A'$ is $\mathscr{I}$-projective $\equiv \mathscr{I}(A_1, A_2)\mathscr{T}(A, A_1) = 0$ (e.g., $A' = EB$)
- $L = EF\colon \mathscr{T} \to \mathscr{T}$ comonad (comonoid of endofunctor)

**Proposition**

*simplicial object $(L^{n+1}A)_{n=0}^{\infty}$ forms an $\mathscr{I}$-projective resolution of $A \in \mathscr{T}$*

key idea: $FL^{\bullet+1}A \to FA \to 0$ exact in $\mathscr{S}$

# what to do with groupoid

$G$: étale groupoid with base $X = G^{(0)}$

- triangulated categories $\mathscr{S} = KK^X$, $\mathscr{T} = KK^G$
- triaongulated functor $E = \mathrm{Ind}_X^G \colon \mathscr{S} \to \mathscr{T}$, $A \mapsto C_0(G) \otimes_{{}_sC_0(X)} A$
- adjoint functor $F = \mathrm{Res}_X^G \colon \mathscr{T} \to \mathscr{S}$;

$$KK^X(B, A) \simeq KK^G(C_0(G) \otimes_{{}_sC_0(X)} B, A)$$

$\rightsquigarrow P_A \in \langle(\ker \mathrm{Res}_X^G)\text{-proj}\rangle$, $N_A \in \ker \mathrm{Res}_X^G$, exact triangle

$$P_A \to A \to N_A \to \Sigma P_A$$

for the stable homological functor $F'(A) = K_*(G \ltimes_r A)$:

$$E^1_{p,q} = K_q(G \ltimes_r P_p) \Rightarrow K_{p+q}(G \ltimes_r P_A)$$

# how to use Baum–Connes conjecture

**Theorem (Tu; groupoid version of Higson–Kasparov)**

*$G$: second countable (or just $\sigma$-compact) Hausdorff locally compact groupoid with Haagerup property ($\exists$ proper conditionally negative definite function; e.g., amenable)*
*then $\exists$ proper $G$-space $Z$, $G \ltimes Z$-$C^*$-algebra $P$*

- *$P$ is a continuous field of nuclear $C^*$-algebras*
- *$P \simeq C_0(X)$ in $KK^G$ ($X = G^{(0)}$)*

**Proposition**

*$G$ étale, torsion free stabilizers, and as above*
*then $P_A \simeq A$ for $KK^X$-nuclear $G$-algebras $A$*
*$P_A$: approximation of $A$ from $\mathscr{I}$-projective objects for*
*$\mathscr{I} = \ker(\mathrm{Res}^G_X \colon KK^G \to KK^X)$*

# main result, again

**Theorem (P.–Y.)**

*$G$: second countable ample groupoid on $X$;*

- *whose stabilizers $G^x_x = \{g \mid sg = rg = x\}$ are torsion free*
- *satisfies the consequence of Tu's theorem ($\exists$ proper $G$-space $Z$, $C_0(Z)$-nuclear $P$ such that $P \simeq_{KK^G} C_0(X)$)*

*$A$: $KK^X$-nuclear $G$-$C^*$-algebra (like $C_0(X)$)*

*then $\exists$ spectral sequence*

$$E^2_{p,q} = H_p(G, K_q(A)) \Rightarrow K_{p+q}(G \ltimes_r A)$$

$H_*(G, K_*(A))$: groupoid homology with coefficients

- $K_*(A)$: $C_c(G, \mathbb{Z})$-module with $C_c(G, \mathbb{Z}) K_*(A) = K_*(A)$
- homology of bar complex $(C_c(G^{(n)}, \mathbb{Z}) \otimes_{C_c(X,\mathbb{Z})} K_*(A))_{n=0}^{\infty}$

# how groupoid homology appears

$G$: étale groupoid, $A$: $G$-C*-algebra

- $L = \operatorname{Ind}_X^G \operatorname{Res}_X^G \colon KK^G \to KK^G$
- $(L^{n+1}A)_{n=0}^{\infty}$: simplicial approximation of $A$ by $(\ker \operatorname{Res}_X^G)$-projective objects
- $(K_*(G \ltimes_r L^{n+1}A))_{n=0}^{\infty}$: simplicial approximation of $K_*(G \ltimes_r A)$
- $K_*(G \ltimes_r L^{n+1}A) \simeq K_*(C_0(G^{(n)}) \otimes_{s C_0(X)} A)$

if $G$ is ample (totally disconnected):

$$K_*(C_0(G^{(n)}) \otimes_{s C_0(X)} A) \simeq C_c(G^{(n)}, \mathbb{Z}) \otimes_{C_c(X,\mathbb{Z})} K_*(A)$$

# Smale space

Smale space $(X, \phi)$:

- $X$: compact metric space
- $\phi$: "hyperbolic" dynamics on $X$

$\exists 0 < \lambda < 1 \forall x \in X$ has small neighborhood $\sim X^s(x, \varepsilon) \times X^u(x, \varepsilon)$

- $X^s(x, \varepsilon) = \{x' \mid d(x, x') < \varepsilon, d(\phi^n x, \phi^n x') \leq \lambda^n d(x, x'), n \in \mathbb{N}\}$ local stable set (contracting direction)
- $X^u(x, \varepsilon) =$ same but $n \leftrightarrow -n$

**Example**

- $X$ totally disconnected $\equiv$ subshift of finite type (symbolic dynamics)
- solenoid $\varprojlim \mathbb{T} = \{(z_0 = z_1^m, z_1 = z_2^m, \ldots)\}$, shift map
- substitution tiling system (Anderson–Putnam)

# Putnam's homology

$(X,\phi)$: Smale space (non-wandering, irreducible)
Putnam (cf. Bowen): open Markov partitions give

- $(Y,\psi)$ with $Y^u(y,\varepsilon)$ totally disconnected
- $f\colon (Y,\psi) \to (X,\phi)$ homeomorphism on stable sets (s-bijective)
- $(Z,\zeta)$ with $Z^s(y,\varepsilon)$ totally disconnected
- $f'\colon (Z,\zeta) \to (X,\phi)$ homeomorphism on unstable sets

"bisimplicial" subshit of finite type

$$\Sigma_{L,M} = \underbrace{Y \times_X \cdots \times_X Y}_{(L+1)} \times_X \underbrace{Z \times_X \cdots \times_X Z}_{(M+1)}$$

(stable) homology $H^s(X,\phi)$ from bicomplex $D^s(\Sigma_{L,M})$ (homological in $L$, cohomological in $M$)
$D^s(\Sigma)$: Krieger's dimension group

# induction from subgroupoid

Want: "induction" $\mathrm{Ind}_H^G \colon KK^H \to KK^G$ for

- $G$: étale groupoid with base $X = G^{(0)}$
- $H$: open subgroupoid with $X = H^{(0)}$
- $G/H$ might not be Hausdorff

working model: $\mathrm{Ind}_H^G A = (C_0(G) \otimes_{{}_s C_0(X)} A) \rtimes H$

$(\mathrm{Ind}_H^G \mathrm{Res}_H^G)^n C_0(X) \rightsquigarrow$ groupoid pullback $H^{\times_G n}$:

arrows of $H^{\times_G n}$:

$$\begin{array}{ccccccc}
x'_1 & \xleftarrow{g'_1} & x'_2 & \xleftarrow{g'_2} & \cdots & \xleftarrow{g'_{n-1}} & x'_n \\
h_1 \uparrow & & h_2 \uparrow & & & & h_n \uparrow \\
x_1 & \xleftarrow[g_1]{} & x_2 & \xleftarrow[g_2]{} & \cdots & \xleftarrow[g_{n-1}]{} & x_n
\end{array}$$

# transversality

want: understand $H^{\times_G n}$ for

- $(Y,\psi)$: Smale space with totally disconnected stable sets
- $X$: transversal for unstable equivalence relation (e.g., $Y^s(y,\varepsilon)$)
- $f\colon \Sigma \to Y$: factor map from SFT, bijective on stable sets
- $G = R^u(Y,\psi)|_X$, $H = R^u(\Sigma,\sigma)|_{f^{-1}X}$

**Proposition**

*given $a^1, \dots, a^n \in \Sigma$, $f(a^k)$ mutually unstably equivalent then $\exists b^k$ unstably equivalent to $a^k$, $f(b^k)$ all equal*

**Corollary**

*Morita equivalence $R^u(\Sigma,\sigma)^{\times_{R^u(Y,\psi)} n} \sim R^u(\Sigma^{\times_Y n}, \sigma^{\times n})$*

# groupoid homology to Putnam homology

**Theorem (P.–Y.)**

$(X, \phi)$: *(irreducible non-wandering) Smale space*
$H^s(X, \phi) \simeq H(R^u(X, \phi)|_T, \mathbb{Z})$ *for any transversal (stable set)* $T \in X$

- fibered product of $(Y, \psi)$: gives resolution of $\underline{\mathbb{Z}} \to C^0 \to C^1 \to$
- Putnam's bicomplex: each row computes $H_p(G, C^k)$

moreover, $K_*(C^* R^u(X, \phi))$ is finite rank