Translation invariant noncommutative Dirichlet forms

based on joint work with A. Viselter

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General aim

Our aim will be a study of a certain class of (noncommutative) probabilistic evolutions, understood as Markov semigroups on von Neumann algebras and associated L^p-spaces associated to locally compact quantum groups and describe their generators via quantum Dirichlet forms with specific invariance properties.

Classical Markov semigroups and Dirichlet forms

(X, μ) – classical measure space

Definition

A Markov semigroup $(P_t)_{t \geq 0}$ on (X, μ) is a family of operators acting on the von Neumann algebra $\mathrm{L}^\infty(X, \mu)$ and satisfying the following conditions:

- $P_0 = I$, $P_{t+s} = P_t \circ P_s$, $s, t \geq 0$;
- $w^* - \lim_{t \to 0^+} P_t(f) = f$, $f \in \mathrm{L}^\infty(X, \mu)$;
- $\forall_{t \geq 0}$ P_t is a contractive positive operator and $\mu \circ P_t \leq \mu$.

Markov semigroups as above yield contractive C_0-semigroups on all spaces $\mathrm{L}^p(X, \mu)$, $p \in [1, \infty)$; we will call the semigroup symmetric if the corresponding operators on $\mathrm{L}^2(X, \mu)$ consists of self-adjoint operators.

Classical Markov semigroups – continued

Markov semigroups can be studied via their $\mathrm{L}^p(X,\mu)$-generators; naturally the easiest case is that of $\mathrm{L}^2(X,\mu)$. Assume that $(P_t)_{t\geq 0}$ is a symmetric Markov semigroup and consider

$$Q(f) = \lim_{t\to 0^+} \frac{1}{t}\int_X \bar{f}(f - P_t f)d\mu, \quad f \in \mathrm{Dom}(Q) \subset \mathcal{L}^2(X,\mu)$$

This is a Dirichlet form: i.e. a densely defined quadratic form, which is closed, real and if $P_\wedge$ denotes the projection from $L^2(X,\mu)_{\mathbb{R}}$ onto $\{f \in L^2(X,\mu)_{\mathbb{R}} : 0 \leq f \leq 1\}$ then

$$Q(P_\wedge f) \leq Q(f), \quad f \in \mathrm{Dom}(Q)_{\mathbb{R}}.$$

Theorem (Beurling-Deny)

There is a 1-1 correspondence between symmetric Markov semigroups on $\mathrm{L}^\infty(X,\mu)$ and Dirichlet forms on $\mathrm{L}^2(X,\mu)$.

Convolution semigroups of measures and Lévy processes

G – locally compact group

A family $(\mu_t)_{t\geq 0_+}$ of probability measures on G is called a convolution semigroup of measures if

1. $\mu_{t+s} = \mu_t \star \mu_s, \quad t, s \geq 0;$
2. $\mu_t(f) \stackrel{t\to 0^+}{\longrightarrow} \mu_0(f) := f(e), \quad f \in \mathrm{C}_0(G).$

These are precisely distributions of Lévy processes (processes with independent, identically distributed increments). Further we can define

$$(P_t(f))(g) = \int_G f(gh) d\mu_t(h), \quad f \in \mathrm{L}^\infty(G).$$

to get the Markov semigroup of the process. Corresponding Dirichlet forms are characterised by the translation invariance.

Convolution semigroups of measures – generating functionals

G – locally compact group
$(\mu_t)_{t \geq 0_+}$ – convolution semigroup of measures

Theorem (Lévy-Khintchine, Hunt, Heyer)

The limit

$$\gamma(f) := \lim_{t \to 0^+} \frac{\mu_t(f) - f(e)}{t} = \lim_{t \to 0^+} \frac{\int_G (f(g) - f(e)) d\mu_t(g)}{t}$$

exists for all $f \in \mathrm{Dom}(\gamma) \supset C_c^2(G)$ (compactly supported 'twice differentiable' functions). The functional γ determines $(\mu_t)_{t \geq 0_+}$. If G – compact, we can replace $C_c^2(G)$ by $\mathrm{Pol}(G)$ – coefficients of finite-dimensional representations of G. In particular the domain of γ always contains a dense *-subalgebra of $C_0(G)$.

Quantum Markov semigroups

M - von Neumann algebra, with a fixed normal semifinite faithful weight ϕ

Definition

An operator $T : \mathrm{M} \to \mathrm{M}$ will be called Markov if it is a positive contraction and in addition we also have the condition

$$\phi \circ T \leq \phi.$$

We will call T completely Markov, if the same properties hold for $T \otimes \mathrm{id}_{M_n}$ acting on $(\mathrm{M} \otimes M_n, \phi \otimes \mathrm{tr}_n)$ for each $n \in \mathbb{N}$.

Quantum Markov semigroups – L^p-versions

Tracial case – L^p-spaces are certain completions of M, with the $\|x\|_p := (\tau(|x|^p))^{\frac{1}{p}}$.

Non-tracial state case – L^p-spaces are either interpolation spaces (which requires embedding M into $L^1(M) = M_*$) (Araki, Kosaki, Izumi) or certain concrete spaces of operators (Connes, Hilsum, Haagerup).

Non-tracial weight case:

$L^p(M, \phi)$ – Haagerup L^p-space. We consider symmetric embeddings $\iota_p : M^{(p)} \to L^p(M)$: these are informally defined as

$$\iota_p(x) = D^{\frac{1}{2p}} x D^{\frac{1}{2p}}, \quad x \in M^{(p)}.$$

Here $M^{(p)} \subset M$ is 'the set of p-integrable elements' and D can be thought of as 'the density matrix of the weight' – and formally is the unbounded generator of the unitary group implementing the modular automorphism group of ϕ in the core of M.

Quantum Markov semigroups – L^p-versions continued

Given a map $T : \mathrm{M} \to \mathrm{M}$ we will say it is KMS-symmetric if its *KMS implementation*, the map $T^{(2)} : \iota_2(\mathrm{M}^{(2)}) \to \mathrm{L}^2(\mathrm{M}, \phi)$

$$T^{(2)}(\iota_2(x)) = \iota_2(Tx)$$

extends to a *bounded self-adjoint* operator.
Note that we require in particular that $T^{(2)}(\iota_2(\mathrm{M}^{(2)})) \subset \iota_2(\mathrm{M}^{(2)})$.

Definition

A (quantum) KMS-symmetric Markov semigroup is a pointwise weak*-continuous semigroup $(T_t)_{t \geq 0^+}$ of normal KMS-symmetric Markov maps on M.

In fact *KMS*-symmetry plus being a positive contraction itself yields the weight inequality. Completely Markov KMS-symmetric semigroups act on all L^p-spaces!

Quantum Dirichlet forms

Definition (Cipriani, Goldstein+Lindsay)

A quantum Dirichlet form for (M, ϕ) is a densely defined closed quadratic form on $\mathrm{L}^2(\mathrm{M})$ which is real (in a suitable sense) and satisfies the condition

$$Q(P_\wedge x) \leq Q(x), \quad x \in \mathrm{Dom}(Q_\mathbb{R})$$

where $P_\wedge$ is the orthogonal projection from $\mathrm{L}^2(\mathrm{M})_\mathbb{R}$ onto the 'interval' $[0, D^{\frac{1}{2}}]$.

Theorem (Cipriani, Goldstein+Lindsay, Viselter+AS)

There is a 1-1 correspondence between completely Markov *KMS*-symmetric semigroups and quantum completely Dirichlet forms.

Quantum Dirichlet forms: example of an application

Most examples of applications of quantum Dirichlet forms appear in the tracial context. But not all!

Theorem (Caspers + AS)

Let M be a σ-finite von Neumann algebra. Then the following are equivalent:

1. M has the Haagerup property;
2. for any (equivalently, every) faithful normal state ϕ the pair (M, ϕ) admits a quantum Dirichlet form 'of compact type':

   $L^2(\mathsf{M}, \phi)$ admits an orthonormal basis $(e_n)_{n\in\mathbb{N}}$ and a non-decreasing sequence of non-negative numbers $(\lambda_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} \lambda_n = +\infty$ and the prescription

$$Q(\xi) = \sum_{n=1}^{\infty} \lambda_n |\langle e_n, \xi\rangle|^2, \quad \xi \in \mathrm{Dom}\, Q,$$

   where $\mathrm{Dom}\, Q = \{\xi \in H_\phi : \sum_{n=1}^{\infty} \lambda_n |\langle e_n, \xi\rangle|^2 < \infty\}$, defines a completely Dirichlet form.

LCQGs

$\mathbb{G}$ – locally compact quantum group à la Kustermans-Vaes

$\mathsf{L}^\infty(\mathbb{G})$ – the von Neumann algebra, together with the *coproduct* (carrying all the information about $\mathbb{G}$)

$$\Delta : \mathsf{L}^\infty(\mathbb{G}) \to \mathsf{L}^\infty(\mathbb{G}) \overline{\otimes} \mathsf{L}^\infty(\mathbb{G})$$

and a canonical *right Haar weight* ϕ

$\mathsf{C}_0(\mathbb{G})$ – the corresponding (reduced) C^*-object

$\mathsf{C}_0^u(\mathbb{G})$ – the universal version of $\mathsf{C}_0(\mathbb{G})$, with the counit $\epsilon : \mathsf{C}_0^u(\mathbb{G}) \to \mathbb{C}$

$\mathsf{L}^2(\mathbb{G})$ – the GNS Hilbert space of the right invariant Haar weight ϕ on $\mathsf{L}^\infty(\mathbb{G})$

$\mathsf{L}^1(\mathbb{G})$ – predual of $\mathsf{L}^\infty(\mathbb{G})$, with a natural Banach algebra structure.

$$\mathsf{C}_0(\mathbb{G}) \subset \mathsf{L}^\infty(\mathbb{G})$$

$$\mathsf{L}^2(\mathbb{G}) \approx \mathsf{L}^2(\mathsf{L}^\infty(\mathbb{G}), \phi)$$

Dual groups

Each LCQG $\mathbb{G}$ admits the dual LCQG $\widehat{\mathbb{G}}$.

$\mathrm{L}^\infty(\widehat{\mathbb{G}})$, $\mathrm{C}_0(\widehat{\mathbb{G}})$ – subalgebras of $B(\mathrm{L}^2(\mathbb{G}))$

In particular for G – locally compact group

$$\mathrm{L}^\infty(\widehat{G}) = \mathrm{VN}(G), \quad \mathrm{C}_0(\widehat{G}) = C_r^*(G), \quad \mathrm{C}_0^u(\widehat{G}) = C^*(G)$$

Simplifications in the compact case

Definition

$\mathbb{G}$ is said to be compact if $\mathsf{C}_0(\mathbb{G})$ is unital (so written as $\mathsf{C}(\mathbb{G})$), equivalently, the weight ϕ is a state.

Any compact quantum group can be described purely algebraically via the Hopf *-algebra $\mathrm{Pol}(\mathbb{G}) \subset \mathsf{C}(\mathbb{G})$, with the counit ϵ.

$$\Delta : \mathrm{Pol}(\mathbb{G}) \to \mathrm{Pol}(\mathbb{G}) \odot \mathrm{Pol}(\mathbb{G}),$$

$$\epsilon : \mathrm{Pol}(\mathbb{G}) \to \mathbb{C}.$$

Convolution semigroups of states on compact quantum groups

A family $(\mu_t)_{t\geq 0_+}$ of states on $\mathrm{Pol}(\mathbb{G})$ is called a convolution semigroup of states if

1. $\mu_{t+s} = \mu_t \star \mu_s := (\mu_t \otimes \mu_s) \circ \Delta, \quad t, s \geq 0;$
2. $\mu_t(a) \stackrel{t\to 0^+}{\longrightarrow} \mu_0(a) := \epsilon(a), \quad a \in \mathrm{Pol}(\mathbb{G}).$

We associate to it a convolution semigroup of operators $(P_{\mu_t})_{t\geq 0_+}$ on $\mathrm{Pol}(\mathbb{G})$:

$$P_{\mu_t} := (\mathrm{id} \otimes \mu_t) \circ \Delta$$

These extend to operators on $\mathrm{L}^\infty(\mathbb{G})$ which form a completely Markov semigroup.

The corresponding Dirichlet forms contain $\mathrm{Pol}(\mathbb{G})$ in the domain and can be characterised/studied in the purely algebraic manner (see Cipriani, Franz, Kula).

Generating functionals: compact case

Convolution semigroups of states $(\mu_t)_{t\geq 0_+}$ admit generating functionals:

$$\gamma(a) = \lim_{t\to 0^+} \frac{\mu_t(a) - \epsilon(a)}{t}, \quad a \in \mathrm{Pol}(\mathbb{G}).$$

These have an internal characterisation (as hermitian, normalised, conditionally positive functionals) and determine the semigroup uniquely.

One can further define and study probabilistic properties of the semigroups (such as Gaussianity) in terms of the generating functionals: this turns out to be closely related to cohomology of $\mathrm{Pol}(\mathbb{G})$ and naturally exploits the fact that $\mathrm{Pol}(\mathbb{G})$ is an algebra.

Convolution semigroups of states revisited

$\mathbb{G}$ – locally compact quantum group

A family $(\mu_t)_{t\geq 0_+}$ of states on $C_0^u(\mathbb{G})$ is called a convolution semigroup of states if

1. $\mu_{t+s} = \mu_t \star \mu_s := (\mu_t \otimes \mu_s) \circ \Delta, \quad t, s \geq 0;$
2. $\mu_t(a) \stackrel{t\to 0^+}{\longrightarrow} \mu_0(a) := \epsilon(a), \quad a \in C_0^u(\mathbb{G}).$

Such convolution semigroups admit generating functionals

$$\gamma(a) = \lim_{t\to 0^+} \frac{\mu_t(a) - \epsilon(a)}{t}, \quad a \in \mathrm{Dom}(\gamma).$$

Theorem (Lindsay +AS)

The domain of the generating functional as above is a dense *-subspace of $C_0^u(\mathbb{G})$; moreover γ determines the semigroup $(\mu_t)_{t\geq 0_+}$ uniquely.

For $\mathbb{G}$ compact we have $\mathrm{Dom}(\gamma) \supset \mathrm{Pol}(\mathbb{G})$ (usually with a strict containment!).

Convolution semigroups of operators

The following is essentially a consequence of known results of the last 10 or so years, due to Daws, Junge, Neufang, Ruan and others.

Theorem

There exist $1-1$ correspondences between:

1. convolution semigroups $(\mu_t)_{t\geq 0}$ of states of $\mathrm{C}_0^u(\mathbb{G})$;
2. C_0-semigroups $(T_t^{\mathrm{u}})_{t\geq 0}$ of completely positive maps of norm 1 on $\mathrm{C}_0^u(\mathbb{G})$ that commute with the *left translation operators*;
3. semigroups $(P_t)_{t\geq 0}$ of normal, unital, completely positive maps on $\mathrm{L}^\infty(\mathbb{G})$ that are point–ultraweakly continuous at 0^+, and that satisfy

$$\Delta \circ P_t = (P_t \otimes \mathrm{id}) \circ \Delta, \quad t \geq 0;$$

4. C_0-semigroups $(M_t)_{t\geq 0}$ of norm 1 left module maps on $\mathrm{L}^1(\mathbb{G})$ with completely positive adjoints.

Convolution operators – revisited once again

$C_0^u(\mathbb{G})$ admits a canonical involutive operator R^u, so called universal unitary antipode (playing the role of the inverse operation).

Theorem

Let $\mu \in S(C_0^u(\mathbb{G}))$. The operator $P_\mu : L^\infty(\mathbb{G}) \to L^\infty(\mathbb{G})$, informally given by the formula

$$P_\mu = (\mathrm{id} \otimes \mu) \circ \Delta$$

is unital, completely positive, ϕ-preserving. Its KMS implementation (acting on $L^2(\mathbb{G})$) is always bounded and belongs to $L^\infty(\hat{\mathbb{G}})$. The map P_μ is KMS-symmetric iff $\mu = \mu \circ R^u$.

Main result

We can now add the Dirichlet form part.

Theorem

Let $\mathbb{G}$ be a locally compact quantum group. There exist $1-1$ correspondences between:

1. w^*-continuous convolution semigroups $(\mu_t)_{t\geq 0}$ of R^{u}-invariant states of $\mathrm{C}_0^u(\mathbb{G})$;
2. C_0^*-semigroups $(P_t)_{t\geq 0}$ of normal, unital, completely positive maps on $\mathrm{L}^\infty(\mathbb{G})$ that are KMS-symmetric with respect to ϕ and are translation-invariant: $\Delta \circ P_t = (P_t \otimes \mathrm{id}) \circ \Delta$ for every $t \geq 0$;
3. completely Dirichlet forms Q on $\mathrm{L}^2(\mathbb{G})$ with respect to ϕ that are invariant under $\mathcal{U}(\mathrm{L}^\infty(\hat{\mathbb{G}})')$ (modulo multiplication of forms by a positive number).

Skip examples

Examples

- Commutative case (G-classical): convolution semigroups on $\mathrm{L}^\infty(G)$ correspond to Lévy processes on G, are described via the Lévy-Khintchine formula.
- Dual case (G-classical, $\mathrm{L}^\infty(\hat{G}) = \mathrm{VN}(G)$): convolution semigroups are of the form
$$P_t(\lambda_g) = e^{t\psi(g)}\lambda_g, \quad g \in G$$
where $\psi : G \to \mathbb{R}$ is a conditionally positive-definite function. Corresponding Dirichlet form on $\mathrm{L}^2(\hat{G}) = \mathrm{L}^2(G)$ is
$$Q(f) = \int_G |f(g)|^2 \psi(g) dg, \quad f \in \mathrm{Dom}(Q).$$

Examples continued – cocycle twists

Let $\mathbb{G}$ – locally compact quantum group, $\Omega \in \mathrm{L}^\infty(\mathbb{G})\overline{\otimes}\mathrm{L}^\infty(\mathbb{G})$ be a unitary 2-cocycle on $\mathbb{G}$. Then (by a result of De Commer) we can define $\mathbb{G}_\Omega$ via

$$\mathrm{L}^\infty(\mathbb{G}_\Omega) := \mathrm{L}^\infty(\mathbb{G}),$$

$$\Delta_\Omega(m) = \Omega^* \Delta(m)\Omega, \quad m \in \mathrm{L}^\infty(\mathbb{G}_\Omega).$$

Theorem

If $(P_t)_{t\geq 0^+}$ is a convolution semigroup of operators on $\mathrm{L}^\infty(\mathbb{G})$ in the sense studied earlier, and $(P_t \otimes \mathrm{id})(\Omega) = \Omega$, then $(P_t)_{t\geq 0^+}$ is also a convolution semigroup of operators on $\mathrm{L}^\infty(\mathbb{G}_\Omega)$.

Example of application: start from $\mathbb{G} = \hat{G}$, with G containing an abelian subgroup H admitting a non-trivial 2-cocycle and use a conditionally positive-definite function on G which vanishes on H.

Specifically: we can build interesting convolution semigroups on quantized Heisenberg groups or on quantized $SL_2(\mathbb{C})$.

Domain of the generator

Examples skipped

Recall: a convolution semigroup of states $(\mu_t)_{t\geq 0_+}$ of states on $C_0^u(\mathbb{G})$ admits a densely defined 'generating functional'

$$\gamma(a) = \lim_{t\to 0^+} \frac{\mu_t(a) - \epsilon(a)}{t}, \quad a \in \mathrm{Dom}(\gamma).$$

$\mathcal{D}_+ = \{\tau^u_{\frac{i}{4}}((\omega \otimes \mathrm{id})(\widehat{W})) : \omega \in C_0^u(\widehat{\mathbb{G}})^*_+\}$
'Fourier transforms of positive measures on the dual group'

Theorem

Let $(\mu_t)_{t\geq 0}$ be a convolution semigroup of R^{u}-invariant states on $C_0^u(\mathbb{G})$. Then $\mathrm{span}(\mathcal{D}_+ \cap \mathrm{Dom}(\gamma))$ is a dense R^{u}-invariant *-subalgebra of $C_0^u(\mathbb{G})$.

This opens for example some cohomological perspectives!

From the generating functional to the convolution semigroup

Theorem

Suppose that $\mathcal{A}$ is a dense, unital $\mathrm{R^u}$-invariant *-subalgebra of the unitisation of $C_0^u(\mathbb{G})$ and that $\gamma : \mathcal{A} \to \mathbb{C}$ is a hermitian, normalised and conditionally positive $\mathrm{R^u}$-invariant functional. If in addition:

1. $\mathcal{A}$ is 'spanned' by Fourier transforms as before;
2. $\mathcal{A}$ contains 'sufficiently many in the L^2-sense' Fourier transforms;
3. γ is lower semicontinuous in the natural sense;
4. $\mathcal{A}$ 'interacts properly' with the Dirichlet property,

then (an extension of) γ generates a convolution semigroup of positive functionals.

If γ is a generating functional, then the unitisation of $\mathrm{span}(\mathcal{D}_+ \cap \mathrm{Dom}(\gamma))$ satisfies the above. If $\mathbb{G}$ is compact, $\mathrm{Pol}(\mathbb{G})$ satisfies the above (for any conditionally positive functional).

Perspectives

- Cipriani and Sauvageot showed that in the tracial case all quantum Dirichlet forms (subject to technical conditions) arise canonically from certain derivations; this takes a simpler form in the case of convolution semigroups on compact quantum groups, and is related to Lévy-Khintchine decomposition. Is there such a result in the state/weight case?
- classical results of Hunt for Lie groups and then Heyer for general lc groups show that for each convolution semigroup say on $C_0(G)$ the domain of its generator contains a canonical subalgebra. Can we have a result of this form for locally compact quantum groups?

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