Dual spaces of operator systems

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Outline

1. Aims and motivations
2. MOS and quasi-operator systems
3. Duality for operator systems
4. Biduals of operator systems

• A “unital operator system” is a self-adjoint subspace of $\mathcal{B}(\mathfrak{H})$ containing the identity, equipped with the induced matrix cone.

Theorem

(Choi-Effros) Let E be a finite dimensional unital operator system. For the dual space E^*, when equipped with the dual matrix cone, one can find an order unit, so that E^* be identified with a unital operator system.

• Recent application to graph systems and dual graph systems in quantum information theory.

• There is some interests in infinite dimensional operator systems in quantum information theory.

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- Some issues:

1. The dual cone of the dual space of a unital operator system needs not admits an order unit: e.g., the dual space of c (the C^*-algebra of all convergence sequences) is ℓ^1, and the dual cone (which is the set of all positive summable sequences) does not has an order unit.
$\Rightarrow$ Need to consider "general operator systems"; i.e. self-adjoint subspaces of $\mathcal{B}(\mathfrak{H})$, equipped with the induced matrix cones and the induced matrix norms.

2. The dual space of a unital operator system needs not be an operator system under the dual matrix cone and the dual matrix norm.
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• Operator system tensor products: Kavruk-Paulsen-Todorov-Tomforde (unital case); Li-Ng (non-unital case).

• If one wants to study duality for operator system tensor products, then one wants to ask

Question 1': Is there a canonical / universal to find equivalent matrix norm on the dual space of an operator system T, under which the dual space becomes an operator system?

• If Q1' has a positive answer, we denoted the "dual operator system" by T^{d}. A natural question is

Question 2: If T^{d} exists, does $(T^{\mathrm{d}})^{\mathrm{d}}$ exists?

• If Q2 has a positive answer, there is one more question.

• It is not hard to see that the usual bidual T^{**} of T is also an operator system in the canonical way. An other question is

Question 3: If Q2 has a positive answer, is the canonical map from T^{**} to $(T^d)^d$ preserves the operator system structures?

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Theorem

Let T be an oper. sys.

(a) $\exists$ an equivalent matrix norm on T^* turning it into a dual oper. sys. $\Leftrightarrow$ the "matrix ordered matrix normed space" T satisfies a form of "bounded decomposition property".

In this case,

$$\|f\|^{\mathrm{d}} := \sup\left\{\left\|[f_{kl}(x_{ij})]_{i,j,k,l}\right\| : x \in M_n(T)_+; \|x\| \leq 1; n \in \mathbb{N}\right\}$$

is the largest equivalent matrix norm dominated by the dual matrix norm on T^* that turns T^* into a dual oper. sys. We denote the resulting oper. sys. $(T^*, \|\cdot\|^{\mathrm{d}})$ by T^{d}.

(b) If T^* can be turned into a dual oper. sys. under an equivalent matrix norm, then so is $(T^{\mathrm{d}})^*$.

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Suppose that T is a C^*-algebra or a unital operator system.

(a) $\exists$ an equivalent matrix norm on T^* turning it into a dual oper. sys.

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• According to Weaver, a unital oper. sys. $T \subseteq \mathcal{B}(\mathfrak{H})$ is called a graph system if it is closed under the weak-*-topology.

Corollary

Let $T \subseteq \mathcal{B}(\mathfrak{H})$ be a graph sys. and

$$T_\# := \{\omega|_T : \omega \in \mathcal{B}(\mathfrak{H})_*\} \subseteq T^*.$$

Then $T_\#$ admits a largest matrix norm $\|\cdot\|_\#$ that is domin. by and equiv. to the dual matrix norm $\|\cdot\|^*$ such that $T_\#$ becomes an oper. sys. under $\|\cdot\|_\#$.

• The resulting operator system in the above can be regarded as a predual graph system.

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• X vector space. For $n \in \mathbb{N}$, see $M_n(X) \subseteq M_{n+1}(X)$. Denote

$$M_\infty(X) := \bigcup_{n\in\mathbb{N}} M_n(X).$$

• $M_\infty(X)$ is a bimodule over $M_\infty := M_\infty(\mathbb{C})$.

• Y another vector space. For linear map $\varphi : X \to Y$, define

$$\varphi^{(\infty)}([x_{ij}]_{i,j}) := \big[\varphi(x_{ij})\big]_{i,j} \quad ([x_{ij}]_{i,j} \in M_\infty(X))$$

and $\varphi^{(n)} := \varphi^{(\infty)}|_{M_n(X)}$

• X is an operator space if $\exists$ a norm $\|\cdot\|$ on $M_\infty(X)$ (called the matrix norm) s.t. $\forall x_i \in M_\infty(X)$ and $a_i, b_i \in M_\infty \subseteq \mathcal{B}(\ell^2)$

$$\Big\|\sum_{k=1}^n a_k^* x_k b_k\Big\| \leq \Big\|\sum_k a_k^* a_k\Big\|^{1/2}\Big\|\sum_k b_k^* b_k\Big\|^{1/2} \max_k \|x_n\|.$$

• $\varphi : X \to Y$ is s.t.b. completely bounded if $\varphi^{(\infty)}$ is bounded. Moreover, φ is called a

++ complete isometry if $\varphi^{(\infty)}$ is isometric;

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- For a normed space E, we set $B_E := \{x \in E : \|x\| \leq 1\}$.
- If $f = [f_{ij}]_{i,j} \in M_\infty(E^*)$, we define $\theta_f \in \mathcal{B}(E; M_\infty)$ by
$$\theta_f(x) := [f_{ij}(x)]_{i,j} \quad (x \in E).$$
Then $\theta : M_n(E^*) \to \mathcal{B}(E; M_n)$ is bij.
- Similarly, $\exists$ (not nec. surj.) a map $\theta : M_\infty(E) \to \mathcal{B}(E^*; M_\infty)$ given by $\theta_x(f) := [f(x_{ij})]$, for every $x = [x_{ij}] \in M_\infty(E)$.
- X a SMOS. For $f \in M_\infty(X^*)$, we set $(f^*)_{ij} := (f_{ji})^*$ and
$$\|f\| = \sup\left\{\left\|\theta_f^{(\infty)}(x)\right\| : x \in B_{M_\infty(X)}\right\}.$$
Denote $M_\infty(X^*)^+ := \{f \in M_\infty(X^*) : \theta_f \text{ is comp. pos.}\}$.
Then X^* becomes a SMOS.

Proposition

Let X be a SMOS. Then X^* is a MOS $\Leftrightarrow$ the linear span of $M_\infty(X)^+$ is norm-dense in $M_\infty(X)$.

- For a normed space E, we set $B_E := \{x \in E : \|x\| \leq 1\}$.
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Let X be a SMOS. Then X^* is a MOS $\Leftrightarrow$ the linear span of $M_\infty(X)^+$ is norm-dense in $M_\infty(X)$.

- A dual operator system is a weak-*-closed subspace of $\mathcal{B}(\mathfrak{H})$ equipped with the induced structure ("non-unital graph system").

- If X is a SMOS, then X^* is called a dual quasi-operator system if $\exists$ a compl. order monomor. compl. embed. and $\Gamma : X^* \to \mathcal{B}(\mathfrak{H})$ s.t. $\Gamma : X^* \to \Gamma(X^*)$ is a weak-*-homeom.

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• $\forall n \in \mathbb{N}$, denote by $\mathcal{WQ}_n^{X^*}$ all $\sigma(X^*, X)$-cont. compl. pos. compl. contract. from X^* to M_n.

• Consider the von Neumann algebra

$$N(X^*) := \bigoplus_{n\in\mathbb{N}}^{\ell^\infty} \ell^\infty(\mathcal{WQ}_n^{X^*}; M_n).$$

• Define $\mu_{X^*} : X^* \to N(X^*)$ by

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(a) μ_{X^*} is a weak-*-cont. compl. order monom. compl. contract.

(b) The following are equivalent.

1. μ_{X^*} is a compl. bounded below.
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(c) V a dual oper. sys. and $\Phi : X^* \to V$ is a weak-*-cont. compl. pos. compl. contract. $\Rightarrow$ $\exists$ a unique weak-*-cont. compl. pos. compl. contract. $\overline{\Phi} : \widetilde{X^*} \to V$ with $\Phi = \overline{\Phi} \circ \mu_{X^*}$.

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- $\widetilde{X^*}$ is "universal dual oper. sys. cover" of X^*.
- For $f \in M_m(X^*)$, one has $\|\mu_{X^*}^{(m)}(f)\| = \sup\left\{\left\|\theta_f^{(n)}(x)\right\| : x \in M_n(X)_+; \|x\| \leq 1; n \in \mathbb{N}\right\}$.

Note that $\theta_f^{(n)}(x) = [f_{kl}(x_{ij})]_{i,j,k,l}$.

Corollary

T a dual quasi-oper. sys. (resp. dual oper. sys.). If $\check{T}$ is the dual oper. sp. T equip with a smaller weak-*-cl. matrix cone, then $\check{T}$ is a dual quasi-oper. sys. (resp. dual oper. sys.).

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Let X be a SMOS. Denote

$$U_X := \{u - v : u, v \in M_\infty(X)^+ \cap B_{M_\infty(X)}\}.$$

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(b) In the case when X^* is a dual quasi-oper. sys.,

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is the largest element in $\mathcal{N}_X^{\mathrm{sys}}$.

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• Let T be a oper. sys. such that T^* is a dual quasi-oper. sys. $\Rightarrow \mu_{T^*} : T^* \to T^{\mathrm{d}}$ is a weak-*-homeom. compl. order monom. compl. embedd.

• $(T^{\mathrm{d}})^*$ is a dual quasi-oper. sys. $\Rightarrow \mu_{(T^{\mathrm{d}})^*} : (T^{\mathrm{d}})^* \to (T^{\mathrm{d}})^{\mathrm{d}}$ is a weak-*-homeom. compl. order monom. compl. embedd.

• $\mu_{(T^{\mathrm{d}})^*} \circ (\mu^*_{T^*})^{-1} : T^{**} \to (T^{\mathrm{d}})^{\mathrm{d}}$ is a weak-*-homeom. compl. order monom. compl. embedd.

• The following is an answer to Q3:

Theorem

Let T be a C^*-algebra or a unital operator system. Then $\mu_{(T^{\mathrm{d}})^*} \circ (\mu^*_{T^*})^{-1}$ is a complete isometry.

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Reference:

C.K. Ng,
Dual spaces of operator systems,
preprint (arXiv:2105.11112).