Topological recursion of scalar fields in noncommutative geometry

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Introduction

This project started in 1998 as an attempt to understand quantum field theories on noncommutative geometries.

- No interacting and mathematically consistent QFT is known in 4 dimensions.
- The hope was that the situation could improve on noncommutative spaces. Renormalisation and improvement in β-function were established.

Since 2009 we accumulated hints that something special is behind our computations, but we were unable to locate it.

Topological recursion

... is this special structure. It governs a remarkable variety of research lines in mathematics and physics and establishes beautiful connections between different fields.

From field theories on NCG to matrix models

- Finite-dimensional noncommutative algebras are matrix algebras. Matrices play a key role in the Connes-Chamseddine spectral action of the standard model coupled to gravity.
- The algebra there is $C^\infty(M) \otimes$ matrices, which together is still infinite-dimensional.
- When we investigate quantum field theory of such objects, the appearance of divergences forces us (at least in intermediate steps) to restrict to finite-dimensional spaces.
- It is natural to take them even in finite-dimensional algebras, i.e. matrices.

Examples for action functionals

1. $S(\Phi) = \mathrm{Tr}(P(\Phi))$, $\quad \Phi = \Phi^*$, $\quad P(\Phi) = \sum_{n=2}^{K} c_n \Phi^k$
(Hermitian one-matrix model)

2. $S(\Phi_1, \Phi_2) = \mathrm{Tr}(P(\Phi_1) + P(\Phi_2) + \Phi_1\Phi_2)$, $\quad \Phi_i = \Phi_i^*$
(Hermitian two-matrix model)

3. $S_J(\Phi) = \mathrm{Tr}(P(\Phi) + J\Phi)$, $\quad \Phi = \Phi^*$,
(external field model)

4. Dirac $\mathcal{D} = \mathrm{D} \pm J\mathrm{D}J^{-1}$, Laplace $\Delta = \mathcal{D}^2 = \mathrm{D}^2 + J\mathrm{D}^2 J^{-1}$
$S_{\mathrm{D}}(\Phi) = \mathrm{Tr}(\mathrm{D}^2\Phi^2 + \frac{\lambda}{3}\Phi^3)$, $\quad \Phi = \Phi^*$
(Kontsevich model = matrix Airy function)

5. $S_{\mathrm{D}}(\Phi) = \mathrm{Tr}(\mathrm{D}^2\Phi^2 + \frac{\lambda}{4}\Phi^4)$, $\quad \Phi = \Phi^*$
(quartic Kontsevich model = NC-Φ^4 model)

All these are one of four faces of a beautiful structure!

Four faces

- (A) Matrix models, enumerative geometry, noncommutative geometry
- (B) Complex geometry, topological recursion
- (C) Intersection theory on the moduli space $\overline{\mathcal{M}}_{g,n}$ of stable complex curves, algebraic geometry
- (D) Integrable hierarchies

(Labelling inspired and material taken from: R. Belliard, S. Charbonnier, B. Eynard, E. Garcia-Failde, "Topological recursion for generalised Kontsevich graphs and *r*-spin intersection numbers," arXiv:2105.08035)

Selected historical notes

- [Tutte, 60s] found generating functions for the number of maps of polygonal countries on surfaces.
- [Brézin-Itzykson-Parisi-Zuber 78] gave a dual formulation by introducing the Hermitian 1-matrix model (A).
- [Gross-Migdal, Brézin-Kazakov, Douglas-Shenker 90] established integrability of the 1-matrix model (A-D).
- [Witten 90] conjectured from 2D-quantum gravity that generating function of ψ-classes on $\overline{\mathcal{M}}_{g,n}$ is τ function for KdV hierarchy (C-D).
- Proved by [Kontsevich 91] who related both structures to the matrix Airy function (A-C & A-D).
- [Chekhov-Eynard-Orantin 06] found structures for complex curves which govern Hermitian 1- and 2-matrix models (A-B).
- [Eynard-Orantin 07] established topological recursion as a universal structure to relate (A,B,D), later with (B-C).

The four faces of the Kontsevich model

- C: Let ψ_i be first Chern class of line bundle $\mathcal{L}_i \to \overline{\mathcal{M}}_{g,n}$ whose fibre is $T^*_{c_i}\Sigma$, for c_i a maked point of $\Sigma \in \overline{\mathcal{M}}_{g,n}$. Define
$$\omega_{g,n}(z_1,...,z_n) := 2^{2-2g-n} \sum_{\substack{d_1+...+d_n\\=3g-3+n}} \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1}\cdots\psi_n^{d_n} \prod_{i=1}^n \frac{(2d_i-1)!!}{z_i^{2d_i+2}}$$
- A: $\omega_{g,n}(z_1,...,z_n)$ is complexification of $G^{(g)}_{k_1,...,k_n}$,
$$\sum_{g=0}^{\infty} N^{2-2g-2n} G^{(g)}_{k_1,...,k_n} := \Big\langle \int \frac{d\Phi}{\mathcal{Z}} \Phi_{k_1k_1}\cdots\Phi_{k_nk_n} e^{-N\,\mathrm{Tr}(D^2\Phi^2+\frac{\lambda}{3}\Phi^3)}\Big\rangle_c$$
Connection by expansion into ribbon graphs which provide cell decomposition of $\mathcal{M}_{g,n}\times(\mathbb{R}_+)^n$ via Strebel differentials
- D: Partition function $\mathcal{Z}$ of matrix model gives rise to Virasoro constraints $L_i\mathcal{Z}=0$ and string equation. Translate to KdV.
- B: $\omega_{g,n}$ computed by topological recursion for spectral curve $(x:\hat{\mathbb{C}}\to\hat{\mathbb{C}},\ x(z)=z^2,\ y(z)=z,\ \omega_{0,2}(z_1,z_2)=\frac{dz_1\,dz_2}{(z_1-z_2)^2})$

Topological recursion [Eynard-Orantin 07]

Starting from a spectral curve (B) consisting of

- a ramified covering $x:\Sigma\to\Sigma_0$ of Riemann surfaces,
- meromorphic differentials $\omega_{0,1}=ydx$ on Σ and $\omega_{0,2}$ on $\Sigma\times\Sigma$,

recursively construct family $\omega_{g,n}$ of meromorphic differentials on Σ^n, with poles at zeros of dx (ramification points), by

$$\omega_{g,n}(z_1,...,z_n)=\sum_i \operatorname*{Res}_{q\to\beta_i} K(z_1,q,\sigma_i(q))dz\Big(\omega_{g-1,n+1}(q,\sigma_i(q),z_2,...,z_n) + \sum_{\substack{g_1+g_2=g,\ (g_j,I_j)\neq(0,\emptyset)\\ I_1\uplus I_2=\{z_2,\dots,z_n\}}} \omega_{g_1,|I_1|+1}(q,I_1)\omega_{g_2,|I_2|+1}(\sigma_i(q),I_2)\Big)$$

[sum over ramification points β_i of x; local involution $x(q)=x(\sigma_i(q))$ near β_i; recursion kernel $K(z_1,z_2,z_3)=\frac{\frac{1}{2}\int_{z'=z_3}^{z_2}\omega_{0,2}(z_1,z')}{\omega_{0,1}(z_2)-\omega_{0,1}(z_3)}$]

- $\omega_{g,n}$ are combinations of intersection numbers on $\overline{\mathcal{M}}_{g,n}$ (C).
- There is a systematic way to obtain a Hirota equation (D).

Further examples for topological recursion

- $(\hat{\mathbb{C}} \to \hat{\mathbb{C}},\ x(z) = \alpha + \gamma(z+\frac{1}{z}),\ y(z) = \sum_j u_j(z^j - z^{-j}), \omega_{0,2} = B)$ Hermitian 1MM, generates enumerations of maps [Tutte 60s]
- [Mirzakhani 07] recursion for Weil-Petersson volumes of moduli spaces of bordered Riemann surfaces captured by $(\hat{\mathbb{C}} \to \hat{\mathbb{C}},\ x(z) = z^2,\ y(z) = \frac{2}{\pi}\sin(\pi z),\ \omega_{0,2} = B)$
- For $\mathfrak{X}$ a toric Calabi-Yau 3-fold, let $\mathcal{S}$ be the singular locus of its mirror Calabi-Yau. Then $\omega_{g,n}$ of $\mathcal{S}$ generate Gromov-Witten invariants of $\mathfrak{X}$ (which classify stable genus-g maps into $\mathfrak{X}$) [Bouchard-Mariño-Klemm-Pasquetti 07]
- Lambert curve $(\hat{\mathbb{C}} \to \hat{\mathbb{C}},\ x(z) = -z + \log z,\ y(z) = z, \omega_{0,2} = B)$ generates Hurwitz numbers, i.e. number of coverings of $\hat{\mathbb{C}}$ with specified ramification profile at ∞ [Bouchard-Mariño 07]. [Ekedahl-Lando-Shapiro-Vainshtein 00]-formula relates Hurwitz numbers to integral of Hodge class.

The quartic Kontsevich model = NC $\lambda\Phi^4$-model

- renormalised action $S(\Phi) = \mathrm{Tr}((\mathrm{D}^2 + \mu_{bare}^2)Z\Phi^2 + \frac{\lambda}{4}Z^2\Phi^4)$
- sequence of measures for approximating $N \times N$-matrices

$$d\mu(\Phi) = \frac{1}{\mathcal{Z}} d\Phi\, e^{-NS(\Phi)}$$

- Fact: all moments/cumulants have formal $\frac{1}{N}$-expansion

  e.g. 2-point function $N \int_{H_N} d\mu(\Phi)\, \Phi_{kl}\Phi_{lk} = \sum_{g=0}^{\infty} N^{-2g} G^{(g)}_{|kl|}$

Theorem [Grosse-W 09]

Closed non-linear eq. for complexified planar 2-point function

$$\Big(\zeta + \eta + \mu_{bare}^2 + \lambda \int_0^\infty dt\; \varrho_0(t)\, ZG^{(0)}(\zeta, t)\Big) ZG^{(0)}(\zeta, \eta)$$

$$= 1 + \lambda \int_0^\infty dt\; \varrho_0(t)\, \frac{ZG^{(0)}(t, \eta) - ZG^{(0)}(\zeta, \eta)}{t - \zeta}$$

where $\varrho_0(t) = \sum_k \frac{r_k}{N} \delta(t - e_k)$ if D^2 has eigenvalues $\{e_k\}$ of multiplicities $\{r_k\}$.

Solution

Ansatz $ZG^{(0)}(x,y) = \dfrac{e^{\mathcal{H}_x[\tau_y(\bullet)]}\sin\tau_y(x)}{\lambda\pi\varrho_0(x)}$ with Hilbert transform $\mathcal{H}$

Theorem [Panzer-W 18]

The τ-equation is for $\varrho_0(t)\equiv 1$ solved by

$$\tau_y(x) = \operatorname{Im}\log\big(y + I(x+\mathrm{i}\epsilon)\big)$$

$$I(\zeta) := \lambda W_0\big(\tfrac{1}{\lambda}e^{\frac{1+\zeta}{\lambda}}\big) - \lambda\log\big(1-\lambda W_0\big(\tfrac{1}{\lambda}e^{\frac{1+\zeta}{\lambda}}\big)\big)$$

W_0 = principal branch of Lambert-W.

Theorem [Grosse-Hock-W 19]

Ansatz $I(\zeta) = -R(-\mu^2 - R^{-1}(\zeta))$ solves general τ-equation if

- $R(z) = z - \lambda(-z)^{D/2}\int \frac{dt\ \varrho_\lambda(t)}{(\mu^2+t)^{D/2}(t+\mu^2+z)}$
- ϱ_λ is implicit solution of $\varrho_0(R(x)) = \varrho_\lambda(x)$
- $\varrho_\lambda(x) = x\ {}_2F_1\big({}^{\alpha_\lambda,1-\alpha_\lambda}_{\quad 2}\big|-x\big)$ for $\varrho_0(x)=x$, $\alpha_\lambda = \frac{\arcsin(\lambda\pi)}{\pi}$

Direct solution for finite N

Theorem ([Schürmann-W 19], inspired by [Hock-Grosse-W 19])

Let $(\varepsilon_k, \varrho_k)$ be implicitly defined by $e_k = R(\varepsilon_k),\ r_k = R'(\varepsilon_k)\varrho_k$

for $R(z) = z - \frac{\lambda}{N}\sum_{k=1}^{d} \frac{\varrho_k}{z+\varepsilon_k}$.

Then $G^{(0)}(\zeta,\eta) = \mathcal{G}^{(0)}(z,w)$ for $R(z)=\zeta$, $R(w)=\eta$ and

$$\boxed{\mathcal{G}^{(0)}(z,w) = \frac{1 - \frac{\lambda}{N}\sum_{k=1}^{d} \frac{r_k \prod_{j=1}^{d} \frac{R(w)-R(-\widehat{\varepsilon_k}^j)}{R(w)-R(\varepsilon_j)}}{(R(z)-R(\varepsilon_k))(R(\varepsilon_k)-R(-w))}}{R(w)-R(-z)}}$$

where $u \in \{z, \hat{z}^1, \dots, \hat{z}^d\}$ are all solutions of $R(u) = R(z)$.

(The symmetry $\mathcal{G}^{(0)}(z,w) = \mathcal{G}^{(0)}(w,z)$ is automatic)

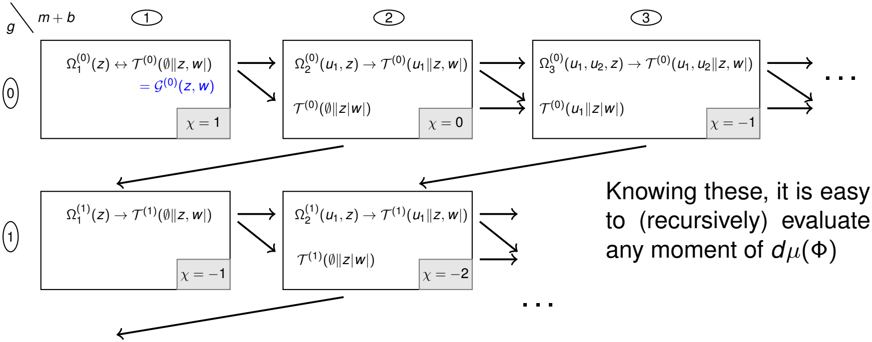
Thus, planar 2-point function solved by the composition of a rational function $\mathcal{G}^{(0)}$ with inverse of another rational function R.

Solution procedure [Branahl-Hock-W 20]

Recall that $d\mu(\Phi)$ depends on spectral values $e_1, \dots, e_d$. Define

$$\sum_{g=0}^{\infty} N^{2-2g-n} \Omega^{(g)}_{q_1,\dots,q_n} := \frac{\partial^{n-1}\Big(N \sum_{k=1}^{N} \int_{H_N} d\mu(\Phi) \Phi_{q_1 k}\Phi_{k q_1}\Big)}{\partial e_{q_2} \cdots \partial e_{q_n}} + \frac{\delta_{n,2}}{(e_{q_1}-e_{q_2})^2}$$

Their complexification forms with two families $\mathcal{T}(I\|z,w|)$, $\mathcal{T}(I\|z|w|)$ a system of equations to solve in decreasing χ:



Knowing these, it is easy to (recursively) evaluate any moment of $d\mu(\Phi)$

Contact with topological recursion [BHW 20]

- Pass to meromorphic differentials
  $\omega_{g,m}(z_1,...,z_m) = \lambda^{2-2g-m}\Omega^{(g)}_m(z_1,...,z_m)\prod_{k=1}^m dR(z_i)$
- Intermediate steps of solution scheme extremely lengthy, but final result simple and structured:

$$\omega_{0,2}(u,z) = \frac{du\,dz}{(u-z)^2} + \frac{du\,dz}{(u+z)^2}$$

$$\omega_{0,3}(u_1,u_2,z) = -\sum_{i=1}^{2d} \frac{\Big(\frac{1}{(u_1-\beta_i)^2}+\frac{1}{(u_1+\beta_i)^2}\Big)\Big(\frac{1}{(u_2-\beta_i)^2}+\frac{1}{(u_2+\beta_i)^2}\Big)du_1\,du_2\,dz}{R'(-\beta_i)R''(\beta_i)(z-\beta_i)^2}$$
$$+\Big[d_{u_1}\Big(\frac{\omega_{0,2}(u_2,u_1)}{(dR)(u_1)}\frac{dz}{R'(-u_1)(z+u_1)^2}\Big)+u_1\leftrightarrow u_2\Big]$$

where $\beta_1,...,\beta_{2d}$ are the ramification points of R, i.e. $dR(\beta_i)=0$

Observation

The blue terms are exactly those of topological recursion, the red terms are a consistent extension called blobbed topological recursion [Borot-Shadrin 15].

Quartic Kontsevich model obeys BTR!

Proposition $(g, m) \in \{(0,3), (0,4), (0,5), (1,1)\}$ / Conjecture

Let $R : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be ramified cover identified in $\mathcal{G}^{(0)}(z, w)$ with ramification points $\beta_1, ..., \beta_{2d}$. Define $\omega_{0,1}(z) = -R(-z)dR(z)$ and for $2 - 2g - m \leq 0$ the $\omega_{g,m}$ as before from $\Omega_m^{(g)}$.
Then parts $\mathcal{P}\omega_{g,m}$ containing the poles at ramification points:

$$\mathcal{P}_z\omega_{g,m}(u_1, ..., u_{m-1}, z)$$
$$= \sum_{i=1}^{2d} \operatorname*{Res}_{q\to\beta_i} \frac{\frac{1}{2}\int_{q'=\sigma(q)}^{q'=q} B(z, q')}{\omega_{0,1}(q) - \omega_{0,1}(\sigma_i(q))} \Bigg(\omega_{g-1,m+1}(u_1, ..., u_{m-1}, q, \sigma_i(q)) + \sum_{\substack{I_1 \uplus I_2 = \{u_1,...,u_{m-1}\} \\ g_1+g_2=g \\ (I_1,g_1)\neq(\emptyset,0)\neq(I_2,g_2)}} \omega_{g_1,|I_1|+1}(I_1, q)\omega_{g_2,|I_2|+1}(I_2, \sigma_i(q))\Bigg)$$

where σ_i = local Galois involution near β_i, i.e. $R(z) = R(\sigma_i(z))$, $\sigma_i(\beta_i) = \beta_i$, $\sigma_i \neq \mathrm{id}$ and $B(u, z) = \frac{du\,dz}{(u-z)^2}$ Bergman kernel.

Proof for genus $g = 0$ [Hock-W 21]

When trying to prove the conjecture for $g = 0$ we noticed surprising identities between $\omega_{0,m+1}(u_1, \dots, u_m, -z)$ and $\omega_{0,k+1}(u_1, \dots, u_k, z)$. They are of independent interest:

Definition

Let $x : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a ramified covering with ramification points $\beta_1, \dots, \beta_r$. For a global involution $\iota : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, which neither fixes nor permutes the β_i, let $y(z) := -x(\iota z)$. Then a family $\{\omega_{0,n}\}_{n\geq 2}$ of meromorphic differentials is introduced by

$$\omega_{0,2}(w,z) = \frac{1}{2}\frac{dw\,dz}{(w-z)^2} + \frac{1}{2}\frac{d(\iota w)\,d(\iota z)}{(\iota w - \iota z)^2} - \frac{1}{2}\frac{dw\,d(\iota z)}{(w-\iota z)^2} - \frac{1}{2}\frac{d(\iota w)\,dz}{(\iota w - z)^2}$$

and for $m \geq 2$ by the involution identity

$$\omega_{0,m+1}(u_1, \dots, u_m, z) + \omega_{0,m+1}(u_1, \dots, u_m, \iota z) = \sum_{s=2}^{m} \sum_{I_1 \uplus \dots \uplus I_s = \{u_1, \dots, u_m\}} \frac{1}{s} \operatorname*{Res}_{w \to z} \left(\frac{dy(z)dx(w)}{(y(z)-y(w))^s} \prod_{i=1}^{s} \frac{\omega_{0,|I_i|+1}(I_i, w)}{dx(w)} \right).$$

Theorem [Hock-W 21]

The involution identity has the (under mild assuptions unique) solution

$$
\begin{aligned}
&\omega_{0,m+1}(u_1,\dots,u_m,z)\\
&=\sum_{i=1}^{r}\operatorname*{Res}_{q\to\beta_i} K_i(z,q)\sum_{I_1\uplus I_2=\{u_1,\dots,u_m\}}\omega_{0,|I_1|+1}(I_1,q)\omega_{0,|I_2|+1}(I_2,\sigma_i(q))\\
&-\sum_{k=1}^{m} d_{u_k}\Big[\operatorname*{Res}_{q\to\iota u_k}\sum_{I_1\uplus I_2=\{u_1,\dots,u_m\}}\tilde{K}(z,q,u_k)d_{u_k}^{-1}\big(\omega_{0,|I_1|+1}(I_1,q)\omega_{0,|I_2|+1}(I_2,q)\big)\Big]
\end{aligned}
$$

where the recursion kernels are given by

$$
K_i(z,q):=\frac{\frac{1}{2}\big(\frac{dz}{z-q}-\frac{dz}{z-\sigma_i(q)}\big)}{dx(\sigma_i(q))(y(q)-y(\sigma_i(q)))}\;,\;\tilde{K}(z,q,u):=\frac{\frac{1}{2}\big(\frac{d(\iota z)}{\iota z-\iota q}-\frac{d(\iota z)}{\iota z-u}\big)}{dx(q)(y(q)-y(\iota u))}.
$$

The solution implies symmetry $z\mapsto\iota z$ of the rhs of the involution identity.

Back to quartic Kontsevich model

Theorem

For the choice

$$\iota z = -z \,, \qquad x(z) = R(z) := z - \frac{\lambda}{N} \sum_{k=1}^{d} \frac{\varrho_k}{\varepsilon_k + z} \,,$$

the solution of the involution identity coincides with the solution of the system for $(\Omega_n^{(0)}, \mathcal{T}^{(0)})$ found in [Branahl-Hock-W 20].

- Many more surprising combinatorial identites found on the way. Some conjectures were proved by Maciej Dołęga.
- There are a few examples for blobbed topological recursion (e.g. multitrace Hermitian matrix model, stuffed maps), but recursion kernel for blob is rather special.
- We have some ideas to extend the involution identity to higher g. The difficulty is more the $(\Omega_n^{(g)}, \mathcal{T}^{(g)})$-system.

Outlook

- We established another correspondence (A-B) between a matrix model (NCG) and [blobbed] topological recursion.
- The natural question is about faces C and D.

Fact [Borot-Shadrin 15]

- Forms $\omega_{g,m}$ which satisfy BTR encode intersection numbers on the moduli space $\overline{\mathcal{M}}_{g,m}$ of stable complex curves.
- These could be interesting or not. The deep rôle of the involution $z \mapsto -z$ makes us confident that they will encode some geometric structure.

Integrability

Is not known in BTR. But this model with involution $z \mapsto -z$ and blob given by recursion kernel is special. We are optimistic.