Index Theory for Fourier Integral Operators and the Connes-Moscovici Local Index Formula

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The Setting

- M smooth manifold (closed or $\mathbb{R}^n$).
- G discrete group
- $g \mapsto \Phi_g$ representation (unitary) by quantized canonical transformations on $\mathscr{L}(L^2(M))$.

We are interested in the index theory for operator algebras of the form

$$D = \sum D_g \Phi_g;$$

the sum is finite, the D_g are pseudodifferential operators on M.

Includes the following situations:

- D ψdo ($G = \{1\}$): Classical index problem
- $D = \Phi_g$: Atiyah-Weinstein index problem
- $D =$ Dirac on Lorentzian time slice with APS/anti-APS b.c.
- Φ_g given by diffeomorphisms of M ('shift operators') Antonevich-Lebedev, Connes-Moscovici, Dave, Perrot, Ponge ...

Canonical Transformations

$T_0^* M = T^* M \setminus 0$ cotangent bundle without zero section

Definition

A canonical transformation C is a diffeomorphism $C : T_0^* M \to T_0^* M$ such that

- C preserves the symplectic form ω (symplectomorphism)
- C is one-homogeneous in the fiber:
  $(x, \xi) = C(y, \eta) \Rightarrow (x, \lambda\xi) = C(y, \lambda\eta)$, for $\lambda > 0$.

Theorem (Duistermaat-Singer)

Any *order preserving* automorphism of $\Psi^\infty(M)$ is given by conjugation with a quantized canonical transformation.

$\Rightarrow$ setting almost maximal.

Canonical Transformations

Example 1

Let $\alpha : M \to M$ be a diffeomorphism. Then α induces a canonical transformation C_α by

$$C_\alpha : T_0^* M \to T_0^* M;\ \ C_\alpha(y, \eta) = (\alpha^{-1}(y), (\partial\alpha)^t(\alpha^{-1}(y))\eta).$$

Example 2

Let $H = H(y, \eta) \in C^\infty(T_0^* M, \mathbb{R})$ be homogeneous of degree 1 in η.
$t \mapsto F_H(t)$ flow of the associated Hamiltonian vector field on $T_0^* M$.
Then, for each t, the map $F_H(t) : T_0^* M \to T_0^* M$ defines a canonical transformation.

Quantizing Canonical Transformations

Observation 1

The (twisted) graph of a canonical transformation C, i.e. the set

$$\Lambda = \{((x,\xi),(y,-\eta)) \in T_0^*M \times T_0^*M : (x,\xi) = C(y,\eta)\}$$

is a Lagrangian subspace of $T_0^*M \times T_0^*M$.

Observation 2

Microlocally, Λ can be described by a 1-homogeneous phase function, i.e. for $((x_0,\xi_0);(y_0,\eta_0))$ in the graph there exist $U_{x_0}, V_{y_0}, \Gamma \subset \mathbb{R}^d$ and $\varphi : U_{x_0} \times V_{y_0} \times \Gamma \to \mathbb{R}$:

$$\Lambda \overset{\mu\text{loc}}{=} \{((x,\partial_x\varphi(x,y,\theta)),(y,-\partial_y\varphi(x,y,\theta))) : \partial_\theta(x,y,\theta) = 0\}.$$

Observation 3

From this phase, we can define a Fourier integral operator (‘quantized canonical transformation’): For $a \in S^0$ consider the kernel $K(x,y) = \int \exp(i\varphi(x,y,\theta))a(x,y,\theta)\,d\theta$.

Ellipticity and the Fredholm Property

When is $D = \sum D_g \Phi_g$ a Fredholm operator?

- Assume all D_g are of order zero (use order reduction)
- Denote by C_g the canonical transformation associated with g.
- Then G acts on S^*M due to the homogeneity of the C_g.

Definition

The symbol of the operator $D = \sum_g D_g \Phi_g$ is the element

$$\sigma(D) = (\sigma(D_g))_{g \in G} \in C(S^*M) \rtimes G.$$

Call D elliptic, if $\sigma(D)$ is invertible.

Ellipticity and the Fredholm Property

Theorem

Elliptic operators $D : L^2(M) \longrightarrow L^2(M)$ are Fredholm.

Problem

Representation might not be unique and lead to a different symbol. Sufficient condition. Necessary?

Does not arise, if G is amenable and the action is topologically free (Antonevich & Lebedev).

Theorem

Can define localized analytic and algebraic indices. Both coincide

Theorem (Gorokhovsky, de Kleijn, Nest)

Obtain algebraic index theorem for equivariant operators.

Connection between both theorems?

More Concrete Examples. Case 1

Key idea: On $\mathbb{R}^n$ one can do the explicit computations.
Result: Index theorem for operators associated with metaplectic group

- $M = \mathbb{R}^n$
- $\Psi = \Psi(\mathbb{R}^n)$ Classical Shubin type ψdos on $\mathbb{R}^n$.
  Symbols of order m satisfy

  $$\partial_\xi^\alpha \partial_x^\beta a(x,p) \lesssim (1 + |x| + |p|)^{m-|\alpha|-|\beta|}.$$

  Principal symbols are functions on $\mathbb{S}^{2n-1} \subseteq T^*\mathbb{R}^n$.
- We identify $T^*\mathbb{R}^n \cong \mathbb{C}^n$ via $(x,p) \mapsto p - ix$ (complex structure).
- G is a discrete subgroup of $\mathcal{U}(n)$. It is represented on $L^2(\mathbb{R}^n)$ by metaplectic operators.

What are metaplectic operators?

The Metaplectic Group

Definition

$Mp(n) \subseteq \mathscr{L}(L^2(\mathbb{R}^n))$ is the group generated by all operators

$$A = e^{i\widehat{H}},$$

where $\widehat{H} = \mathrm{op}^w(H)$ (Weyl quantization) and $H = H(x,p)$ is a homogeneous real quadratic form on $T^*\mathbb{R}^n$.
The complex metaplectic group $Mp^c(n)$ is the group generated by all

$$A = e^{i(\widehat{H}+\phi)},$$

with H as before and $\phi \in \mathbb{R}$.

Since the Weyl quantization of real symbols yields self-adjoint operators, all metaplectic operators are unitary on $L^2(\mathbb{R}^n)$.

Relation to the Symplectic Group

Definition

The symplectic group $Sp(n) \subset GL(2n, \mathbb{R})$ is the group of linear canonical transformations of $T^*\mathbb{R}^n \cong \mathbb{R}^{2n}$, i.e., linear transformations that preserve the symplectic form $dp \wedge dx$.

Properties

- $Sp(n)$ is generated by the canonical transformations arising from the flow at time $t = 1$, of the Hamiltonian system

$$\dot{x} = \partial_p H, \quad \dot{p} = -\partial_x H$$

  where $H = H(x, p)$ is a homogeneous real quadratic form as above.
- There is a natural projection $\pi : Mp(n) \to Sp(n)$, taking a metaplectic operator to the corresponding canonical transformation. This projection is a nontrivial double covering of $Sp(n)$. So we can not represent the symplectic matrices by metaplectic operators. However, ...

From Unitary Matrices to Metaplectic Operators

...we can lift the unitary operators as follows:

- Identifying $(x, p) \in T^*\mathbb{R}^n$ with $z = p - ix \in \mathbb{C}^n$ gives $T^*\mathbb{R}^n$ a complex structure.
- Under this identification, $\mathcal{U}(n) \cong Sp(n) \cap O(2n)$.
- $\mathcal{U}(n)$ is generated by matrices $\exp(B + iA)$ with A a real symmetric $n \times n$ matrix and B a skew-symmetric real $n \times n$ matrix.

Proposition

The mapping $R : \mathcal{U}(n) \longrightarrow Mp^c(n)\ ,\ g \mapsto \pi^{-1}(g)\sqrt{\det g}$, defined near I in $\mathcal{U}(n)$, where π^{-1} is the section for $\pi : Mp(n) \to Sp(n)$ with $\pi^{-1}(I) = I$ and $\sqrt{1} = 1$, extends to $\mathcal{U}(n)$ as a monomorphism of groups. In terms of Hamiltonians: For A symmetric, B is skew-symmetric, and

$$H(x, p) = \frac{1}{2}(x, p)\begin{pmatrix} A & -B \\ B & A \end{pmatrix}\begin{pmatrix} x \\ p \end{pmatrix},$$

$$R\left(\exp(B + iA)\right) = \exp(-i\widehat{H})\sqrt{\det(\exp(B + iA))} = \exp(-i\widehat{H})\exp(i\,\mathrm{tr}\,A/2).$$

A More Explicit Description

$\mathcal{U}(n)$ is generated by $O(n)$ and $\mathcal{U}(1)$, interpreted as a subgroup of $\mathcal{U}(n)$ via $e^{i\varphi} \mapsto \mathrm{diag}(e^{i\varphi}, 1, \ldots, 1)$. We then obtain

$$\begin{aligned} R_g u(x) &= u(g^{-1}(x)), \qquad g \in O(n) \\ R_g u(x) &= e^{i\varphi(\frac{1}{2} - \widehat{H}_1)} u(x), \quad g = \mathrm{diag}(e^{i\varphi}, 1, \ldots, 1), \varphi \neq 0. \end{aligned}$$

where $\widehat{H}_1 = \frac{1}{2}(x_1^2 - \partial_{x_1}^2)$.
In this case, R_g is a so-called fractional Fourier transform. Mehler's formula shows that for $0 < \varphi < 2\pi$, $\varphi \neq \pi$ (reflection in 1st variable),

$$\begin{aligned} & R_g u(x) \\ = & \sqrt{\frac{1 - i \operatorname{ctg} \varphi}{2\pi}} \int \exp\left(i \left((x_1^2 + y_1^2) \frac{\operatorname{ctg} \varphi}{2} - \frac{x_1 y_1}{\sin \varphi} \right) \right) u(y_1, x_2, ..., x_n) dy_1 . \end{aligned}$$

Fourier integral operator with quadratic phase.
R_g is the Fourier transform in 1st variable for $\varphi = \pi/2$.

Symbols

$\mathcal{U}(n)$ acts on Ψ by conjugation: $\mathcal{U}(n) \ni g \mapsto (A \mapsto R_g A R_g^{-1})$, $A \in \Psi$.

Egorov Type Theorem

The principal symbol $\sigma : \Psi \to C(\mathbb{S}^{2n-1})$ satisfies

$$\sigma(R_g A R_g^{-1}) = (g^{-1})^* \sigma(A), \quad A \in \Psi.$$

Symbol Map

To $D = \sum D_g R_g$ associate the symbol $(\sigma(D_g))_g \in C(\mathbb{S}^{2n-1}) \rtimes G$.
NB G discrete subgroup of $\mathcal{U}(n)$.

Ellipticity

Operators Acting in Ranges of Projections

P_1, P_2: $N \times N$ matrix projections over $C^*(G)$. Consider the triple (D, P_1, P_2), where $D = \sum_g A_g R_g$, $A_g \in \Psi^0$

$$D : \operatorname{im} P_1 \to \operatorname{im} P_2, \quad \operatorname{im} P_1, P_2 \subseteq L^2(\mathbb{R}^n, \mathbb{C}^N).$$

Ellipticity

Call (D, P_1, P_2) elliptic, if $\exists r \in \mathrm{Mat}_N(C(\mathbb{S}^{2n-1}) \rtimes G)$ s.t.

$$P_1 r \sigma(D) = P_1, \quad \sigma(D) r P_2 = P_2.$$

Lemma

Ellipticity implies the Fredholm property.

Twisted Traces

Let $s \in \mathcal{U}(n)$ acting on $\mathbb{C}^n \cong T^*\mathbb{R}^n$. Define

$$\begin{array}{rcl} \tau_s : C_c^\infty(\mathbb{C}^n, \Lambda(\mathbb{C}^n)) & \to & \mathbb{C} \\ \omega & \mapsto & \displaystyle\int_{L_s} \omega_{|L_s}, \end{array}$$

where L_s is the fixed point set of s with the complex orientation. For each s we get a closed graded trace

$$\tau_{\langle s \rangle} : C_c^\infty(\mathbb{C}^n, \Lambda(\mathbb{C}^n)) \rtimes \mathcal{U}(n) \to \mathbb{C}$$

from $\tau_{\langle s \rangle}((a_g)_g) = \sum_{g \in \langle s \rangle} \tau_g(a_g)$.

The Index Theorem

The Topological Index $\text{ind}_t : K_0(C_0(\mathbb{C}^n) \rtimes G) \to \mathbb{C}$

P_0, P_1 projections in the smooth crossed product $\text{Mat}_N(C^\infty(\mathbb{C}^n) \rtimes \mathbb{C})$, $P_0 = P_1$ at infinity.

$$\text{ind}_t([P_1] - [P_0]) = \sum_{\langle s\rangle \subseteq G} \frac{1}{\det(1 - s|_{L_s^\perp})}$$
$$\times \text{tr}\, \tau_{\langle s\rangle} \left(P_1 \exp\left(-\frac{dP_1 dP_1}{2\pi i}\right) - P_0 \exp\left(-\frac{dP_0 dP_0}{2\pi i}\right)\right).$$

Index Theorem

Let $G \subset \mathcal{U}(n)$ be discrete of polynomial growth, $D = \sum D_g R_g$, P_1, P_2 in the smooth crossed product and let (D, P_1, P_2) be elliptic. Then

$$\text{ind}\ D = \text{ind}_t[\sigma(D)]$$

Key Ingredient: The Euler Operator

The action of the R_g extends to forms (via pullback with g^{-1}).

$$\mathcal{E} = d + d^* + xdx\wedge + (xdx\wedge)^* : \mathcal{S}(\mathbb{R}^n, \Lambda^{ev}(\mathbb{C}^n)) \longrightarrow \mathcal{S}(\mathbb{R}^n, \Lambda^{odd}(\mathbb{C}^n)).$$

$\mathcal{E}$ is $\mathcal{U}(n)$equivariant, elliptic and Fredholm of index 1. Let $\mathcal{E}_0$ be the associated zero-order operator

$$\mathcal{E}_0 = (\mathcal{E}\mathcal{E}^* + 1)^{-1/2}\mathcal{E}.$$

Theorem

The mapping

$$\begin{array}{rcl} \beta : K_0(C^*(G)) & \longrightarrow & K_0(C_0(T^*\mathbb{R}^n) \rtimes G) \\ P & \longmapsto & [(\sigma(\mathcal{E}_0 \otimes 1_N), 1 \otimes P, 1 \otimes P)] \end{array} \quad (1)$$

is an isomorphism of Abelian groups.

More Concrete Examples. Case 2

The Connes-Moscovici local index formulae for operators associated with he affine metaplectic group
As before:

- $M = \mathbb{R}^n$
- $\Psi = \Psi(\mathbb{R}^n)$ Classical Shubin type ψdos on $\mathbb{R}^n$.
- We identify $T^*\mathbb{R}^n \cong \mathbb{C}^n$ via $(x, p) \mapsto p - ix$ (complex structure).
- $\mathcal{U}(n)$ represented on $L^2(\mathbb{R}^n)$ by metaplectic operators R_g.
- $\mathbb{C}^n$ is represented on $L^2(\mathbb{R}^n)$ by Heisenberg-Weyl operators. For $z = a - ik$,

  $$T_z u(x) = e^{ikx - iak/2} u(x - a).$$

  Note that

  $$T_z T_w = e^{-i \mathrm{Im}\langle z, w\rangle/2} T_{z+w}.$$

  and

  $$R_g T_z R_g^{-1} = T_{gz}.$$

Noncommutative Geometry

Definition (Connes): Spectral Triple

Triple $(\mathscr{A}, \mathscr{H}, D)$, where

- $\mathscr{H}$ is a complex Hilbert space
- $\mathscr{A}$ is a selfadjoint algebra of bounded operators on $\mathscr{H}$, and
- $D : \mathscr{D}(D) \to \mathscr{H}$ is an unbounded selfadjoint operator on $\mathscr{H}$
  1. $(I + D^2)^{-1}$ is compact on $\mathscr{H}$
  2. Each $a \in \mathscr{A}$ maps $\mathscr{D}(D)$ to itself and $[D, a]$ extends to a bounded operator on $\mathscr{H}$.

Graded, p-summable Triples

Assume additionally that

- $\mathscr{H} = \mathscr{H}_+ \oplus \mathscr{H}_-$ is a graded Hilbert space (grading $\gamma \in \mathscr{L}(\mathscr{H})$, $\gamma = \gamma^*$, $\gamma^2 = 1$).
- $D = \begin{pmatrix} 0 & D_+ \\ D_- & 0 \end{pmatrix}$ is an odd operator: $D\gamma = -\gamma D$
- D is p-summable: $(I + D^2)^{-p/2}$ is trace class
- $a\gamma = \gamma a$, $a \in \mathscr{A}$.

Consequences (Connes)

- Can define Chern-Connes character $\mathrm{ch}(\mathscr{A}, \mathscr{H}, D) \in HP^{ev}(\mathscr{A})$ (even periodic cyclic cohomology)
- If P is a projection in $Mat_N(\mathscr{A})$, then $\mathrm{ind}\, P(D \otimes I_N)P : P\mathscr{H}_+^N \to P\mathscr{H}_-^N = \langle \mathrm{ch}(\mathscr{A}, \mathscr{H}, D), [P] \rangle$, where $[P] \in K_0(\mathscr{A})$ is the K-class of P.

Connes-Moscovici

Problem. Formula not suitable for explicit computation.
Idea. Find a representative of $\mathrm{ch}(\mathscr{A}, \mathscr{H}, D)$ which is easier to handle.

- Assume $(\mathscr{A}, \mathscr{H}, D)$ is **regular**: For $a \in \mathscr{A}$, both a and $[D, a]$ belong to the domains of all iterated commutators with $|D|$.
- **Define** $\Psi(\mathscr{A}, \mathscr{H}, D)$: smallest algebra of linear operators in $\mathscr{H}^\infty = \cap_{j \geq 1} \mathrm{Dom}|D|^j$ that contains $\mathscr{A}$ and $[D, \mathscr{A}]$ and is closed under taking commutators with D^2.
  For $B \in \Psi(\mathscr{A}, \mathscr{H}, D)$ consider the **zeta functions**
  $$\zeta_B(z) = \mathrm{Tr}(B|D|^{-2z}), \quad \mathrm{Re}(z) \text{ large.}$$
- Assume $(\mathscr{A}, \mathscr{H}, D)$ has **simple dimension spectrum**: $\exists F \subset \mathbb{C}$ discrete, such that $\zeta_B(z)$ extends meromorphically to $\mathbb{C}$ with **at most simple poles** in the set $F + \mathrm{ord}\, B$.

Connes Moscovici

Theorem (Connes-Moscovici)

The Chern–Connes character $\mathrm{ch}(\mathscr{A}, \mathscr{H}, D) \in HP^{ev}(\mathscr{A})$ has a representative $(\Psi_0, \Psi_2, \Psi_4, ..., \Psi_{2k}, ...)$, where

$$\Psi_0(a_0) = \mathrm{res}_{z=0}\, z^{-1}\, \mathrm{tr}_s\left(a_0|D|^{-2z}\right),$$

and, for $k \geq 1$,

$$\begin{aligned}&\Psi_{2k}(a_0, a_1, ..., a_{2k})\\ &\quad = \sum_\alpha c_{k,\alpha}\, \mathrm{res}_{z=0}\, \mathrm{tr}_s\left(a_0[D, a_1]^{[\alpha_1]} \ldots [D, a_{2k}]^{[\alpha_{2k}]}|D|^{-2(|\alpha|+k+z)}\right),\end{aligned}$$

$\alpha = (\alpha_1, \alpha_2, ..., \alpha_{2k})$ is a multi-index, $B^{[j]}$ stands for the j-th iterated commutator of the operator B with D^2, and

$$c_{k,\alpha} = (-1)^{|\alpha|}\frac{\Gamma(|\alpha| + k)}{\alpha!(\alpha_1 + 1)...(\alpha_1 + ... + \alpha_{2k} + 2k)},$$

The Spectral Triple

Definition: $\mathscr{A}$, $\mathscr{H}$

- $\mathscr{A}$: algebra generated by the operators T_z, $z \in \mathbb{C}^n$, and R_g, $g \in \mathcal{U}(n)$. Elements have the form $\sum_k a_k T_{z_k} R_{g_k}$.
- The Hilbert space $\mathscr{H}$ is $L^2(\mathbb{R}^n, \Lambda(\mathbb{C}^n))$. It is naturally graded by the degree of forms (even/odd).
- The Dirac operator is the Euler operator

$$D = \mathcal{E} = (d + d^* + xdx \wedge +(xdx\wedge)^*) : \\ \mathcal{S}(\mathbb{R}^n, \Lambda^{ev}(\mathbb{C}^n)) \longrightarrow \mathcal{S}(\mathbb{R}^n, \Lambda^{odd}(\mathbb{C}^n)).$$

Spectral Triple?

Theorem

$(\mathscr{A}, \mathscr{H}, D)$ is a spectral triple meeting Connes-Moscovici's assumptions: It is regular, finitely summable and has simple dimension spectrum.

For $\Psi(\mathscr{A}, \mathscr{H}, D)$ we choose the algebra of all operators

$$B = \sum D_k T_{z_k} R_{g_k}, \quad D_k \text{ Shubin } \psi\text{do}, z_k \in \mathbb{C}^n, g_k \in \mathcal{U}(n).$$

This algebra is larger than the one required by CM (makes things better).

Theorem: Connes-Moscovici Cyclic Cocycles

Let $a_j = T_{z_j} R_{g_j}$, $z_j \in \mathbb{C}^n, g_j \in \mathcal{U}(n)$. Then for $\Psi = (\Psi_0, \Psi_2, \ldots)$
$\Psi_{2k}(a_0, a_1, ..., a_{2k}) =$

$$= 0, \text{ if the mapping } w \mapsto gw + z \text{ has no fixed points or } k > \dim \mathbb{C}^n_g$$

$$= \frac{i^{-k}}{(2k)!} e^{i\varepsilon} \prod_{j=1}^{m} e^{\frac{i}{4}|(z,e_j)|^2 \operatorname{ctg} \varphi_j/2} \int_{\mathbb{C}^n_g} \sigma(w_1) \wedge \sigma(w_2) \wedge ... \wedge \sigma(w_{2k}) \wedge e^{-\omega}$$

Here:

- $\mathbb{C}^n_g \subset \mathbb{C}^n$ is the fixed point set of $g = g_0 g_1 ... g_{2k}$; $m = n - \dim \mathbb{C}^n_g$;
- $e^{i\varphi_j}$ for $j = 1 \ldots m$: eigenvalues of $g \neq 1$, $e_j \in \mathbb{C}^n$ eigenvectors;
- $z = w_0 + w_1 + ... w_{2k}$, where $w_j = (g_0 g_1 ... g_{j-1}) z_j$;
- $e^{i\varepsilon} = T_{w_0} T_{w_1} T_{w_2} \ldots T_{w_{2k}} T_z^{-1}$;
- $\sigma(a - ik) = adp - kdx \in \Lambda^1(\mathbb{R}^{2n})$;
- $\omega = \sum_{j=1}^{n} dp_j \wedge dx_j$ is the symplectic form on $\mathbb{R}^{2n}$.
- $\int_{\mathbb{C}^n_g} : \Lambda(\mathbb{C}^n_g) \to \mathbb{C}$: Berezin integral
  $\lambda \mapsto$ coeff. of $dp_1 \wedge dx_1 \wedge \ldots \wedge dp_n \wedge dx_n$ in λ

Application: Noncommutative Tori

Let $v_j \in \mathbb{C}^n$, $j = 1, \ldots, N$, linearly independent over $\mathbb{Q}$. Generate lattice

$$\Big\{ \sum_j n_j v_j : n_j \in \mathbb{Z} \Big\} \subseteq \mathbb{C}^n.$$

Define $\mathscr{A}_v \subseteq \mathscr{A}$ 'algebra of functions on the noncommutative torus' by

$$\mathscr{A}_v = \Big\{ \sum_k c_k T_{v_1}^{k_1} \cdots T_{v_N}^{k_N} : c_k \in \mathbb{C}, k = (k_1, k_2, ..., k_N), k_j \in \mathbb{Z} \Big\}.$$

We have the commutation relations

$$T_{v_k} T_{v_l} = e^{-iIm(v_k, v_l)} T_{v_l} T_{v_k}.$$

For the Chern–Connes character $\Psi = (\Psi_0, \Psi_2, \ldots)$ we obtain

$$\Psi_{2k}(a_0, ..., a_{2k}) = \frac{i^{-k}}{(2k)!} \int_{\mathbb{C}^n} a_0 da_1 ... da_{2k} \wedge e^{\omega}, \quad k \leq n, \quad a_j \in \mathscr{A}_v$$

with $dT_z = \sigma(z)\wedge$. For $n = 1$ we obtain Connes' cyclic cocycles.

Application: Noncommutative $\mathbb{Z}_4$-orbifolds

Choose $z_1 = k, z_2 = ik$, $k > 0$ and $g = i \in \mathcal{U}(1)$. Define square lattice

$$L = \{n_1 z_1 + n_2 z_2 \in \mathbb{C} : n_1, n_2 \in \mathbb{Z}\}.$$

The group $\mathbb{Z}_4 = \{i^\beta : \beta \in \mathbb{Z}\}$ acts on L by rotations.
Associate unitary operators $U = T_{z_1}, V = T_{z_2}$, $R = R_g$. We obtain the commutation relations:

$$VU = e^{i\theta} UV, \quad RUR^{-1} = V, \quad RVR^{-1} = U^{-1}, \quad \text{where } \theta = -k^2.$$

The algebra generated by U and V is the noncommutative torus $\mathscr{A}_\theta$, that generated by U, V, R is the crossed product $\mathscr{A}_\theta \rtimes \mathbb{Z}_4$ with respect to the action of the generator of $\mathbb{Z}_4$ on the generators $U, V \in \mathscr{A}_\theta$ as:

$$U \longmapsto RUR^{-1} = V, \quad V \longmapsto RVR^{-1} = U^{-1}.$$

This crossed product is known as a 'noncommutative orbifold for the group $\mathbb{Z}_4$' Studied earlier by Farsi, Watling, Walters, Echterhoff, Lück, Phillips ....

Application: Noncommutative $\mathbb{Z}_4$-orbifolds

Elements $f \in \mathscr{A}_\theta \rtimes \mathbb{Z}_4$ can be written as

$$f = \sum_{(z,\alpha)\in L\times\mathbb{Z}_4} f(z,\alpha) T_z R_i^\alpha .$$

For each $(z,\alpha) \in L \times \mathbb{Z}_4$ we have cyclic cocycles

$$\Phi_{2l;z,\alpha} \in HC^{2l}(\mathscr{A}_\theta \rtimes \mathbb{Z}_4), \quad l = 0, 1.$$

- If $\alpha = 0$, then the cocycles are nontrivial only if $z = 0$. Then the fixed point set of $w \mapsto i^\alpha w + z$ equals $\mathbb{C}$ and

  $$\Phi_{0;0,0}(f) = f(0,0), \quad \Phi_{2;0,0}(f_0, f_1, f_2) = \int_{\mathbb{C}} (f_0 df_1 df_2)(0,0),$$

  with the Berezin integral $\int_{\mathbb{C}}$.

Application: Noncommutative $\mathbb{Z}_4$-orbifolds

- If $\alpha \neq 0$, then the fixed point set of $w \mapsto i^\alpha w + z$ is always nontrivial and the fixed point set of the rotation $w \mapsto i^\alpha w$ is the origin. Hence, the cocycle $\Phi_{2;z,\alpha}$ is trivial.

  For the cocycle $\Phi_{0;z,\alpha}$, a direct computation shows that the conjugacy class $\langle (z, i^\alpha) \rangle \subset \mathbb{Z}^2 \rtimes \mathbb{Z}_4$ is equal to

$$\langle (z, i^\alpha) \rangle = \{ i^\beta z + w(1 - i^\alpha) \mid (w, \beta) \in L \times \mathbb{Z}_4 \} \times \{ i^\alpha \}.$$

  Hence, the cyclic cocycle is equal to

$$\Phi_{0;z,\alpha}(f) = \sum_{z' \in i^{\mathbb{Z}} z + L(1 - i^\alpha)} \exp\left(\frac{i}{4} |z'|^2 \operatorname{ctg} \frac{\pi\alpha}{4} \right) f(z', \alpha).$$

Noncommutative $\mathbb{Z}_6$-orbifolds can be handled similarly.

Thank You for Your Attention!

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