Differentiable Characters and Hopf Cyclic Cohomology

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Introduction

In the classical theory of the characteristic classes of codim$=n$ foliations, the cohomology of the Haefliger classifying space $B\Gamma_n$ serves as a universal repository but the actual metadata for their geometric construction is stored in the differentiable cohomology $H_d^*(B\Gamma_n)$. In turn the latter is computed by means of the Gelfand-Fuks cohomology $H_{\mathrm{c}}^*\left(\mathfrak{a}_n, O_n\right)$ of the Lie algebra of formal vector fields on $\mathbb{R}^n$, or equivalently by the cohomology $H^*\left(W(\mathfrak{gl}_n, O_n)_n\right)$ of the truncated Weil algebra of $\mathfrak{gl}_n$.

My goal here is to illustrate how a similar pattern is present in the noncommutative approach to the geometry of the leaf-spaces of codim$=n$ foliations, with $HP^\bullet(\mathcal{H}_n, O_n; \mathbb{C}_\delta)$ assuming the role of $H_{\mathrm{c}}^*\left(\mathfrak{a}_n, O_n\right)$ and $HP^\bullet(\Omega^*(\mathrm{GL}_n), d)_n$ that of $H^*\left(W(\mathfrak{gl}_n, O_n)_n\right)$, and also to propose a candidate for the differentiable cyclic cohomology of the convolution algebra $C_c^\infty(\Gamma_n)$. Although the basic constructs are not new (*cf.* next slide), this viewpoint serves to unify them and at the same time brings to light an avatar of the Thom isomorphism in Hopf cyclic cohomology.

[1] A. Connes, *Cyclic cohomology and the transverse fundamental class of a foliation*. In: Geometric Methods in Operator Algebras (Kyoto, 1983).

[2] A. Connes and H. Moscovici, *Hopf algebras, cyclic cohomology and the transverse index theorem*, Commun. Math. Phys. **198** (1998).

[3] A. Connes and H. Moscovici, *Differentiable cyclic cohomology and Hopf algebraic structures in transverse geometry*, In: Essays on Geometry and Related Topics (Genève, 2001).

[4] A. Gorokhovsky, *Secondary characteristic classes and cyclic cohomology of Hopf algebras*, Topology **41** (2002).

[5] H. Moscovici and B. Rangipour, *Hopf cyclic cohomology and transverse characteristic classes*, Adv. Math. **227** (2011).

[6] H. Moscovici, *Geometric construction of Hopf cyclic characteristic classes*, Adv. Math. **274** (2015).

GF isomorphism

- Lie algebra $\mathfrak{a}_n = \left\{ \sum_{i=1}^{n} f_i(x_1, \ldots, x_n) \frac{\partial}{\partial x_i};\ f_i \in \mathbb{R}[x_1, \ldots, x_n]] \right\}$;
- Weil algebra $W(\mathfrak{gl}_n) = \wedge \mathfrak{gl}_n^* \otimes S(\mathfrak{gl}_n^*)$;
- $W(\mathfrak{gl}_n)_m = W(\mathfrak{gl}_n)/S^{m+1}(\mathfrak{gl}_n^*) W(\mathfrak{gl}_n)$ truncated Weil algebra ;
- $W(\mathfrak{gl}_n, O_n)_m$ DG subalgebra of O_n-basic elements in $W(\mathfrak{gl}_n)_m$.

• The canonical connection and curvature forms $\vartheta^i_j \in C^1_c(\mathfrak{a}_n)$ and $R^i_j = d\vartheta^i_j + \vartheta^i_k \wedge \vartheta^k_j \in C^2_c(\mathfrak{a}_n)$ generate **quasi-isomorphic** DGSA $CW^*(\mathfrak{a}_n) \subset C^*_c(\mathfrak{a}_n)$, resp. $CW^*(\mathfrak{a}_n, O_n) \subset C^*_c(\mathfrak{a}_n, O_n)$,

• Sending the canonical basis $\theta = (\theta^i_j)$ of $\mathfrak{gl}_n^*$ to $\vartheta = (\vartheta^i_j)$ gives rise to $W(\mathfrak{gl}_n)_n \cong CW^*(\mathfrak{a}_n)$ and $W(\mathfrak{gl}_n, O_n)_m \cong CW^*(\mathfrak{a}_n, O_n)$, which in turn induce $H^*\big(W(\mathfrak{gl}_n)_n\big) \cong H^*_{\mathrm{c}}\big(\mathfrak{a}_n\big)$, resp. $H^*\big(W(\mathfrak{gl}_n, O_n)_n\big) \cong H^*_{\mathrm{c}}\big(\mathfrak{a}_n, O_n\big)$.

Jet bundle and invariant forms

- $\{\Omega^*(F^\infty M)^{\mathcal{G}}, d\}$ = de Rham complex of $\mathrm{Diff}(M)$-invariant forms on the frame bundle of the smooth manifold M^n, $\mathcal{G} = \mathrm{Diff}(M)$,

$$F^k M \equiv J_0^k M = k-\text{jets at } 0 \quad \text{of local diffeos} \quad \rho : \mathbb{R}^n \to M$$

$$M \leftarrow FM \equiv F^1 M \leftarrow F^2 M \leftarrow \cdots$$

$$P^k M = F^k M / O_n \qquad M \leftarrow P^1 M \leftarrow P^2 M \leftarrow \cdots$$

$$\phi \in \mathcal{G} \quad \text{acts by} \quad \phi \cdot j_0^\infty(\rho) := j_0^\infty(\phi \circ \rho).$$

- For $v \in \mathfrak{a}_n$, $v = j_0^\infty \left(\frac{d}{dt} \mid_{t=0} \psi_t \right)$, with $t \mapsto \psi_t : \mathbb{R}^n \to \mathbb{R}^n$, let $\tilde{v} \mid_{j_0^\infty(\rho)} = j_0^\infty \left(\frac{d}{dt} \mid_{t=0} (\rho \circ \psi_t) \right)$; for $\omega \in C_c^\bullet(\mathfrak{a}_n)$, let

$$\tilde{\omega}(\tilde{v}_1, \ldots, \tilde{v}_m) = \omega(v_1, \ldots, v_m).$$

- The assignment $\omega \mapsto \tilde{\omega}$ induces DGA isomorphisms

$$\{C_c^\bullet(\mathfrak{a}_n), d\} \cong \{\Omega^*(F^\infty M)^{\mathcal{G}}, d\},$$

$$\{C_c^\bullet(\mathfrak{a}_n, O_n), d\} \cong \{\Omega^*(P^\infty M)^{\mathcal{G}}, d\}.$$

- Simplicial manifold $\triangle_{\mathcal{G}} M = \{\triangle_{\mathcal{G}} M[p] = \mathcal{G}^p \times M\}_{p\geq 0}$,

$$\partial_0(\phi_1,\dots,\phi_p,\, x) \;=\; (\phi_2,\dots,\phi_p,\, x),$$
$$\partial_i(\phi_1,\dots,\phi_p,\, x) \;=\; (\phi_1,\dots,\phi_i\phi_{i+1},\dots,\phi_p, x), \quad 1\leq i<p,$$
$$\partial_p(\phi_1,\dots,\phi_p,\, x) \;=\; (\phi_1,\dots,\phi_{p-1},\, \phi_p(x)),$$
$$\sigma_i(\phi_1,\dots,\phi_p,\, x) = (\phi_1,\dots,\phi_i, 1, \phi_{i+1},\dots,\phi_p, x)\,, \quad 0\leq i\leq p.$$

- Bott complex $\{C^{\bullet}(\mathcal{G}, \Omega^{\bullet}(M)), \delta, d\}$ $\omega(\phi_1,\dots,\phi_p)\in\Omega^q(M)$

$$\delta\omega(\phi_1,\dots,\phi_{p+1}) = \omega(\phi_2,\dots,\phi_{p+1}) +$$
$$\sum_{i=1}^{p}(-1)^i\omega(\phi_1,\dots,\phi_i\phi_{i+1},\dots\phi_{p+1}) + (-1)^{p+1}\phi_{p+1}^{*}\omega(\phi_1,\dots,\phi_p).$$

- Geometric "thick" realization $|\triangle_{\mathcal{G}} M| = \coprod_{p\geq 0}\Delta^p\times\triangle_{\mathcal{G}} M[p]/\sim$

- Dupont complex of compatible forms $\{\Omega^*|\triangle_{\mathcal{G}} M|, d\}$

$$\omega = \{\omega_p\}_{p\geq 0}, \quad \omega_p\in\Omega^{\bullet}(\Delta^p\times\triangle_{\mathcal{G}} M[p]) \quad \text{such that}$$
$$(\mu_{\bullet}\times \mathrm{Id})^*\omega_q \;=\; (\mathrm{Id}\times\mu^{\bullet})^*\omega_p, \qquad \forall\,\mu\in\Delta(p,q).$$

Differentiability := continuity w.r.t. jet topology

- Differentiable Bott cochain: locally

$$\omega(\phi_1, \ldots, \phi_p)(x) = \sum f_I \left(x, j^{k_1}_{x_1}(\phi_1), \ldots, j^{k_p}_{x_p}(\phi_p)\right) dx^I,$$

where $f_{I,J}$, resp. f_I depend smoothly on finite order jets, and $x_1 = \phi_2(x_2), \ldots, x_{p-2} = \phi_{p-1}(x_{p-1})$, $x_{p-1} = \phi_p(x_p)$, $x_p = x$.

- Differentiable Dupont form: similarly,

$$\omega_p(\mathbf{t}; \phi_1, \ldots, \phi_p, x) = \sum f_{I,J} \left(\mathbf{t}; x, j^{k_1}_{x_1}(\phi_1), \ldots, j^{k_p}_{x_p}(\phi_p)\right) dt^I \wedge dx^J,$$

Cf. [6] (after Dupont)

The chain map $\oint_{\Delta^\bullet} : \Omega^\bullet_{\mathrm{d}}(|\triangle_{\mathcal{G}} M|) \to C^\bullet_{\mathrm{d}}(\mathcal{G}, \Omega^*(M))$ induces an isomorphism $H^\bullet_{\mathrm{d}}(|\triangle_{\mathcal{G}} M|, \mathbb{R}) \cong H^\bullet_{\mathrm{d}}(M_{\mathcal{G}}, \mathbb{R})$.

• Choice of a torsion free connection ∇ gives rise to a cross-section to $\pi_1 : F^\infty M \to FM$, namely $\sigma_\nabla(u) = j_0^\infty(\exp_x^\nabla \circ u)$, $u \in FM$.

• Define $\sigma_p(\mathbf{t}; \phi_1, \ldots, \phi_p, u) = \sigma_{\sum_{i=0}^p t_i \nabla^{\phi_i}}(u)$, where $\phi_0 = 1 \equiv \mathrm{Id}$.
Using $\sigma_{\nabla^\phi} = \phi^{-1} \circ \sigma_\nabla \circ \phi$, $\forall \phi \in \mathcal{G}$, one sees that $\{\sigma_p\}_{p\geq 0}$ defines a map $\hat{\sigma}_\nabla : |\triangle_\mathcal{G} PM| \to P^\infty M$.

Cf. [3], [6]

• The chain map $\omega \in C_{\mathrm{c}}^*(\mathfrak{a}_n, \mathrm{O}_n) \mapsto \hat{\sigma}_\nabla^*(\tilde{\omega}) \in \Omega_{\mathrm{d}}^*(|\triangle_\mathcal{G} PM|)$ induces quasi-iso of DG-algebras, and therefore so is the map so the map
$\omega \in C_{\mathrm{c}}^*(\mathfrak{a}_n, \mathrm{O}_n) \mapsto \mathcal{D}(\omega) = \oint_{\Delta^\bullet} \hat{\sigma}_\nabla^*(\tilde{\omega}) \in C_{\mathrm{d}}^{\mathrm{tot}\,\bullet}(\mathcal{G}, \Omega^*(PM))$.

Connes' transfer map to cyclic cohomology

DGA: $\mathcal{B}(M,\mathcal{G}) = \Omega^*_c(M)\hat{\otimes} \bigwedge^* \mathbb{C}[\mathcal{G}']$, $\mathcal{G}' = \{\epsilon_\varphi ; \varphi \in \mathcal{G}, \text{ but } \epsilon_1 = 0\}$;
differential: $d(\omega\otimes\epsilon) = d(\omega)\otimes\epsilon$, $\epsilon \in \bigwedge^* \mathbb{C}[\mathcal{G}']$.

BBDGA: $\mathcal{C}(M,\mathcal{G}) = \mathcal{B}(M,\mathcal{G}) \rtimes \mathcal{G}$ with multiplication dictated by the rules
$U^*_\varphi \omega U_\varphi = \varphi^*\omega$, $U^*_\varphi \epsilon_\psi U_\varphi = \epsilon_{\psi\phi} - \epsilon_\varphi$, and with **differentials**:
$d(bU_\varphi) := (db)U_\varphi$, $\delta(bU_\varphi) := (-1)^{\deg b}(b\epsilon_\varphi)U_\varphi$, $\mathbf{d} := d + \delta$.

- Given $\gamma \in C^q_\wedge(\mathcal{G},\Omega^p(M))$ define $\hat{\gamma} \in \mathcal{C}(M,\mathcal{G})^*$ by $\hat{\gamma}(bU^*_\varphi) = 0$ if $\varphi \neq 1$,

and if $\varphi = 1$ by $\hat{\gamma}(\omega\otimes\epsilon_{\varphi_1}\ldots\epsilon_{\varphi_q}) = \int_M \bar{\gamma}(1,\varphi_1\ldots,\varphi_q)\wedge\omega$, with
$\bar{\gamma} \in \bar{C}^q_\wedge(\mathcal{G},\Omega^p(M))$ the totally antisymmetric homogeneous lift of γ.

[A. Connes, *Noncommutative Geometry* (1994), Lemma III.2.13]

$$\hat{\gamma}[\alpha,\beta] = (-1)^{\deg\alpha}\widehat{\delta\gamma}(\alpha\delta\beta), \quad \hat{\gamma}(\delta\alpha) = 0 \quad \text{and} \quad \widehat{d\gamma}(\alpha) = \hat{\gamma}(d\alpha).$$

Definition of $\Phi : C_{\wedge}^{p,q}(\mathcal{G}, \Omega^*(M)) \to CC^m(C_c^\infty(M) \rtimes \mathcal{G})$

$$\Phi(\gamma)(a^0, \ldots, a^m) = \frac{p!}{(m+1)!} \sum_{j=0}^{m} (-1)^{j(m-j)} \hat{\gamma}(\mathbf{d}a^{j+1} \cdots \mathbf{d}a^m \, a^0 \mathbf{d}a^1 \cdots \mathbf{d}a^j),$$

where $\mathbf{d} = d + \delta$, $m = \dim M - p + q$ and $a^0, \ldots, a^m \in C_c^\infty(M) \rtimes \mathcal{G}$.

[A. Connes, *Noncommutative Geometry* (1994), Theorem III.2.14]

$\Phi : C_{\wedge}^{*,*}(\mathcal{G}, \Omega^*(M)) \to CC^*(C_c^\infty(M) \rtimes \mathcal{G}, b + B)$ is a chain map whose image lands in the subcomplex $\{CC^\bullet(C_c^\infty(\mathcal{G})), b + B\}_{[1]}$ localized at the units. The induced map in cohomology gives a canonical isomorphism

$$\sum_{q \equiv \bullet (\mathrm{mod} 2)} H_q(M_{\mathcal{G}}) \cong HP^\bullet(C_c^\infty(M) \rtimes \mathcal{G})_{[1]}.$$

Remark

This holds more generally for any étale smooth groupoid $\boldsymbol{\Gamma}$, with the subcomplex $C_{\wedge}^{*,*}$ replaced by the subcomplex of normalized cochains $C_{\nu}^{*,*}$.

Empirical definition of algebra $\mathcal{H}_n$

- $M = F\mathbb{R}^n := \mathbb{R}^n \times \mathrm{GL}_n$, $P\mathbb{R}^n = F\mathbb{R}^n / O_n$, $\nabla = d$, $\mathcal{G}_n = \mathrm{Diff}\,\mathbb{R}^n$, $X_k = \sum_\mu y^\mu_k \frac{\partial}{\partial x^\mu}$ horizontal and $Y^j_i = \sum_\mu y^\mu_i \frac{\partial}{\partial y^\mu_j}$ vertical vector fields, dual to the canonical forms θ^k and connection forms ω^i_j.

- Given $\gamma \in \mathrm{Im}(\mathcal{D})$, $\Phi(\gamma)(a^0, \ldots, a^\ell) = \sum_\alpha \tau(a^0 h^1_\alpha(a^1) \ldots h^q_\alpha(a^q))$ with $\sum_\alpha h^1_\alpha \otimes \ldots \otimes h^q_\alpha \in \mathcal{H}_n^{\otimes q}$ (defined below and on next slide).

- $h^q_\alpha \in \mathcal{H}_n :=$ unital subalgebra of $\mathcal{L}\left(C^\infty(F\mathbb{R}^n \rtimes \mathcal{G}_n)\right)$ generated by

$$Y^j_i(fU^*_\phi) = Y^j_i(f)U^*_\phi, \quad X_k(fU^*_\phi) = X_k(f)U^*_\phi, \quad \delta^i_{jk}(fU^*_\phi) = \gamma^i_{jk}(\phi)fU^*_\phi,$$

with $\gamma^i_{jk}(\phi)(x, \mathbf{y}) := \left(\mathbf{y}^{-1} \cdot \phi'(x)^{-1} \cdot \partial_\mu \phi'(x) \cdot \mathbf{y}\right)^i_j \mathbf{y}^\mu_k$.

How they appear:

$$df = \sum_{k=1}^n X_k(f)\theta^k + \sum_{i,j=1}^n Y^i_j(f)\,\omega^j_i\,,$$

$$U_\phi\, df\, U^*_\phi = \sum_{k=1}^n (X_k(f)\circ\phi)\,\theta^k + \sum_{i,j=1}^n (Y^i_j(f)\circ\phi)\left(\omega^i_j + \gamma^i_{jk}(\phi)\,\theta^k\right).$$

$\mathcal{H}_n$ as Hopf algebra

• $\mathcal{H}_n$ is a Hopf algebra, with counit $\epsilon(h) = h(1)$, coproduct uniquely determined by the Leibniz rule: $\forall\, a, b \in C^\infty(F\mathbb{R}^n \rtimes \mathcal{G}_n)$

$$\Delta(h) = \sum h_{(1)} \otimes h_{(2)} \quad \text{iff} \quad h(ab) = \sum h_{(1)}(a) h_{(2)}(b),$$

and antipode $\quad S(Y_i^j) = -Y_i^j, \quad S(X_k) = -X_k + \delta_{jk}^i Y_i^j, \quad S(\delta_{jk}^i) = -\delta_{jk}^i.$

• $\delta : \mathcal{H}_n \to \mathbb{C}$ with $\delta(Y_i^j) = \delta_i^j,\ \delta(X_k) = 0,\ \delta(\delta_{jk}^i) = 0$ is a character.

The canonical trace $\quad \tau : C_c^\infty(F\mathbb{R}^n \rtimes \mathcal{G}_n) \to \mathbb{C}$

$$\tau(fU_e) = \int_{F\mathbb{R}^n} f \,\mathrm{vol} \quad \text{and} \quad \tau(fU_\varphi) = 0 \quad \text{if } \varphi \neq e \equiv \mathrm{Id},$$

is δ-invariant with respect to the action of $\mathcal{H}_n$:

$$\tau(h(a)) = \delta(h)\,\tau(a), \qquad \alpha \in C_c^\infty(F\mathbb{R}^n \rtimes \mathcal{G}_n),\ h \in \mathcal{H}_n.$$

Hopf cyclic complex

Tautological definition of Hopf cyclic complex

The map $\chi_q : \mathcal{H}_n^{\otimes q} \to CC^q\left(C_c^\infty(F\mathbb{R}^n \rtimes \mathcal{G}_n)\right)$ defined by

$$\chi_q(h^1 \otimes \ldots \otimes h^q)(a_0, a_1, \ldots, a_q) = \tau\left(a_0 h^1(a_1) \cdots h^q(a_q)\right)$$

is a faithful map of cyclic complexes.

$$C^q(\mathcal{H}_n; \mathbb{C}_\delta) = \mathcal{H}_n^{\otimes q}, \; q \geq 0,$$

$$\partial_0(h^1 \otimes \ldots \otimes h^{q-1}) = 1 \otimes h^1 \otimes \ldots \otimes h^{q-1},$$

$$\partial_j(h^1 \otimes \ldots \otimes h^{q-1}) = h^1 \otimes \ldots \otimes \Delta h^j \otimes \ldots \otimes h^{q-1}$$

$$\partial_n(h^1 \otimes \ldots \otimes h^{q-1}) = h^1 \otimes \ldots \otimes h^{q-1} \otimes 1$$

$$\sigma_i(h^1 \otimes \ldots \otimes h^{q+1}) = h^1 \otimes \ldots \otimes \varepsilon(h^{i+1}) \otimes \ldots \otimes h^{q+1}$$

$$\tau_q(h^1 \otimes \ldots \otimes h^q) = S_\delta(h^1) \cdot h^2 \otimes \ldots \otimes h^q \otimes 1; \qquad S_\delta := \delta * S;$$

$b = \sum_{i=0}^{q+1} (-1)^i \partial_i$ and $B = \left(\sum_{i=0}^{q} (-1)^{(q-1)i} \tau_q^i\right) \sigma_{q-1}(1 - (-1)^q \tau_q)$.

Explicit quasi-isomorphisms

Theorem (AC & HM, [2])

$\chi^{-1} \circ \Phi \circ \mathcal{D} : C_{\rm c}^{\bullet}(\mathfrak{a}_n) \to CC^{\bullet}(\mathcal{H}_n; \mathbb{C}_\delta)$ is quasi- isomorphism ;

- $HC^q(\mathcal{H}_n; \mathbb{C}_\delta) \cong \sum_{i \geq 0}^{\oplus} H_{\rm c}^{q-2i}(\mathfrak{a}_n; \mathbb{C})$;
- $HP^{\bullet}(\mathcal{H}_n; \mathbb{C}_\delta) \cong \sum_{q \equiv \bullet ({\rm mod} 2)}^{\oplus} H_{\rm c}^{q}(\mathfrak{a}_n; \mathbb{C})$;
- $HP^{\bullet}(\mathcal{H}_n, O_n; \mathbb{C}_\delta) \cong \sum_{q \equiv \bullet ({\rm mod} 2)}^{\oplus} H_{\rm c}^{q}(\mathfrak{a}_n, O_n; \mathbb{C})$.

Corollary

- Let $C_{\mathcal{D},\Lambda}^{\bullet}(\mathcal{G}_n, \Omega^*(P\mathbb{R}^n)) := Im(\mathcal{D}) \subset C_{{\rm d},\Lambda}^{\bullet}(\mathcal{G}_n, \Omega^*(P\mathbb{R}^n))$.

Then both $\mathcal{D} : C_{\rm c}^{\bullet}(\mathfrak{a}_n, O_n) \to C_{\mathcal{D},\Lambda}^{\bullet}(\mathcal{G}_n, \Omega^*(P\mathbb{R}^n))$ and $\chi^{-1} \circ \Phi_{\mathcal{D}} : C_{\mathcal{D},\Lambda}^{\bullet}(\mathcal{G}_n, \Omega^*(P\mathbb{R}^n)) \to CC^{\bullet}(\mathcal{H}_n, O_n; \mathbb{C}_\delta)$ are quasi-iso .

Outcome: isomorphism to differentiable $\mathcal{G}_n$-equivariant cohomology

Since $H_{\rm c}^{\bullet}(\mathfrak{a}_n, {\rm O}_n) \xrightarrow{\mathcal{D}_* \cong} H_{\mathcal{D}}^{\bullet}(\mathcal{G}_n, \Omega^*(P\mathbb{R}^n)) \overset{\iota_* \cong}{\hookrightarrow} H_{\rm d}^{\bullet}(\mathcal{G}_n, \Omega^*(P\mathbb{R}^n))$

$\Longrightarrow \chi_{\rm d} := \iota_* \circ (\chi^{-1} \circ \Phi_{\mathcal{D}})_*^{-1} : HP^{\bullet}(\mathcal{H}_n, O_n; \mathbb{C}_\delta) \xrightarrow{\cong} H_{\rm d}^{\bullet}(\mathcal{G}_n, \Omega^*(P\mathbb{R}^n))$.

DG Hopf cyclic cohomology (Gorokhovsky [4])

• For both DG algebras and DG Hopf algebras the cyclic complex is a triple complex (b, B, d), with the definition of b and B adjusted by appropriate signs, and with the degree 1 differential d commuting with b and B. The cyclic resp. periodic cyclic cohomology is defined as the respective cohomology of the triple complex with finite cochains.

• Let $\mathcal{A} = \bigoplus_{k\in\mathbb{Z}^+} A^k$ be a unital DGA. For $c \in \mathcal{A}^{\otimes q}$ of weight m define

$$\mathcal{R}(c) := \frac{(-1)^{qm+m(m-1)/2}}{m!}\big(d \circ (e_d + E_d)\big)^m(c),$$

where $\quad L_d = [b + B, e_d + E_d]; \quad [b, e_d] = 0 = [B, E_d]).$

$\mathcal{R} : CC^\bullet(\mathcal{A}, b, B, d) \to CC^\bullet(A^0, b, B)$ is quasi-isomorphism.

Cycles over DGA and their characters (cf. [4])

• BBDGA $\mathcal{C}(M,\mathcal{G}) = \mathcal{B}(M,\mathcal{G}) \rtimes \mathcal{G}$ with $\boldsymbol{\gamma} = \{\gamma_{p,q}\}_{m=\dim M - p+q}$ $\in ZC^{\bullet}_{\wedge}(\mathcal{G}, \Omega^{\bullet}(M))$ is a cycle $\mathcal{C}_\gamma = \{\mathcal{C}^{\bullet,\bullet}(M,\mathcal{G}), \delta, d; \gamma\}$ over $\Omega^*_c(M)$. (*Cf. general definition in [4], modeled on Lemma III.2.13*).

Character of $\mathcal{C}_\gamma$: $X_\gamma = \{\hat{\gamma}_{p,q}\}$ is cocycle $\in CC^{\bullet}\big(\Omega^*_c(M) \rtimes \mathcal{G}), b, B, d\big)$

$$\hat{\gamma}_{p,q}(\theta_0, \theta_1, \dots, \theta_p) = (-1)^{\sum_{i=0}^{p}(p-i)\deg\theta_i}\, \hat{\gamma}_{p,q}(\theta_0\, \delta\theta_1 \cdots \delta\theta_p).$$

Indeed, by Lemma III 2.13, $B\hat{\gamma}_{p,q} = 0$, $b\hat{\gamma}_{p,q} + d\hat{\gamma}_{p+1,q} = 0$ and $\hat{\gamma}_{p,q}(\theta_p, \theta_1, \dots, \theta_{p-1}) = (-1)^{p+\deg\theta_p \sum_{i=0}^{p-1}\deg\theta_i}\hat{\gamma}_{p,q}(\theta_0, \theta_1, \dots, \theta_p)$.

Character interpretation of Φ

For any cocycle $\gamma = \{\gamma_{p,q}\}_{m=\dim M-p+q} \in C^{\bullet}_{\wedge}(\mathcal{G}, \Omega^{\bullet}(M))$, the cocycles $\Phi_*(\gamma) := \{\Phi(\gamma_{p,q})\}$ and $\mathcal{R}_*(X_\gamma) := \{\mathcal{R}(\hat{\gamma}_{p,q})\}$ are cohomologous, determining the same class $\Phi_*[\gamma] = \mathcal{R}_*[X_\gamma] \in HP^{\bullet}\left(C^\infty_c(M) \rtimes \mathcal{G}\right)$.

Hopf DGA $\Omega^*(\mathrm{GL}_n)$ and its cyclic cohomology (cf. [4])

• $\Omega^*(\mathrm{GL}_n) \cong C^\infty(\mathrm{GL}_n) \otimes \bigwedge \mathfrak{gl}_n^*$ with the differential d is a (Fréchet) Hopf DGA, with the product given by exterior multiplication, the coproduct induced by the product of GL_n, the antipode induced by inversion.

• **Cyclic complex**: $C^q(\Omega^*(\mathbb{R}^n)) := \Omega^*(\mathrm{GL}_n)^{\hat{\otimes} q} = \Omega^*(\mathrm{GL}_n \times \cdots \times \mathrm{GL}_n)$,
$b = \sum_{i=0}^{q+1} (-1)^i \partial_i \qquad B = \left(\sum_{i=0}^{q} (-1)^{(q-1)i} \tau_q^i\right) \sigma_{q-1}(1 - (-1)^q \tau_q)$,
$d(\alpha_1 \otimes \cdots \otimes \alpha_q) = \sum_{i=1}^{q} (-1)^{\deg \alpha_1 + \cdots + \deg \alpha_q} \alpha_1 \otimes \cdots d\alpha_i \cdots \otimes \alpha_q)$.
• $CC^\bullet(\Omega^*(\mathbb{R}^n), b, B, d)_m := CC^\bullet(\Omega^*(\mathbb{R}^n), b, B, d)/F^m\Omega^*(\mathbb{R}^n)$,
where $F^m\Omega^*(\mathbb{R}^n) = \{\sum \alpha_1 \otimes \cdots \otimes \alpha_q \mid \deg \alpha_1 + \cdots + \deg \alpha_q > m\}$.

Theorem (A. Gorokhovsky)

$$HC^q(\Omega^*(\mathrm{GL}_n), d)_m \cong \sum_{i \geq 0}^{\oplus} H^{q-2i}(W(\mathfrak{gl}_n, O_n)_m), \qquad \forall\, m \in \mathbb{Z}^+.$$

Idea of proof: Hochschild complex of $\Omega^*(\mathrm{GL}_n)$ (as coalgebra) $\equiv$ simplicial de Rham complex of the nerve of GL_n.
Note also the analogy to Van Est Thm. $\quad H^*_{\mathrm{d}}(\mathrm{GL}_n) \cong H^*(\mathfrak{gl}_n, O_n)$.

Action and characteristic map

- **Groupoid homomorphism:** $\mathcal{J} : \mathbb{R}^n \rtimes \mathcal{G}_n \to \mathrm{GL}_n(\mathbb{R})$,
$$\mathcal{J}(x, \varphi) = \varphi'(x) \equiv \left(\frac{\partial \varphi^i}{\partial x^j}\right) \in \mathrm{GL}_n(\mathbb{R})$$
- **DG Hopf action:** $\Omega^*(\mathrm{GL}_n) \otimes \Omega^*_c(\mathbb{R}^n \rtimes \mathcal{G}_n) \to \Omega^*_c(\mathbb{R}^n \rtimes \mathcal{G}_n)$,
$$\alpha(\varpi) = \mathcal{J}^*(\alpha)\varpi, \qquad \alpha \in \Omega^*(\mathrm{GL}_n) \quad \varpi \in \Omega^*_c(\mathbb{R}^n \rtimes \mathcal{G}_n)$$
- **Closed graded trace:** $\oint \varpi := \int_{\mathbb{R}^n} \iota^* \varpi, \quad \iota : \mathbb{R}^n \times \mathbf{1} \hookrightarrow \mathbb{R}^n \rtimes \mathcal{G}_n$
- $\Omega^*(\mathrm{GL}_n)$**-invariance:** $\oint \alpha(\varpi) = \epsilon(\alpha) \oint \varpi$

Tautological characterization of DG Hopf cyclic cohomology

$$\kappa_q(\alpha^1 \otimes \alpha^2 \otimes \cdots \otimes \alpha^q)(\varpi_0, \varpi_1, \ldots, \varpi_q) = (-1)^{\sum_{i<j} \deg \alpha^i \deg \varpi_j} \oint \varpi_0\, \alpha^1(\varpi_1) \cdots \alpha^q(\varpi_q)$$

is a chain map, inducing $\kappa_* : HC^*(\Omega^*(\mathrm{GL}_n), d)_n \to HC^*(\Omega^*_c(\mathbb{R}^n \rtimes \mathcal{G}_n))$.

DG counterpart of Φ (A. Gorokhovsky)

M is smooth manifold and $\mathcal{G}$ is a subgroup of $\mathrm{Diff}\, M$.

For $\varphi_0 \cdots \varphi_{p-1}\varphi_p = 1$, $\Psi(\gamma)(\omega_0 U_{\varphi_0}, \omega_1 U_{\varphi_1}, \ldots, \omega_p U_{\varphi_p}) =$

$$(-1)^{p(\dim M - q)} \int_M \bar{\gamma}(1, \varphi_0, \varphi_0\varphi_1, \ldots, \varphi_0 \cdots \varphi_{p-1})\omega_0 \omega_1^{\varphi_0} \cdots \omega_p^{\varphi_0 \cdots \varphi_{p-1}},$$

otherwise $\Psi(\gamma)(\omega_0 U_{\varphi_0}, \omega_1 U_{\varphi_1}, \ldots, \omega_p U_{\varphi_p}) = 0.$

$\Psi : C_{\wedge}^{p,q}(\mathcal{G}, \Omega^*(M)) \to CC^p\left(\Omega_c^*(M) \rtimes \mathcal{G}\right)$ is a chain map and

$$\Phi_\bullet = \mathcal{R}_\bullet \circ \Psi_\bullet : \sum_{q \equiv \bullet (\mathrm{mod} 2)} H_q(M_{\mathcal{G}}) \xrightarrow{\cong} HP^\bullet\left(C_c^\infty(M) \rtimes \mathcal{G}\right)_{[1]}.$$

DG counterpart of χ_{d} isomorphism

$$H^\bullet(W(\mathfrak{gl}_n, O_n)_n) \xrightarrow{\mathcal{E}_* \cong} H^\bullet_{\mathcal{E}}(\mathcal{G}_n, \Omega^*(\mathbb{R}^n)) \overset{\iota_* \cong}{\hookrightarrow} H^\bullet_{\mathrm{d}}(\mathcal{G}_n, \Omega^*(\mathbb{R}^n)) \implies$$

$$\kappa_{\mathrm{d}} := \iota_* \circ (\kappa^{-1} \circ \Psi_{\mathcal{D}})_*^{-1} : HP^*(\Omega^*(\mathrm{GL}_n), d)_n \xrightarrow{\cong} H^\bullet_{\mathrm{d}}(\mathcal{G}_n, \Omega^*(\mathbb{R}^n)).$$

Definition: Differentiable cyclic cohomology

- A cycle $\mathcal{C}_\gamma$ is differentiable if $\gamma = \{\gamma_{p,q}\}_{m=\dim M-p+q} \in C^\bullet_{\wedge\,\mathrm{d}}(\mathcal{G}, \Omega^\bullet(M))$.

- The characters $X_\gamma = \{\Psi(\hat{\gamma}_{p,q})\}$ with γ running through $C^\bullet_{\wedge\,\mathrm{d}}(\mathcal{G}, \Omega^\bullet(M))$ generate the differentiable cyclic cohomology $HP^\bullet_{\mathrm{d}}\big(\Omega^*_c(M) \rtimes \mathcal{G}\big)$, while the retracted characters $Y_\gamma := \mathcal{R}(X_\gamma) = \{\Phi(\hat{\gamma}_{p,q})\}$ generate the differentiable cyclic cohomology $HP^\bullet_{\mathrm{d}}\big(C^\infty_c(M) \rtimes \mathcal{G}\big)$.

Recall the isomorphisms

$$\chi_{\rm d} := \iota_* \circ (\chi^{-1} \circ \Phi_{\mathcal{D}})_*^{-1} : HP^\bullet(\mathcal{H}_n, O_n; \mathbb{C}_\delta) \xrightarrow{\cong} H_{\rm d}^\bullet(\mathcal{G}_n, \Omega^*(P\mathbb{R}^n))$$
$$\kappa_{\rm d} := \iota_* \circ (\kappa^{-1} \circ \Psi_{\mathcal{D}})_*^{-1} : HP^*(\Omega^*(\mathrm{GL}_n), d)_n \xrightarrow{\cong} H_{\rm d}^\bullet(\mathcal{G}_n, \Omega^*(\mathbb{R}^n)).$$

Analogue of the Thom isomorphism

$$HP^\bullet(\mathcal{H}_n, O_n; \mathbb{C}_\delta) \xrightarrow{\cong} HP^*(\Omega^*(\mathrm{GL}_n), d)_n$$

Proof The inclusion $C_{\rm d}^p(\mathcal{G}_n; \Omega^q(\mathbb{R}^n)) \hookrightarrow C_{\rm d}^p(\mathcal{G}_n; \Omega^q(P^\infty\mathbb{R}^n))$ (induced by $\Omega^q(\mathbb{R}^n) \to \Omega^q(P^\infty\mathbb{R}^n))$ is a quasi-isomorphism for the associated total complexes. Therefore

$$H_{\rm d}^*\big(\mathcal{G}_n, \Omega^*(P\mathbb{R}^n)\big) \cong H_{\rm d}^*\big(\mathcal{G}_n, \Omega^*(\mathbb{R}^n)\big),$$

which combined with the two isomorphisms above yields the stated isomorphism.

Godbillon-Vey example

- Godbillon-Vey: $gv := \frac{1}{y_1^2} dx \wedge dy_1 \wedge dy_2 \in C^0(\mathcal{G}_1, \Omega^3(F^\infty S^1))$;

- $\gamma(\varphi) := \mathcal{D}(gv)(\varphi) = d\left(\frac{d}{dx}(\log \varphi'(x)) \log y_1 \, dx\right)$;

- $\Phi_{FS^1}(\gamma) = \chi(\delta_1)$ where δ_1 is the generator of $HP^1(\mathcal{H}_1, \mathbb{C}_\delta)$, defined by $\delta_1(fU_\varphi^*) = y_1 \frac{d}{dx}(\log \varphi'(x)) \, fU_\varphi^*$, which is algebraic.

- $\gamma \in C^1(\mathcal{G}_1, \Omega^2(FS^1)) \sim \beta \in C^2(\mathcal{G}_1, \Omega^1(S^1))$ (Bott-Thurston cocycle) $\beta(\varphi, \psi) := d(\log \varphi' \circ \psi) \log \psi'$, which is transcendental.

- $\Psi_{S^1}(\beta) = \Phi_{S^1}(\beta) \in HC^1\left(C^\infty(S^1 \rtimes \mathcal{G}_1)\right)$ is Connes' G-V cyclic cocyle.

The general case (cf. [6])

- Let $\omega = (\omega^i_j)$, $\omega^i_j := (\mathbf{y}^{-1})^i_\mu d\mathbf{y}^\mu_j$, be the flat connection form;
- $\phi^*(\omega^i_j) = \omega^i_j + \gamma^i_{jk}(\phi)\,\theta^k$, which generates curvature on $|\triangle_{\mathcal{G}_n} FM|$;
- the simplicial connection and curvature Dupont's de Rham complex is:

$$\hat\omega_p(\mathbf{t};\varphi_0,\dots,\varphi_p) := \textstyle\sum_{i=0}^p t_i\varphi_i^*(\omega), \quad \hat\Omega := d\hat\omega + \hat\omega\wedge\hat\omega, \quad \in \Omega^2_{\mathrm{d}}(\bar\triangle_{\mathcal{G}_n} FM);$$

$$\hat\Omega_p(\mathbf{t};\varphi_0,\dots,\varphi_p) = \textstyle\sum_{i=0}^p dt_i\wedge\varphi_i^*(\omega) + \sum_{i=0}^p t_i\big(\varphi_i^*(\Omega) - \varphi_i^*(\omega)\wedge\varphi_i^*(\omega)\big) + \sum_{i,j=0}^p t_i t_j\,\varphi_i^*(\omega)\wedge\varphi_j^*(\omega).$$

The Chern-Weil algorithm produces Chern forms $c_k(\hat\Omega)$ and Chern-Simons forms $Tc_k(\hat\omega)$ on $|\triangle_{\mathcal{G}} P\mathbb{R}^n|$; using graded commutativity one builds a Vey basis of $H^*_{\mathrm{d}}(|\triangle_{\mathcal{G}} P\mathbb{R}^n|)$, which via $\oint_{\Delta^\bullet}$ yields a basis of $H^*_{\mathrm{d}}(\mathcal{G}_n, \Omega^*(P\mathbb{R}^n))$.