The spectral action expanded in Yang-Mills and Chern-Simons forms

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Mathematical Physics – IMAPP
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Part 0:
The Context

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Connes–Chamseddine 2006:

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   Questions arise:
   - For which functions f is $\frac{d^n}{dt^n} f(D+tA)\big|_{t=0}$ trace-class?
   - Can we obtain a formula like Connes–Chamseddine?

Part 1:
Multiple Operator Integrals

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Examples:

1. $H = \Delta$, the Laplacian
2. $H = D$, the generalized Dirac operator

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$$T^H_{f^{[n]}} : \mathcal{B}(\mathcal{H}) \times \cdots \times \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}),$$

depending on f and H, such that in particular

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One definition is

$$T^H_{f^{[n]}}(V_1, \ldots, V_n) := \int_{\Delta_n} \int_{\mathbb{R}} e^{its_0 H} V_1 e^{its_1 H} \cdots V_n e^{its_n H} \widehat{f^{(n)}}(t)\, dt\, d\sigma(s)$$

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If $n = 1$, $H = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$, $V = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$,

$$T^H_{f^{[1]}}(V) = \int_{s_0+s_1=1} ds_0\, ds_1 \int_{\mathbb{R}} dt\, e^{its_0 \left(\begin{smallmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{smallmatrix}\right)} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} e^{its_1 \left(\begin{smallmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{smallmatrix}\right)} \widehat{f'}(t)$$

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&= \int_{s_0+s_1=1} ds_0\,ds_1 \int_{\mathbb{R}} dt \begin{pmatrix}0 & e^{its_0\lambda_1+its_1\lambda_2}\\ e^{its_0\lambda_2+its_1\lambda_1} & 0\end{pmatrix} \widehat{f'}(t)\\
&= \begin{pmatrix}0 & \frac{f(\lambda_1)-f(\lambda_2)}{\lambda_1-\lambda_2}\\ \frac{f(\lambda_2)-f(\lambda_1)}{\lambda_2-\lambda_1} & 0\end{pmatrix} =: \begin{pmatrix}0 & f^{[1]}(\lambda_1,\lambda_2)\\ f^{[1]}(\lambda_1,\lambda_2) & 0\end{pmatrix}
\end{aligned}$$

By using the various defining expressions, Multiple Operator Integration can solve a lot of analytical issues arising from geometrical problems.
Recently, Anna Skripka and I sharpened a few facts surrounding:

1. When the MOI is **trace-class**, so

$$\mathrm{Tr}(|T^{H}_{f^{[n]}}(V_1,\ldots,V_n)|) < \infty\,.$$

2. Existence of a **spectral shift function** η such that

$$\mathrm{Tr}\left(f(H+V) - \sum_{k=0}^{n}\frac{d^n}{dt^n}f(H+V))\Big|_{t=0}\right) = \int_{\mathbb{R}}\eta(t)f^{(n)}(t)\,dt\,.$$

$$T^H_{f^{[n]}}(V_1, \ldots, V_n) := \int_{\Delta_n} \int_{\mathbb{R}} e^{its_0 H} V_1 e^{its_1 H} \cdots V_n e^{its_n H} \widehat{f^{(n)}}(t)\, dt\, d\sigma(s)$$

When $V_1, \ldots, V_n \in \mathcal{S}^n$, then $T^H_{f^{[n]}}(V, \ldots, V) \in \mathcal{S}^1$. Often, V is noncompact, but we only have

$$\tilde{V} := V(H - i)^{-1} \in \mathcal{S}^n.$$

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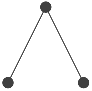
Lemma

For $V_1,\ldots,V_n\in\mathcal{B}(\mathcal{H})$, we have

$$\begin{aligned}T^H_{f^{[n]}}(V_1,\ldots,V_n) =& T^H_{(fu)^{[n]}}(V_1,\ldots,\tilde{V}_j,V_{j+1},\ldots,V_n)\\ &- T^H_{f^{[n-1]}}(V_1,\ldots,\tilde{V}_jV_{j+1},\ldots,V_n),\end{aligned}$$

where $u(t):=t-i$.



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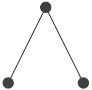

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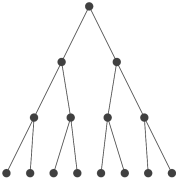
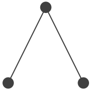


Proposition

For $n \in \mathbb{N}$, $f \in \mathcal{S}(\mathbb{R})$, $V \in \mathcal{B}(\mathcal{H})$, we have

$$T^H_{f^{[n]}}(V,\dots,V) = \sum_{p=0}^{n} \sum_{\substack{j_1,\dots,j_p \geqslant 1,\\ j_{p+1}\geqslant 0,\\ j_1+\dots+j_{p+1}=n}} (-1)^{n-p}\, T^H_{(fu^p)^{[p]}}(\tilde{V}^{j_1},\dots,\tilde{V}^{j_p})\tilde{V}^{j_{p+1}}.$$



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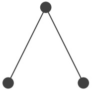

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In particular, if $\tilde V = V(H-i)^{-1}\in\mathcal{S}^n$, then $\frac{1}{n!}\frac{d^n}{dt^n} f(H+tV)\big|_{t=0} = T^H_{f^{[n]}}(V,\dots,V)\in\mathcal{S}^1$.

Part 2:

Perturbations of the Spectral Action

Let $(\mathcal{A}, \mathcal{H}, D)$ be a unital spectral triple with $(D-i)^{-1} \in \mathcal{S}^n$ (i.e. finitely summable/finite dimensional). Our goal is to Taylor-expand

$$\mathrm{Tr}(f(D+V)),$$

for $V = \sum a_j[D, b_j] \in \Omega^1_D(\mathcal{A})_{\text{s.a.}}$ and obtain forms that depend on 1-form $A = \sum a_j db_j \in \Omega^1(\mathcal{A})$.

Let $\Omega^\bullet \subseteq T_\mathcal{A}\Omega$ be the algebra of **universal differential forms.** where $d : \Omega^\bullet \to \Omega^{\bullet+1}$ is the univeral differential, defined by

$$d(a) := 1 \otimes a - a \otimes 1 \qquad (a \in \mathcal{A} = \Omega^0)\,,$$

and the Leibniz rule. With d, $\Omega^\bullet$ becomes a cohomologically graded differential algebra.
A representation of $\Omega^1 = \Omega^1(\mathcal{A})$ is given by

$$\Omega^1_D(\mathcal{A}) := \left\{ \sum a_j[D, b_j] : \ a_j, b_j \in \mathcal{A} \right\} \subseteq \mathcal{B}(\mathcal{H})$$

$$\mathrm{Tr}(f(D+V)-f(D)) = \sum_{n=1}^{\infty} \frac{1}{n!} \frac{d^n}{dt^n} \mathrm{Tr}(f(D+tV))\Big|_{t=0}$$
$$= \sum_{n=1}^{\infty} \mathrm{Tr}(T^{D}_{f^{[n]}}(V,\ldots,V)).$$

For $\hat{f} \in \mathcal{S}$ of exponential decrease,

$$\begin{aligned}\operatorname{Tr}(f(D+V)-f(D)) &= \sum_{n=1}^{\infty} \frac{1}{n!} \frac{d^n}{dt^n} \operatorname{Tr}(f(D+tV))\Big|_{t=0} \\ &= \sum_{n=1}^{\infty} \operatorname{Tr}(T^D_{f^{[n]}}(V,\ldots,V)).\end{aligned}$$

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Defining

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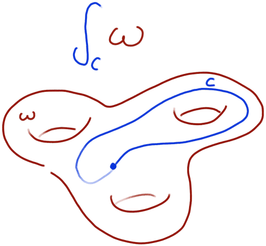
we get $\mathrm{Tr}(f(D+V)-f(D)) = \sum_{n=1}^{\infty} \frac{1}{n}\langle V,\ldots,V\rangle_n$.

Can we identify universal forms in
$\mathrm{Tr}(f(D+V)-f(D))=\sum_{n=1}^{\infty}\frac{1}{n}\langle V,\dots,V\rangle_n$? We define

$$\int_{\phi_n} a_0\,da_1\cdots da_n := \phi_n(a^0,\dots,a^n)$$
$$:= \langle a^0[D,a^1],[D,a^2],\dots,[D,a^n]\rangle_n\,.$$



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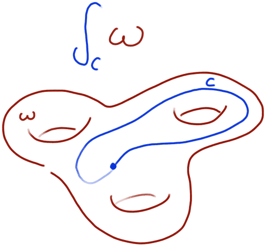

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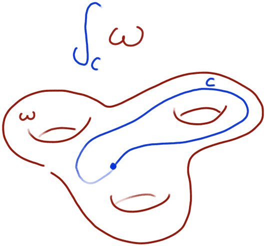

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$$\psi_n := c_n(\phi_n - \tfrac{1}{2}B_0\phi_{n+1})$$

For the right c_n, $(\psi_1,\psi_3,\psi_5,\dots)$ forms a (b,B)-**cocycle.**

The only properties of $\langle \cdot, \ldots, \cdot \rangle_n$ that we need are

1. $\langle V_1, \ldots, V_j, aV_{j+1}, \ldots, V_n \rangle_n - \langle V_1, \ldots, V_j a, V_{j+1}, \ldots, V_n \rangle_n$
$= \langle V_1, \ldots, V_j, [D, a], V_{j+1}, \ldots, V_n \rangle_{n+1}$,

2. $\langle V_1, \ldots, V_n \rangle_n = \langle V_n, V_1, \ldots, V_{n-1} \rangle_n$.

From this, one obtains $b\phi_{2k} = 0$ and $b\psi_{2k-1} + B\psi_{2k+1} = 0$, when choosing the right normalization.

Using just the rule

$$\langle V_1, \dots, V_j, aV_{j+1}, \dots, V_n\rangle_n - \langle V_1, \dots, V_j a, V_{j+1}, \dots, V_n\rangle_n \\ = \langle V_1, \dots, V_j, [D,a], V_{j+1}, \dots, V_n\rangle_{n+1},$$

we obtain, when $V = a[D,b]$,

$$\langle a[D,b]\rangle_1 = \int_{\phi_1} A$$

$$\langle a[D,b], a[D,b]\rangle_2 = \int_{\phi_2} A^2 + \int_{\phi_3} AdA$$

$$\langle a[D,b], a[D,b], a[D,b]\rangle_3 = \int_{\phi_3} A^3 + \int_{\phi_4} AdAA + \int_{\phi_5} AdAdA$$

etcetera.

For $V \in \Omega^1_D$, written as $V = \sum_j a_j[D, b_j]$ introduce $A := \sum_j a_j db_j \in \Omega^1$ and its curvature

$$F := A^2 + dA \in \Omega^2 .$$

Using $F_t := t^2A^2 + tdA$, we have the Chern–Simons forms

$$\mathrm{cs}_{2k-1}(A) := \int_0^1 A(F_t)^{n-1} dt .$$

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Theorem

Let $(\mathcal{A}, \mathcal{H}, D)$ be a finitely summable spectral triple, and $V \in \Omega^1_D(\mathcal{A})_{s.a.}$ the gauge field corresponding to $A \in \Omega^1(\mathcal{A})$. Then,

$$\mathrm{Tr}(f(D+V) - f(D)) \sim \sum_{k=1}^{\infty} \left(\int_{\psi_{2k-1}} \mathrm{cs}_{2k-1}(A) + \frac{1}{2k} \int_{\phi_{2k}} F^k \right) .$$

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Theorem

Let $(\mathcal{A}, \mathcal{H}, D)$ be a finitely summable spectral triple, and $V \in \Omega^1_D(\mathcal{A})_{s.a.}$ the gauge field corresponding to $A \in \Omega^1(\mathcal{A})$. Then, there exists δ such that for all $|t| < \delta$:

$$\mathrm{Tr}(f(D + tV) - f(D)) = \sum_{k=1}^{\infty} \left(\int_{\psi_{2k-1}} \mathrm{cs}_{2k-1}(tA) + \frac{1}{2k} \int_{\phi_{2k}} F_t^k \right),$$

and the series converges absolutely.

The terms in the Taylor expansion can be reordered using MOI-techniques and form something quite simple. Recently, Walter van Suijlekom and I proved the following.

Theorem

Let $(\mathcal{A}, \mathcal{H}, D)$ be a finitely summable spectral triple, and $V \in \Omega^1_D(\mathcal{A})_{s.a.}$ the gauge field corresponding to $A \in \Omega^1(\mathcal{A})$. Then, there exists δ such that for all $|t| < \delta$:

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and the series converges absolutely.

This extends [Connes–Chamseddine 2006], applications in physics? **Maybe in 2021 :)**