Generalized inductive limits with asymptotically order zero maps

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Part I: Inductive limits of C^*-algebras

Inductive limits of C^*-algebras

An *inductive system* of C^*-algebras consists of a sequence $(A_n)_n$ of C^*-algebras together with connecting *-homomorphisms

$$A_0 \xrightarrow{\rho_{1,0}} A_1 \xrightarrow{\rho_{2,1}} A_2 \to \dots.$$

For each $k \geq 0$, the quotient map $\pi : \prod_n A_n \to {}^{\prod_n A_n}\!/\!_{\bigoplus_n A_n}$ induces a *-homomorphism $\rho_k : A_k \to {}^{\prod_n A_n}\!/\!_{\bigoplus_n A_n}$ by

$$\rho_k(a) := \pi\big(\big(\rho_{n,k}(a)\big)_{n>k}\big), \ \forall\ a \in A_k.$$

The *inductive limit* of the system $(A_n, \rho_{m,n})$ is the C^*-algebra

$$A := \overline{\bigcup_{k\geq 0} \rho_k(A_k)} \subset \prod_n A_n \Big/ \bigoplus_n A_n.$$

Inductive limits of C^*-algebras

This construction has provided many interesting examples of C^*-algebras:

Example

- Approximately Finite (AF) algebras
(inductive limits of finite dimensional C^*-algebras)
- Approximately Circle (A$\mathbb{T}$) algebras
(inductive limits of direct sums of matrix algebras over $C(\mathbb{T})$)
- Approximately Homogeneous (AH) algebras
(inductive limits of homogeneous C^*-algebras,
i.e., corners of matrix algebras over $C(X)$ for some compact Hausdorff X.)

Inductive limits of C^*-algebras

It has also shed light on the structure of many naturally occurring C^*-algebras:

Theorem

A simple, separable, unital C^*-algebra A fits into the Elliott classification program (meaning it can be classified by its K-theory and traces) if it is

- *[Elliott] AF,*
- *[Elliott] $A\mathbb{T}$ with Real Rank Zero (i.e., $A_{s.a.} = \overline{GL(A) \cap A_{s.a.}}$) ([Elliot-Evans] This includes irrational rotation algebras), or*
- *[Dadarlat–Elliott–Gong] AH with Real Rank Zero and "slow dimension growth."*

Generalized inductive limits

Working with these inductive systems, one realizes that on-the-nose behavior at each step is often superfluous.

Asymptotic behavior is what really matters.

Following this philosophy, Blackadar and Kirchberg introduced geneneralized inductive systems of C^*-algebras, where the connecting maps only *asymptotically* behave like *-homomorphisms. They showed that the limits of such systems form important classes of C^*-algebras.

Ignoring the full generality of their constructions, we focus on their so-called *NF systems*.

NF systems

Definition

An *NF system* consists of a sequence $(F_n)_n$ of finite dimensional C^*-algebras together with asymptotically multiplicative completely positive contractive (cpc) maps

$$F_0 \xrightarrow{\rho_{1,0}} F_1 \xrightarrow{\rho_{2,1}} F_2 \to \dots .$$

Asymptotically multiplicative means that for any $k \geq 0$, $x, y \in F_k$, and $\varepsilon > 0$, there exists an $M > k$ such that for all $m > n > M$,

$$\|\rho_{m,n}\big(\rho_{n,k}(x)\rho_{n,k}(y)\big) - \rho_{m,k}(x)\rho_{m,k}(y)\| < \varepsilon.$$

The inductive limit of an NF system is formed the same as before: $\overline{\bigcup_k \rho_k(F_k)} \subset \prod_n F_n / \bigoplus F_n$ where $\rho_k : F_k \to \prod_n F_n / \bigoplus F_n$ are the induced cpc maps.

NF Algebras

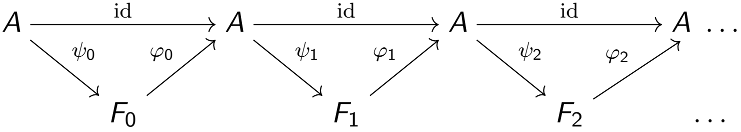
Since this quotient only cares about what happens asymptotically, the limit is still a C^*-algebra, which we call an *NF Algebra*.

Theorem (BK)

A separable C^*-algebra A is NF iff it admits a cpc approximation with asymptotically multiplicative maps, i.e., there exists a sequence of finite dimensional C^*-algebras $(F_n)_n$ and cpc maps $A \xrightarrow{\psi_n} F_n \xrightarrow{\varphi_n} A$ so that for all $a, b \in A$,

$$\|\varphi_n \circ \psi_n(a) - a\| \to 0 \textit{ and}$$
$$\|\psi_n(a)\psi_n(b) - \psi_n(ab)\| \to 0.$$

In particular, these C^*-algebras are nuclear, and we have an approximately commutative diagram.



NF Algebras

Theorem (BK)

NF C^*-algebras are exactly the separable nuclear C^*-algebras which are quasidiagonal.

Definition

A separable C^*-algebra A is *quasidiagonal* if it admits a sequence of cpc maps $\psi_n : A \to F_n$ which are

- asymptotically multiplicative ($\|\psi_n(a)\psi_n(b) - \psi_n(ab)\| \to 0$) &
- asymptotically isometric ($\|a\| = \lim_n \|\psi_n(a)\|$).

But what if we just want to get at nuclear C^*-algebras?

To do so, we must relax the asymptotically multiplicative assumption, but then the inductive limit fails to be an algebra.

We need to relax multiplicativity without losing the C^*-structure.

Part II: Order Zero Maps

Order Zero Maps

Definition

A cp map $\psi : A \to B$ between C^*-algebras is called *order zero* if it is orthogonality preserving:

$$ab = 0 \implies \psi(a)\psi(b) = 0, \ \forall \ a, b \in A_+.$$

Theorem (Winter-Zacharias)

Let A and B be C^*-algebras with A unital. A cp map $\psi : A \to B$ is order zero iff

$$\psi(a)\psi(b) = \psi(1_A)\psi(ab), \ \forall \ a, b \in A.$$

Remark

Note that if $\psi(1_A) = 1_B$, then ψ is a $*$-homomorphism.

Order Zero Maps

These are a natural step-down from *-homomorphisms, and they actually retain a lot of the same nice properties.

Theorem (Wolf)

If $\psi : A \to B$ is a cp order zero map from a unital C^*-algebra A, then $\psi(1_A) \in \psi(A)'$.

Proposition (WZ)

If $\psi : A \to B$ is a cp order zero map, then so are all of its matrix amplifications $\psi^{(r)} : \mathrm{M}_r(A) \to \mathrm{M}_r(B)$.

In other words, an order zero map is completely order zero.

Proposition

If $\psi : A \to B$ is an injective cp order zero map, then $\psi^{-1}(\psi(A) \cap B_+) = A_+$. Moreover, if a cp order zero map is invertible (on its image), its inverse is automatically cp.

Images of order zero maps

A cp order zero map leaves an impression of its structure in its image– even to the point that we can detect when a self-adjoint subspace of a C^*-algebra is the image of some cp order zero map.

This preserved structure is actually enough to allow us to build a C^*-algebra out of the image outright.

In particular, for a cpc order zero map $\psi : A \to B$ from a unital C^*-algebra, setting $X := \psi(A)$ and $e := \psi(1_A)$, we have the following:

1. [W] $e \in X' \cap X$
2. [WZ] $X^2 := \{xy \ : \ x, y \in X\} = \{ez \ : \ z \in X\} =: eX$, and
3. e is an order unit for $X \subset B$.

An order unit

Definition

Given a self-adjoint subspace X of a C^*-algebra B. We say a positive element $e \in X$ is an *order unit* for X if for each $x = x^* \in X$, there exists an $R > 0$ so that $Re \geq x$.
We say e is a *uniform order unit* if $\|x\|e \geq x$ for all $x = x^* \in X$.

Example

- The unit of a unital C^*-algebra A is a uniform order unit for A.
- $\mathrm{id}_{(0,1]}$ is a uniform order unit for $\mathbb{C}\mathrm{id}_{(0,1]} \subset C_0((0,1])$.
- $\mathrm{id}_{(0,1]} \otimes 1_A$ is a uniform order unit for $\mathrm{id}_{(0,1]} \otimes A \subset C_0((0,1]) \otimes A$.
- If $\rho : A \to B$ is a cp map, then $\rho(1_A)$ is an order unit for $\rho(A)$. It is a uniform order unit when ρ is cp order zero and isometric.

A C^*-structure

It turns out these three criteria are enough to define a pre-C^*-structure on a self-adjoint subspace of a C^*-algebra.

Theorem (C.-Winter)

Let B be a C^*-algebra, $X \subset B$ a self-adjoint subspace, and $e \in B_+^1$ a distinguished element satisfying

1. $e \in X' \cap X$
2. $X^2 = eX$, *and*
3. e *is an order unit for* X.

Then there is an associative bilinear map $\bullet : X \times X \to X$ *satisfying*

$$xy = e(x \bullet y) \ \forall \ x, y \in X$$

so that $(X, \bullet)$ *is a* $*$*-algebra with unit* e.
Moreover, there exists a pre-C^**-norm* $\| \cdot \|_\bullet$ *on* $(X, \bullet)$*, and* $X = \overline{X}^{\| \cdot \|_\bullet}$ *already when* $X = \overline{X}^{\| \cdot \|_B}$.

A C^*-structure

For a self-adjoint subspace X of a C^*-algebra B with distinguished element $e \in B^1_+$, we abbreviate the criteria that gave us a pre-C^*-structure on X as follows:

$$(\mathrm{C}^*) \begin{cases} 1.\ e \in X' \cap X \\ 2.\ X^2 = eX, \text{ and} \\ 3.\ e \text{ is an order unit for } X. \end{cases}$$

Whenever (X, e) satisfy (C^*), we can define multiplication $\bullet : X \times X \to X$ and a pre-C^*-norm $\| \cdot \|_\bullet$ on X.

We write $X_\bullet := \overline{(X, \bullet)}^{\| \cdot \|_\bullet}$ for the completion.

Images of cpc order zero maps

For a self-adjoint $X \subset B$ with distinguished $e \in B_+^1$ so that $(X, e) \subset B$ satisfy (C^*), it turns out that the map

$$X_\bullet \supseteq X \xrightarrow{\mathrm{id}_X} X \subset B$$

extends to a cpc order zero map $X_\bullet \to B$.

Theorem (CW)

The following are equivalent for a self-adjoint subspace $X \subset B$ of a C^*-algebra B with distinguished $e \in B_+^1$.

a. *There exists a unital C^*-algebra A and cpc order zero map $\psi : A \to B$ such that $X = \psi(A)$ and $e = \psi(1_A)$.*

b. *(X, e) satisfies (C^*) and $X = \overline{X}^{\|\cdot\|_\bullet}$ (i.e., $X = X_\bullet$ as sets).*

Remark

For any unital C^*-algebra A with injective cp order zero map $\psi : A \to B$ with $\psi(A) = X$ and $\psi(1_A) = e$, the map $\mathrm{id}_X^{-1} \circ \psi : A \to X_\bullet$ becomes a *-isomorphism.

Closed images of cpc order zero maps

One case where we always are guaranteed an injective cp order zero map is when X is closed in B.

Theorem (CW)

The following are equivalent for a **closed** *self-adjoint subspace $X \subset B$ of a C^*-algebra B with distinguished $e \in B_+^1$.*

a. *There exists a unital C^*-algebra A and an* **injective** *cp order zero map $\psi : A \to B$ such that $X = \psi(A)$ and $e = \psi(1_A)$.*
b. *(X, e) satisfies (C^*).*

This comes from the fact that $X = \overline{X}^{\|\cdot\|_\bullet}$ when $X \subset B$ is closed, which means $\mathrm{id}_X : X_\bullet \to X \subset B$ is already a cpc order zero map, which is a complete order isomorphism by virtue of being injective.

Part III: Generalized NF Systems

Back to generalized inductive limits

Recall that our goal was to relax the "asymptotic multiplicative" requirement from the NF systems:

Definition

An *NF system* consists of a sequence $(F_n)_n$ of finite dimensional C^*-algebras together with *asymptotically multiplicative* completely positive contractive (cpc) maps

$$F_0 \xrightarrow{\rho_{1,0}} F_1 \xrightarrow{\rho_{2,1}} F_2 \to \dots$$

But the issue was that, without asymptotic multiplicativity, the limit need not be a C^*-algebra.

Now we are equipped to overcome that hurdle.

Generalizing generalized inductive limits

Given a sequence $(F_n)_n$ of finite dimensional C^*-algebras together cpc connecting maps

$$F_0 \xrightarrow{\rho_{1,0}} F_1 \xrightarrow{\rho_{2,1}} F_2 \to \ldots,$$

we still have induced cpc maps $\rho_k : F_k \to \prod_n F_n / \bigoplus_n F_n =: F_\infty$, and we can still form the limit

$$X = \overline{\bigcup_k \rho_k(F_k)} \subset F_\infty.$$

Though X may not be a C^*-algebra, if we can guarantee that there some $e \in (F_\infty)^1_+$ so that (X, e) satisfy (C^*), then it will be completely order isomorphic to the C^*-algebra $X_\bullet$ via the injective cpc order zero map $\mathrm{id}_X : X_\bullet \to X \subset F_\infty$.

Encoding (C^*)

The task is to encode (C^*) into a system

$$F_0 \xrightarrow{\rho_{1,0}} F_1 \xrightarrow{\rho_{2,1}} F_2 \to \dots$$

of cpc maps between finite dimensional C^*-algebras.

We want conditions on the system which guarantee that we have an element $e \in (F_\infty)^1_+$ so that the limit X together with e satisfy

(C^*)

1. $e \in X' \cap X$
2. $X^2 = eX$, and
3. e is an order unit for X.

An approximately central order unit

To find a positive contraction $e \in X \cap X'$, we need a sequence $(e_n)_n \in \prod_n (F_n)_+^1$ that is

- *asymptotically coherent*, which guarantees that $(\rho_n(e_n))_n \subset X$ is Cauchy, and
- *asymptotically central*, which guarantees that $e := \lim_n \rho_n(e_n)$ commutes with X.

To ensure e is an order unit for X, we require that $(e_n)_n$ is an

- *asymptotic uniform order unit*, which guarantees that $\|\rho_n(x)\| e \geq \rho_n(x)$ for every $n \geq 0$, $x = x^* \in F_n$.

Under these three assumptions, we get a positive contraction $e \in X' \cap X$ that is an order unit for X.

Let's call such a sequence an *asymptotically central order unit*.

Asymptotically order zero

With a designated "unit," we want our system to capture the unital definition of order zero ($\psi(a)\psi(b) = \psi(1)\psi(ab)\ \forall\ a, b \in A$), which translates to $X^2 = eX$.

We arrange for this by requiring that our system be

- *asymptotically order zero with respect to* $(e_n)_n$.
  This condition tells us how to build, for any $k \geq 0$ and $x, y \in F_k$, an element $z \in X = \overline{\bigcup_n \rho_n(F_n)}$ so that $ez = \rho_k(x)\rho_k(y)$.

It turns out this is enough to get $X^2 = eX$.

Remark

Just as with order zero maps, if these maps are asymptotically unital (i.e. $\|e_n - 1_{F_n}\| \to 0$), then the resulting sequence is asymptotically multiplicative, and we land back in the NF setting.

Generalized NF systems

(Working title)

Definition (CW)

A *generalized NF system* $(F_n, \rho_{m,n}, e_n)$ consists of a sequence $(F_n)_n$ of finite dimensional C^*-algebras with cpc connecting maps

$$F_0 \xrightarrow{\rho_{1,0}} F_1 \xrightarrow{\rho_{2,1}} F_2 \to \ldots,$$

that are asymptotically order zero with respect to an asymptotically central order unit $(e_n)_n \in \prod_n (F_n)^1_+$.

Example (BK, WZ, Brown-Carrión-White, CW)

Any separable, unital, nuclear C^*-algebra A admits a cpc approximation $A \xrightarrow{\psi_n} F_n \xrightarrow{\varphi_n} A$ so that $(F_n, \psi_m \circ ... \circ \varphi_n, \psi_n(1_A))$ forms a generalized NF system.

Generalized NF systems from cpc approximations

A cpc approximation $A \xrightarrow{\psi_n} F_n \xrightarrow{\varphi_n} A$ of a unital C^*-algebra with *asymptotically order zero maps* $(\psi_n : A \to F_n)_n$

$$\big(\text{i.e., } \|\psi_n(1_A)\psi_n(ab) - \psi_n(a)\psi_n(b)\| \to 0,\ \forall\ a, b \in A\big)$$

induces a completely isometric cp order zero map $\psi : A \to F_\infty$:

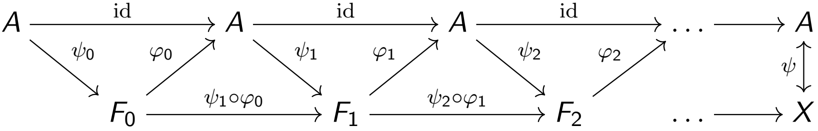
$$\begin{array}{ccccccccc}
A & \xrightarrow{\mathrm{id}} & A & \xrightarrow{\mathrm{id}} & A & \xrightarrow{\mathrm{id}} & \dots & \longrightarrow & A \\
{\scriptstyle \psi_0} \searrow & & \nearrow {\scriptstyle \varphi_0} \quad {\scriptstyle \psi_1} \searrow & & \nearrow {\scriptstyle \varphi_1} \quad {\scriptstyle \psi_2} \searrow & & \nearrow {\scriptstyle \varphi_2} & & {\scriptstyle \psi} \updownarrow \\
& F_0 & & F_1 & & F_2 & \dots & & F_\infty
\end{array}$$

Generalized NF systems from cpc approximations

A cpc approximation $A \xrightarrow{\psi_n} F_n \xrightarrow{\varphi_n} A$ of a unital C^*-algebra with *asymptotically order zero maps* $(\psi_n : A \to F_n)_n$

$$\big(\text{i.e., } \|\psi_n(1_A)\psi_n(ab) - \psi_n(a)\psi_n(b)\| \to 0, \ \forall \ a, b \in A\big)$$

induces a completely isometric cp order zero map $\psi : A \to F_\infty$:



After passing to a subsystem, we can guarantee that $\psi(A) = \varinjlim(F_n, \psi_m \circ ... \circ \varphi_n) =: X$.

The fact that ψ is cpc order zero will imply that the system is asymptotically order zero with respect to the asymptotically central order unit $(\psi_n(1_A))_n$. ($\rightsquigarrow \psi(1_A)$ is the central order unit for X.)

Limits of generalized NF systems

By encoding (C^*) into our definition of generalized NF systems, we have guaranteed that the inductive limit $X \subset F_\infty$ along with $e := \lim_n \rho_n(e_n)$ satisfy (C^*).

Hence, we have the following.

Theorem (CW)

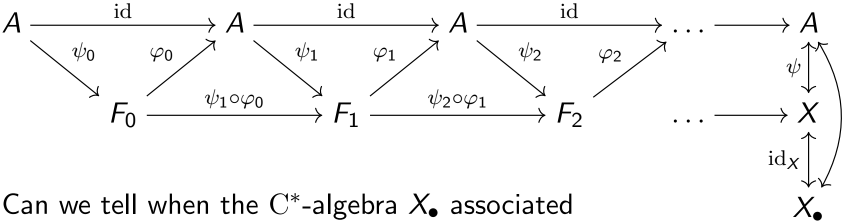
The inductive limit X of a generalized NF system is completely order isomorphic to a unital C^*-algebra $X_\bullet$ via an injective cpc order zero map $\mathrm{id}_X : X_\bullet \to X \subset F_\infty$.

Moreover, if A is a unital C^*-algebra and $\psi : A \to F_\infty$ is an injective cpc order zero map with $\psi(A) = X$ and $\psi(1_A) = e$, then $A \simeq X_\bullet$.

Limits of generalized NF systems from cpc approximations

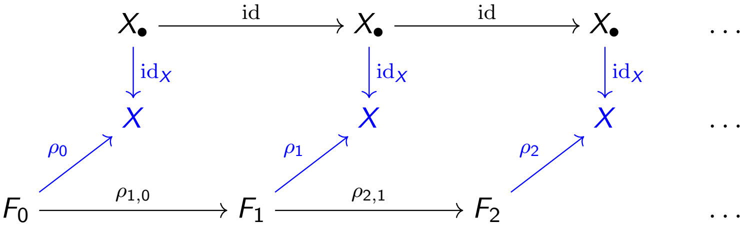
Corollary (CW)

Let A be a separable, unital, nuclear C^*-algebra and $A \xrightarrow{\psi_n} F_n \xrightarrow{\varphi_n} A$ a cpc approximation so that $(F_n, \psi_m \circ \ldots \circ \varphi_n, \psi_n(1_A))$ forms a generalized NF system. Then A is *-isomorphic to $X_\bullet$.



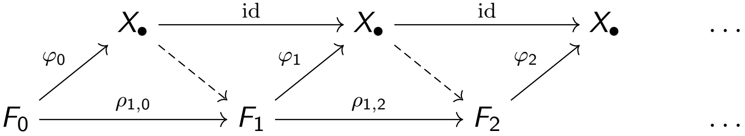
Can we tell when the C^*-algebra $X_\bullet$ associated to a generalized NF system if nuclear?

CPAP for a generalized NF system?



Notice that id_X^{-1} is cp, and so $\varphi_n := \mathrm{id}_X^{-1} \circ \rho_n$ are cpc.

CPAP for a generalized NF system?



Question

Can we come up with the downwards maps to get a completely positive approximation?

[Winter] If we assume the upwards maps are decomposable into a direct sum of a bounded number of cpc order zero maps, then yes.

Without downwards maps

It turns out that we still get nuclearity without the downwards maps by invoking a "one-way CPAP."

Theorem (Sato, Ozawa)

A C^**-algebra is nuclear iff there exists a net* $(\rho_\lambda : F_\lambda \to A)_{\lambda\in\Lambda}$ *of cpc maps from finite dimensional* C^**-algebras such that the induced cpc map*

$$\begin{array}{ccc} \prod_\lambda F_\lambda & \xrightarrow{(\rho_\lambda)_\lambda} & \ell^\infty(\Lambda, A) \\ \downarrow & & \downarrow \\ \prod_\lambda F_\lambda \big/ \bigoplus_\lambda F_\lambda & \xrightarrow{\Phi} & \ell^\infty(\Lambda, A)\big/ c_0(\Lambda, A) \end{array}$$

satisfies $A^1 \subset \Phi\left(\left(\frac{\prod_\lambda F_\lambda}{\bigoplus_\lambda F_\lambda}\right)^1\right)$.

Remark

The proof goes by showing that A^{**} *is hyperfinite, not by constructing the downwards maps.*

Nuclear C^*-algebras from limits of generalized NF systems

Theorem (CW)

The inductive limit X of a generalized NF system is completely order isomorphic to a unital **nuclear** C^**-algebra $X_\bullet$ via an injective cp order zero map* $\mathrm{id}_X : X_\bullet \to X \subset F_\infty$.

Removing quasidiagonality

Recall Blackadar and Kirchberg's characterization of NF algebras as the separable nuclear quasidiagonal C^*-algebras:

Theorem (BK)

The following are equivalent for a separable C^*-algebra A:

1. *A is nuclear and quasidiagonal.*
2. *A is *-isomorphic to an NF algebra.*

By replacing asymptotic multiplicativity with asymptotic order zero, we can drop "quasidiagonal."

Theorem (CW)

The following are equivalent for a separable unital C^*-algebra A:

1. *A is nuclear.*
2. *There exists a generalized NF system $(F_n, \rho_{m,n}, e_n)$ and an injective cp order zero map $\psi : A \to F_\infty$ with $\psi(A) = X$ and $\psi(1_A) = e$.*

Thanks!