Surgery sequences and higher invariants of Dirac operators

Paolo Piazza (Sapienza Università di Roma).

Global Noncommutative Geometry Seminar. March 10th 2021.
Based on joint work with Thomas Schick and Vito Felice Zenobi.

Outline

The Higson-Roe surgery sequence

K-theory invariants

Higher numeric invariants

Higher rho numbers

Details on the delocalized Chern character

Some geometric applications

Invariants of Dirac operator

- We consider a smooth compact manifold without boundary M and a Galois coverings $\Gamma \to \widetilde{M} \to M$.
- We consider a Dirac-type operator D acting between the sections of a vector bundle E and we denote by $\widetilde{D}$ its lift to $\widetilde{M}$.
- $\widetilde{D}$ is a Γ-equivariant operator.
- We are interested in K-theory invariants associated to D and $\widetilde{D}$ and to ways to extract numeric invariants out of these K-theory invariants. We want geometric applications.
- One way to organize these K-theory invariants is through the Higson-Roe analytic surgery sequence.
- This allows to recover known invariants, such as
  - the fundamental class $[D] \in K_*(M)$
  - the index class $\mathrm{Ind}(\widetilde{D}) \in K_*(C_r^*\Gamma)$

  but also to define a new invariant:
  - the rho class of an invertible Dirac operator $\rho(\widetilde{D})$.

The Higson-Roe surgery sequence

The Higson-Roe analytic surgery sequence is the K-theory sequence

$$\longrightarrow K_*(D^*(\widetilde{M})^\Gamma) \longrightarrow K_*(D^*(\widetilde{M})^\Gamma / C^*(\widetilde{M})^\Gamma) \xrightarrow{\partial} K_{*+1}(C^*(\widetilde{M})^\Gamma) \longrightarrow \cdots$$

associated to the short exact sequence of C^*-algebras

$$0 \longrightarrow C^*(\widetilde{M})^\Gamma \longrightarrow D^*(\widetilde{M})^\Gamma \longrightarrow D^*(\widetilde{M})^\Gamma / C^*(\widetilde{M})^\Gamma \longrightarrow 0$$

- $D^*(\widetilde{M})^\Gamma$ is the norm closure of $D^*_c(\widetilde{M})^\Gamma \subset \mathcal{B}(L^2(\widetilde{M}))$
- here $D^*_c(\widetilde{M})^\Gamma$ is the algebra of Γ-equivariant bounded operators on L^2 that are of finite propagation and pseudolocal (i.e. $[f, T]$ is compact for any $f \in C^\infty_c(\widetilde{M})$).
- $C^*(\widetilde{M})^\Gamma$ is the norm-closure of $C^*_c(\widetilde{M}) \subset D^*_c(\widetilde{M})^\Gamma$
- operators in $C^*_c(\widetilde{M})^\Gamma \subset \mathcal{B}(L^2(\widetilde{M}))$ are of finite propagation and locally compact (i.e. fT is compact for any $f \in C^\infty_c(\widetilde{M})$).
- $C^*(\widetilde{M})^\Gamma$ is an ideal in $D^*(\widetilde{M})^\Gamma$

Facts I:

- Example 1: $\Psi^{-\infty}_{\Gamma,c}(\widetilde{M})$ is a subalgebra of $C^*_c(\widetilde{M})^\Gamma$
- Example 2: $\Psi^{0}_{\Gamma,c}(\widetilde{M})$ is a subalgebra of $D^*_c(\widetilde{M})^\Gamma$
- $K_*(D^*(\widetilde{M})^\Gamma / C^*(\widetilde{M})^\Gamma) = K^\Gamma_{*+1}(\widetilde{M}) = K_{*+1}(M)$.
  (Paschke duality)
- $K_*(C^*(\widetilde{M})^\Gamma) = K_*(C^*(\widetilde{M} \times_\Gamma \widetilde{M})) = K_*(C^*_r \Gamma)$
- by definition $\mathrm{S}^\Gamma_*(\widetilde{M}) := K_{*+1}(D^*(\widetilde{M})^\Gamma)$ is the **analytic structure set of $\widetilde{M}$, with** $* = \dim M$.

We can rewrite the Higson-Roe sequence as

$$\cdots \to K_{*+1}(C^*_{red}\Gamma) \to \mathrm{S}^\Gamma_*(\widetilde{M}) \to K^\Gamma_*(\widetilde{M}) \to K_*(C^*_{red}\Gamma) \to \cdots$$

Facts II:

- fix a chopping function χ: a smooth odd function $\to \pm 1$ as $x \to \pm\infty$.
- Can prove that $\chi(\widetilde{D}) \in D^*(\widetilde{M})^\Gamma$ and that $\chi(\widetilde{D})$ is an involution in $D^*(\widetilde{M})^\Gamma / C^*(\widetilde{M})^\Gamma$
- If n is odd then $(\frac{1}{2}(1+\chi(\widetilde{D}))$ is a projection in $D^*(\widetilde{M})^\Gamma / C^*(\widetilde{M})^\Gamma \Rightarrow [\frac{1}{2}(1+\chi(\widetilde{D})] \in K_0(D^*(\widetilde{M})^\Gamma / C^*(\widetilde{M})^\Gamma)$
- if $n = 2k$ we use the grading and get a class in K_1
- these classes corresponds to Kasparov' classes $[D] \in K_*(M)$ under Paschke and we keep the same notation $[D]$
- $\partial[D] \in K_*(C^*(\widetilde{M})^\Gamma)$ is, by def., the coarse index class of $\widetilde{D}$
- it corresponds to the index class $\mathrm{Ind}(\widetilde{D}) \in K_*(C^*(\widetilde{M} \times_\Gamma \widetilde{M}))$ obtained by the Connes-Skandalis projector.
- it also corresponds to the Mishchenko-Fomenko index class $\mathrm{Ind}_{\mathrm{MF}}(\widetilde{D}) \in K_*(C^*_r\Gamma)$

Rho-classes

- assume that $\widetilde{D}$ is L^2-invertible; choose χ equal to the sign function on $\mathrm{spec}(\widetilde{D})$; then $\chi(\widetilde{D}) = \widetilde{D}/|\widetilde{D}|$
- if $\widetilde{D}$ is L^2-invertible then $\partial[D] \equiv \mathrm{Ind}(\widetilde{D}) = 0$ in $K_*((C^*(\widetilde{M})^\Gamma)$
- $\rho(\widetilde{D}) \in K_*(D^*(\widetilde{M})^\Gamma)$ is the natural **lift** of $[D]$.

$$K_{*+1}(D^*(\widetilde{M})^\Gamma) \longrightarrow K_{*+1}(D^*(\widetilde{M})^\Gamma / C^*(\widetilde{M})^\Gamma) \longrightarrow K_*(C^*(\widetilde{M})^\Gamma)$$

$$\rho(\widetilde{D}) \dashleftarrow [D] \xrightarrow{\partial} 0$$

More precisely, for example if dim M is odd,

$$\rho(\widetilde{D}) = [\frac{1}{2}(1 + \frac{\widetilde{D}}{|\widetilde{D}|})] = [\Pi_{>0}(\widetilde{D})] \in K_0(D^*(\widetilde{M})^\Gamma) \equiv \mathrm{S}_1^\Gamma(\widetilde{M})$$

Summary+ Examples+Variants

- Summarizing: if $* = \dim M$ we have recovered the fundamental class $[D] \in K_*(M)$ and the index class $\mathrm{Ind}(\widetilde{D}) \in K_*(C_r^*\Gamma)$ within the Higson-Roe sequence and we have defined the rho class $\rho(\widetilde{D}) \in \mathrm{S}_*^\Gamma(\widetilde{M})$ of an invertible operator $\widetilde{D}$.
- Example 1: if g is a positive scalar curvature metric and M is spin, then we have $\rho(g) := \rho(\widetilde{D}_g^{\mathrm{spin}})$ (use Lichnerowicz)
- Example 2: if $f : X \to Y$ is a homotopy equivalence and $M = X \sqcup (-Y)$ then we have $\rho(f)$ defined via $\widetilde{D}_M^{\mathrm{sign}} + A(f)$, with $A(f)$ the Hilsum-Skandalis perturbation
- there are interesting alternative treatments of the Higson-Roe surgery sequence:
  - Xie-Yu (using Yu's localization algebra)
  - Deeley-Goffeng (à la Baum-Douglas)
  - Zenobi (using the adiabatic groupoid; this will be crucial).

Mapping geometric surgery to analytic surgery

- Higson and Roe proved that one can map the Browder-Novikov-Sullivan-Wall surgery sequence for a smooth manifold M to the Higson-Roe analytic surgery sequence; crucial is the rho class of a homotopy equivalence $\rho(f)$
- P. and Schick proved that one can map the Stolz' surgery sequence for positive scalar curvature metrics to the Higson-Roe surgery sequence; crucial is the rho class of a PSC metric $\rho(g)$
- Further treatments by Xie-Yu and Zeidler (for Stolz), P-Schick, Zenobi and Weinberger-Xie-Yu (for Browder-Novikov-Sullivan-Wall, the last two also for topological manifolds)
- P. and Albin proved that one can map the Browder-Quinn surgery sequence for a Witt pseudomanifold to the Higson-Roe surgery sequence

Higher indices: variations on Connes-Moscovici

- we want to extract numbers out of our K-theory invariants
- for the Index class $\mathrm{Ind}(\widetilde{D})$ we have the seminal work of Connes and Moscovici (here with variations)
- we assume that M is even dimensional
- given $\varphi \in H^k(\Gamma)$ we have a cyclic class $[\tau_\varphi^\Gamma] \in HC^k(\mathbb{C}\Gamma)$ given by the cyclic cocycle: $\tau_\varphi^\Gamma(g_0, g_1, \dots, g_k) = 0$ if $g_0 \cdots g_k \neq e$ $\tau_\varphi^\Gamma(g_0, g_1, \dots, g_k) = \varphi(g_0, g_0 g_1, \dots, g_0 \cdots g_k)$ if $g_0 \cdots g_k = e$
- observe that $[\tau_\varphi^\Gamma] \in HC^*(\mathbb{C}\Gamma, \langle e \rangle)$
- recall that $HC^*(\mathbb{C}\Gamma)$ decomposes as the direct product of $HC^*(\mathbb{C}\Gamma, \langle x \rangle)$ (with $HC^*(\mathbb{C}\Gamma, \langle x \rangle)$ defined requiring $\tau(g_0, g_1, \dots, g_k) = 0$ if $g_0 \cdots g_k \notin \langle x \rangle$)) and that $HC^*(\mathbb{C}\Gamma, \langle x \rangle)$ is explicitly computable (Burghelea)
- there exists a smooth (:= dense holomorphically closed) subalgebra $\mathcal{B}\Gamma$ of $C_r^*\Gamma$ defined by Connes and Moscovici

Higher indices: variations on Connes-Moscovici (cont)

- we can consider $H_*(\mathcal{B}\Gamma)$, the non-commutative de Rham homology of $\mathcal{B}\Gamma$ (more on this later)
- there exists a Chern character $\mathrm{Ch} : K_0(\mathcal{B}\Gamma) \to H_{2*}(\mathcal{B}\Gamma)$ defined à la Chern-Weil
- there is natural pairing $\langle\, , \rangle : H_*(\mathcal{B}\Gamma) \times HC^*(\mathcal{B}\Gamma) \to \mathbb{C}$
- in general we do not know if $\tau^\Gamma_\varphi \in ZC^k(\mathbb{C}\Gamma, \langle e \rangle)$ extends to $\mathcal{B}\Gamma$
- however, if Γ is Gromov hyperbolic then Connes and Moscovici show that there exists a cocycle representative φ such that τ^Γ_φ does extend, defining $[\tau^\Gamma_\varphi] \in HC^k(\mathcal{B}\Gamma)$
- we have higher numeric C^*-indices $\langle \mathrm{Ch}(\mathrm{Ind}^\infty_{\mathrm{MF}}(\widetilde{D})), [\tau^\Gamma_\varphi] \rangle$

Higher rho numbers: general info

- Now we want to extract numbers out of the rho class $\rho(\widetilde{D})$ in $\mathrm{S}^{\Gamma}_{*}(\widetilde{M})$. These are higher rho numbers.
- road map: we map the entire Higson-Roe sequence to a sequence in non-commutative de Rham homology with commutative squares and **then** we pair with the cyclic cohomology of $\mathbb{C}\Gamma$ under additional assumptions
- our results in the article
  Mapping analytic surgery to homology, higher rho numbers and metrics of positive scalar curvature by P.P, Thomas Schick and Vito Felice Zenobi, arXiv 1905.11861
- there is a new version with many applications to PSC metrics
- higher rho numbers treated also in the article
  Delocalized eta invariants, cyclic cohomology and higher rho invariants by Xiaoman Chen, Jinmin Wang, Zhizhang Xie, and Guoliang Yu, arXiv:1901.02378.
- common results by different methods

Preliminaries

• Recall that if A is a unital algebra then $H_*(A)$ is the homology of $\Omega_*(A)_{ab} := \Omega_*(A)/[\Omega_*(A), \Omega_*(A)]$ with $\Omega_*(A)$ the universal differential graded algebra (universal DGA).
• If A is a Fréchet algebra we take projective tensor products and the closure of the graded commutators (denoted $\widehat{\Omega}(A)_{ab}$)
• In particular we have $H_*(\mathbb{C}\Gamma)$ and $H_*(\mathcal{A}\Gamma)$ with $\mathcal{A}\Gamma$ a smooth subalgebra of $C_r^*\Gamma$.
• Can define $H_*(\Omega_\bullet)$ for any DGA $\Omega_\bullet$ over A (that is, $\Omega_0 = A$)
• A basis for $\Omega_*(\mathbb{C}\Gamma)$ is given by $\{g_0 dg_1 \cdots dg_k\}$
• the inclusion $j : \mathbb{C}\Gamma \hookrightarrow \mathcal{A}\Gamma$ induces a morphism of DGA $j : \Omega_*(\mathbb{C}\Gamma) \to \widehat{\Omega}_*(\mathcal{A}\Gamma)$
• $\Omega^e(\mathbb{C}\Gamma)$ is the sub-DGA generated by $\{g_0 dg_1 \cdots dg_k\}$ such that $g_0 g_1 \cdots g_k = e$
• we consider the closure of $j(\Omega^e(\mathbb{C}\Gamma)_{ab})$ in $\widehat{\Omega}_*(\mathcal{A}\Gamma)_{ab}$; get $H_*^e(\mathcal{A}\Gamma)$. Using the quotient complex we also get $H_*^{del}(\mathcal{A}\Gamma)$.

From analytic surgery to homology

- From the previous discussion we get the long exact sequence

$$\cdots H_*(\mathcal{A}\Gamma) \to H_*^{del}(\mathcal{A}\Gamma) \xrightarrow{\delta_\Gamma} H_{*-1}^{e}(\mathcal{A}\Gamma) \to H_{*-1}(\mathcal{A}\Gamma) \to \cdots$$

- Recall the Higson-Roe surgery sequence:

$$\cdots \to K_{*-1}(C^*_{red}\Gamma) \to \mathrm{S}^\Gamma_*(\widetilde{M}) \to K^\Gamma_*(\widetilde{M}) \to K_*(C^*_{red}\Gamma) \to \cdots$$

Theorem

(P-Schick-Zenobi, 2019) For any smooth subalgebra $\mathcal{A}\Gamma$ of $C^*_r\Gamma$ there exist Chern character homomorphisms

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & K_{*-1}(C^*_{red}\Gamma) & \longrightarrow & \mathrm{S}^\Gamma_*(\widetilde{M}) & \longrightarrow & K^\Gamma_*(\widetilde{M}) & \longrightarrow & \cdots \\
 & & \downarrow \mathrm{Ch}_\Gamma & & \downarrow \mathrm{Ch}^{del}_\Gamma & & \downarrow \mathrm{Ch}^{e}_\Gamma & & \\
\cdots & \xrightarrow{j_*} & H_{[*-1]}(\mathcal{A}\Gamma) & \longrightarrow & H^{del}_{[*-1]}(\mathcal{A}\Gamma) & \xrightarrow{\delta} & H^{e}_{[*]}(\mathcal{A}\Gamma) & \xrightarrow{j_*} & \cdots
\end{array}$$

making the diagram commute.

- The theorem is proved using a different realization of the Higson-Roe sequence, based solely on pseudodifferential operators. This is due to Zenobi.

Details: a simplified task

- assume now that M is odd dimensional
- we want to explain $\mathrm{Ch}_\Gamma^{del} : \mathrm{S}_1^\Gamma(\widetilde{M}) \to H_{[\mathrm{ev}]}^{del}(\mathcal{A}\Gamma)$
- consider the C^*-closure of $\Psi_{\Gamma,c}^0(\widetilde{M})$ denoted $\Psi_\Gamma^0(\widetilde{M})$
- thanks to Zenobi's realization there exists a surjection

$$K_0(\Psi_\Gamma^0(\widetilde{M})) \to \mathrm{S}_1^\Gamma(\widetilde{M})$$

with kernel $K_0(C(M))$ and bringing $[\Pi_>(\widetilde{D})]$ into $\rho(\widetilde{D})$.

- the kernel will pair trivially with delocalized cyclic cocycles; for our purposes OK to define $\mathrm{Ch}_\Gamma^{del} : K_0(\Psi_\Gamma^0(\widetilde{M})) \to H_*^{del}(\mathcal{A}\Gamma)$ instead
- let us simplify and define $\mathrm{Ch}_\Gamma^{del} : K_0(\Psi_{\Gamma,c}^0(\widetilde{M})) \to H_*^{del}(\mathbb{C}\Gamma)$
- our approach builds on work of Connes, Lott, Nistor, Gorokhovsky-Lott.

More pseudodifferential operators

- consider $C_c^\infty(\widetilde{M})$
- it is a $\mathbb{C}\Gamma$-module and we denote it by $\mathcal{E}_{\mathbb{C}\Gamma}$
- consider $C^\infty(M, \mathcal{V}_{\mathbb{C}\Gamma})$, $\mathcal{V}_{\mathbb{C}\Gamma} := \widetilde{M} \times_\Gamma \mathbb{C}\Gamma$
- $C^\infty(M, \mathcal{V}_{\mathbb{C}\Gamma}) = C^\infty(\widetilde{M}, \underline{\mathbb{C}\Gamma})^\Gamma$
- $L : C_c^\infty(\widetilde{M}) \to C^\infty(\widetilde{M}, \underline{\mathbb{C}\Gamma})^\Gamma = C^\infty(M, \mathcal{V}_{\mathbb{C}\Gamma})$ associating to κ the element $\sum_\gamma (R_\gamma^* \kappa)\gamma$ is an isomorphism of $\mathbb{C}\Gamma$ modules.
- Ad_L identifies $\Psi^0_{\Gamma,c}(\widetilde{M})$ with a subalgebra $\Psi^0_{\mathbb{C}\Gamma}(\widetilde{M})$ of $\Psi^0_{\mathrm{MF}}(M, \mathcal{V}_{\mathbb{C}\Gamma})$
- elements in $\Psi^0_{\mathbb{C}\Gamma}(\widetilde{M})$ can be written as $\sum_\gamma (R^*_{(\gamma,e)} T)\gamma$ with $T \in \Psi^0_{\Gamma,c}(\widetilde{M})$

More pseudodifferential operators II

- we also have $\mathcal{E}_{\Omega_*(\mathbb{C}\Gamma)} = C^\infty(M, \mathcal{V}_{\mathbb{C}\Gamma} \otimes_{\mathbb{C}\Gamma} \Omega_*(\mathbb{C}\Gamma))$
- we can consider $\mathrm{End}_{\Omega_*(\mathbb{C}\Gamma)}(\mathcal{E}_{\Omega_*(\mathbb{C}\Gamma)})$ and inside this space $\Psi^0_{\Omega_*(\mathbb{C}\Gamma)}(\widetilde{M})$
- An element $\mathsf{T} \in \Psi^0_{\Omega_*(\mathbb{C}\Gamma)}(\widetilde{M})$ is given by definition by

$$\mathsf{T} = \sum_{\lambda,\omega} R^*_{(\lambda,e)} T_\omega \lambda \pi(\omega)^{-1} d\omega$$

with $\omega = g_1 \otimes \cdots \otimes g_k$, $\pi(\omega) = g_1 \cdots g_k$, $d\omega = dg_1 \cdots dg_k$ and $T_\omega \in \Psi^0_{\Gamma,c}(\widetilde{M})$

Lott's connection

• Lott has defined a connection $\nabla^{\mathrm{Lott}} : \mathcal{E}_{\mathbb{C}\Gamma} \to \mathcal{E}_{\Omega_1(\mathbb{C}\Gamma)}$ with curvature $\Theta \in \mathrm{Hom}_{\mathbb{C}\Gamma}(\mathcal{E}_{\mathbb{C}\Gamma}, \mathcal{E}_{\Omega_2(\mathbb{C}\Gamma)})$

• can define a degree-1 derivation $\overline{\nabla}$ on $\Psi^0_{\Omega_*(\mathbb{C}\Gamma)}(\widetilde{M})$ by $\overline{\nabla}(\mathsf{T}) = [\nabla^{\mathrm{Lott}}, \mathsf{T}]$.

• however $(\overline{\nabla})^2 \neq 0$, in fact $(\overline{\nabla})^2(\mathsf{T}) = \Theta\mathsf{T} - \mathsf{T}\Theta$

• but using Connes' trick we can use the triple $(\Psi^0_{\Omega_*(\mathbb{C}\Gamma)}(\widetilde{M}), \overline{\nabla}, \Theta)$ to define a DGA over $\Psi^0_{\Gamma,c}(\widetilde{M}) = \Psi^0_{\mathbb{C}\Gamma}(\widetilde{M})$ and we call it $(\overline{\Omega}_\bullet(\Psi^0_{\Gamma,c}(\widetilde{M})), d)$

The delocalized Chern character Ch_Γ^{del}

- we have the Chern character $K_0(\Psi^0_{\Gamma,c}(\widetilde{M})) \to H_*(\overline{\Omega}_\bullet(\Psi^0_{\Gamma,c}(\widetilde{M})))$
- finally, there is a delocalized trace

$\mathrm{TR}^{del} : \Psi^0_{\Omega_*(\mathbb{C}\Gamma)}(\widetilde{M}) \to \Omega_*^{del}(\mathbb{C}\Gamma)_{ab}$

$$\mathrm{TR}^{del}(\mathrm{T}) = \sum_{\lambda \neq e, \omega} \int_{\mathcal{F}} \mathrm{Tr}\, T_\omega(\widetilde{x}\lambda, \widetilde{x}) d\mathrm{vol}(\widetilde{x}) \lambda \pi(\omega)^{-1} d\omega$$

- e.g. if $T \in \Psi^0_{\mathbb{C}\Gamma}$ then

$\mathrm{TR}^{del}(T) = \sum_{\lambda \neq e} (\int_{\mathcal{F}} \mathrm{Tr}\, T(\widetilde{x}\lambda, \widetilde{x}) d\mathrm{vol}(\widetilde{x}))\lambda \ \in \mathbb{C}\Gamma^{del}_{ab}$

- it induces a chain map $\overline{\Omega}_\bullet(\Psi^0_{\Gamma,c}(\widetilde{M})) \to \Omega_*^{del}(\mathbb{C}\Gamma)_{ab}$ and thus a homomorphism $H_*(\overline{\Omega}_\bullet(\Psi^0_{\Gamma,c}(\widetilde{M}))) \to H_*^{del}(\mathbb{C}\Gamma)$
- the composition

$K_0(\Psi^0_{\Gamma,c}(\widetilde{M})) \xrightarrow{\mathrm{Ch}} H_*(\overline{\Omega}_\bullet(\Psi^0_{\Gamma,c}(\widetilde{M}))) \xrightarrow{\mathrm{TR}^{del}} H_*^{del}(\mathbb{C}\Gamma)$ is our delocalized Chern character

$$\mathrm{Ch}_\Gamma^{del} : K_0(\Psi^0_{\Gamma,c}(\widetilde{M})) \to H_*^{del}(\mathbb{C}\Gamma)$$

In general we use Zenobi's isomorphism

$$\mathrm{S}^{\Gamma}_*(\widetilde{M}) = K_*(C(M) \xrightarrow{\mathfrak{m}} \Psi^0_{\Gamma}(\widetilde{M})) = K_*(C(M) \xrightarrow{\mathfrak{m}} \Psi^0_{\mathcal{A}\Gamma}(\widetilde{M}))$$

with $\Psi^0_{\Gamma}(\widetilde{M})$ the C^*-closure of $\Psi^0_{\Gamma,c}(\widetilde{M})$ and $\Psi^0_{\mathcal{A}\Gamma}(\widetilde{M}) \subset \Psi^0_{\Gamma}(\widetilde{M})$ a smooth subalgebra.
So have to deal with relative K-theory groups ....

Higher rho numbers: Gromov hyperbolic groups

- Let M be odd dimensional. Assume that $\widetilde{D}$ is L^2-invertible.
- We have the rho class $\rho(\widetilde{D}) \in \mathrm{S}_1^\Gamma(\widetilde{M})$ and we've defined $\mathrm{Ch}_\Gamma^{del}(\rho(\widetilde{D})) \in H_{\mathrm{ev}}^{del}(\mathcal{A}\Gamma) \subset H_{\mathrm{ev}}(\mathcal{A}\Gamma)$
- We know that $H_*(\mathcal{A}\Gamma)$ embeds in $HC_*(\mathcal{A}\Gamma)$
- We would like to pair this class with $HC^*(\mathbb{C}\Gamma, \langle x \rangle)$, $x \neq e$.
- This is an **extension** problem as for Connes and Moscovici
- Given $[\tau] \in HC^*(\mathbb{C}\Gamma, \langle x \rangle)$ we would like to extend it to a class in $HC^*(\mathcal{A}\Gamma)$ and then use $H_*(\mathcal{A}\Gamma) \times HC^*(\mathcal{A}\Gamma) \to \mathbb{C}$
- this would give a sense to $\langle \mathrm{Ch}_\Gamma^{del}(\rho(\widetilde{D})), [\tau] \rangle$
- We assume Γ Gromov hyperbolic.

Higher rho numbers: Gromov hyperbolic groups (cont)

Note that this is a difficult problem already for the delocalized trace associated to $\langle x \rangle$, $\tau_{\langle x \rangle}(\sum_\gamma a_\gamma \gamma) = \sum_{g \in \langle x \rangle} a_g$

Theorem

(Puschnigg, 2010) Let Γ be Gromov hyperbolic. There exists a smooth subalgebra $\mathcal{A}\Gamma \subset C^*_r\Gamma$ s. t. $\tau_{\langle x \rangle}$ extends from $\mathbb{C}\Gamma$ to $\mathcal{A}\Gamma$.

Theorem

(P-Schick-Zenobi, 2019) Let Γ be Gromov hyperbolic. Then
(1) *$\forall x \in \Gamma$ there are isomorphisms*

$$HH^*(\mathbb{C}\Gamma, \langle x \rangle) = HH^*_{\mathrm{pol}}(\mathbb{C}\Gamma, \langle x \rangle)\,, \quad HC^*(\mathbb{C}\Gamma, \langle x \rangle) = HC^*_{\mathrm{pol}}(\mathbb{C}\Gamma, \langle x \rangle)$$

(2) *The cyclic cochains of polynomial growth extends to the Puschnigg's algebra $\mathcal{A}\Gamma$ inducing an injection $HC^*(\mathbb{C}\Gamma, \langle x \rangle) \to HC^*(\mathcal{A}\Gamma)$ as a direct summand.*

For (1) we build on results of Dan Burghelea and Ralph Meyer.
For (2) we use heavily the work of Michael Puschnigg.

Higher rho numbers: summary

- Summarizing: for a hyperbolic group we have established the existence of a pairing

$$\langle \ , \ \rangle : \mathrm{S}^{\Gamma}_{*}(\widetilde{M}) \times HC^{*}(\mathbb{C}\Gamma, \langle x \rangle) \to \mathbb{C}$$

  given by $\langle x, \tau \rangle := \langle \mathrm{Ch}^{del}_{\Gamma}(x), \tau \rangle$

- taking $x = \rho(\widetilde{D})$ we can define higher rho numbers

$$\rho^{\tau}(\widetilde{D}) := \langle \mathrm{Ch}^{del}_{\Gamma}(\rho(\widetilde{D})), \tau \rangle\,, \quad \tau \in HC^{*}(\mathbb{C}\Gamma, \langle x \rangle).$$

  For example $\rho^{\tau}(g)$, the higher rho number of the spin-Dirac operator of a PSC metric g.

- There are explicit formulae. For example: if M is odd dimensional and τ is the delocalized trace $\tau_{\langle x \rangle}$ then the higher rho number is Lott's delocalized eta invariant $\eta_{\langle x \rangle}(\widetilde{D})$

More higher rho numbers

- In addition to

$$\langle\ ,\ \rangle : \mathrm{S}^{\Gamma}_{*}(\widetilde{M}) \times HC^{*}(\mathbb{C}\Gamma, \langle x\rangle) \to \mathbb{C}$$

  we also define in our paper a pairing

$$\langle\ ,\ \rangle : \mathrm{S}^{\Gamma}_{*}(\widetilde{M}) \times H^{*}(M \to B\Gamma) \to \mathbb{C}$$

  (under assumptions on Γ, e.g. Gromov hyperbolic)
- this pairing was also defined by different methods by Weinberger-Xie-Yu (2017).
- they use Baum-Connes, we proceed à la Connes-Moscovici
- we employ the higher rho numbers defined by these 2 pairings in the problem of distinguishing PSC metrics
- first pairing useful in presence of torsion elements in Γ
- second pairing OK also for torsion-free groups

Higher rho numbers: some geometric applications

- We use the higher rho numbers $\rho^\tau(g)$ for PSC metrics to study $\pi_0(\mathcal{R}^+(M)/\mathrm{Diffeo}(M))$.
- Consider concordance classes of PSC metrics, $\mathcal{P}^+(M)$; there is an action of $\mathrm{Diffeo}(M)$
- Once we fix a base metric $[g_0]$, $\mathcal{P}^+(M)$ has a group structure
- Consider the coinvariants $\mathcal{P}^+(M)_U$ associated to a finite index subgroup $U \leq \mathrm{Diffeo}(M)$
- there is a surjection $\pi_0(\mathcal{R}^+(M)/U) \to \mathcal{P}^+(M)_U$
- we want to measure the rank of $\mathcal{P}^+(M)_U$

Theorem

Assume that Γ *is Gromov hyperbolic. Then there exists a finite index subgroup* $U \leq \mathrm{Diffeo}(M)$ *such that* $\mathrm{rank}(\mathcal{P}^+(M)_U) \geq$

$$\begin{cases} |\{[\gamma] \subset \Gamma \mid \mathrm{ord}(\gamma) < \infty\}|; & n \equiv -1 \pmod 4 \\ \left|\{[\gamma] \subset \Gamma \mid \mathrm{ord}(\gamma) < \infty, [\gamma] \neq [\gamma^{-1}]\}\right|; & n \equiv 1 \pmod 4 \end{cases}$$

This sharpens a bit results of Xie-Yu (but their result valid for more general groups)

Next result, on the other hand is new:

- Consider $F\Gamma := \{f : \Gamma_{fin} \to \mathbb{C} \mid |\mathrm{supp}(f)| < \infty\}$ with Γ_{fin} denoting the elements of finite order
- Consider $F^p\Gamma = \{f \in F\Gamma \mid f(\gamma) = (-1)^p f(\gamma^{-1})\}$ for $p = 0, 1$
- Consider $F^p_{del}\Gamma := \{f \in F^p\Gamma \mid \sum f(g) = 0\}$

Theorem

(P-Schick-Zenobi) Assume that Γ is Gromov hyperbolic. Assume that $\mathrm{Out}(\Gamma)$ is finite (this is a weak assumption). Then there exists a finite index subgroup $U \leq \mathrm{Diffeo}(M)$ such that

$$\mathrm{rank}(\mathcal{P}^+(M)_U) \geq \sum_{k>0, p\in\{0,1\}} \mathrm{rank}(H^{n+1-4k-2p}(\Gamma; F^p_{del}\Gamma)).$$

THANK YOU!