Nonproduct geometries: the Standard Model and gravity

Noncommutative Geometry Seminar

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Motivation why the Standard Model and gravity?

- Geometry applied to physics that works (good precision).
- Well tested model, geometry which is falsifiable.
- Open for extensions and modifications, plenty of models:

Beyond the Standard Model, Modified Gravity

AB and A.Sitarz, Phys. Rev. D **101**, 075038 (2020)

AB and A.Sitarz, Phys. Rev. D **103**, 044041 (2021)

AB, A.Sitarz and P.Zalecki, in preparation

The Standard Model and gravity: NC geometric perspective

1. The language of spectral triples: $(\mathcal{A}, \mathcal{H}, \mathcal{D}, \gamma, \mathcal{J}), \ldots$
   - $\mathcal{A}$ - $*$-algebra faithfully represented on a Hilbert space $\mathcal{H}$,
   - $\mathbb{Z}/2\mathbb{Z}$-grading γ on $\mathcal{H}$ s.th. $[\gamma, \mathcal{A}] = 0$,
   - real structure: antilinear isometry $\mathcal{J}$
   - (essentially) self-adjoint operator $\mathcal{D}$ on $\mathcal{H}$,
   - compact resolvent, domains, relations ...
2. Spectral action principle:
   - To obtain an effective (classical) action functional (Lagrangian) that is the starting point to compare with the experiment.

How to choose the spectral triple?

- Which axioms are crucial and which can be modified/omitted?
- Which properties can be read from the experiment ?
- Is it possible to have a minimal, unique theory without unnecessary additional conditions ?

How to choose the spectral triple?

- Which axioms are crucial and which can be modified/omitted?
- Which properties can be read from the experiment ?
- Is it possible to have a minimal, unique theory without unnecessary additional conditions ?
- The standard choice: almost-commutative **product** geometries

$$\left(C^\infty(M)\otimes\mathcal{A}_F, L^2(M)\otimes\mathcal{H}_F, \mathcal{D}, J, \gamma\right)$$

  with $\mathcal{D} = \not{D}\otimes 1 + \gamma_5\otimes\mathcal{D}_F$ and $\mathcal{A}_F = \mathbb{C}\oplus\mathbb{H}\oplus M_3(\mathbb{C})$,
- $\mathcal{A}_F$ is chosen to give rise to the gauge group of the Standard Model and $\not{D}$ is the standard Dirac operator

Few minor issues:

- Passing from Euclidean to Lorentzian formulation: how to *rotate* or how to *compute Lorentzian spectral action*?
- Quadrupling of degrees of freedom: too many fermions - one need to use some projection.
- Possibility of $SU(3)$-symmetry breaking: *usual* minimal axioms do not fully eliminate unphysical models.
- Some *predictions* do not agree with experiments, e.g. the value of the Higgs mass.
- Restricted possibility to modify the gravity sector.

Several partial solutions

- Lorentzian formulation [Paschke–Sitarz, 2006] [Barret, 2007] [Eckstein–Franco, 2014] [van den Dungen, 2015] [Brouder–Bizzi–Besnard, 2015][Devastato–Farnsworth–Lizzi–Martinetti, 2018] [B.–Sitarz, 2018] [Martinetti–Singh, 2019], [Dang– Wrochna, 2020]. . .
- Fermion doubling problem [Lizzi–Mangano–Miele–Sparano, 1997] [Gracia-Bondia–Iochum–Schucker, 1998] [D'Andrea–Kurkov–Lizzi, 2016],. . .
- Classification of finite Dirac operators and the problem of leptoquarks [Krajewski, 1998] [Paschke–Sitarz, 1998] [Paschke– Scheck–Sitarz, 1999] [Farnsworth– Boyle, 2014] [Dąbrowski– D'Andrea– Sitarz, 2018] [B.–Sitarz, 2018],. . .
- σ field [Stephan, 2009] [Chamseddine–Connes, 2012], . . .
- Twisted Spectral Triples [Landi–Martinetti, 2016] [Devastato–Martinetti, 2017], . . .

What we explore.

- Do not assume **almost-commutativity**: allow general (nonproduct) spectral triples ($\mathcal{D}$) over the same algebra.
- Our approach: look at the **physical** Standard Model / modifications of gravity – and try to explore what is **the geometry** (in spectral triple language) used to describe it.
- Try to keep the information about the **Lorentzian structure**.
- Last step: pass from the algebraic construction to the analytical framework (compute the spectral action).

Standard Model: Krein-shifted geometry

- Dirac operator for $(1,3)$-Minkowski space: $\mathcal{D} = i\gamma^\mu \partial_\mu$, with $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$.
- Fermionic action: $\int \overline{\psi} \mathcal{D} \psi = \int \psi^\dagger \widetilde{\mathcal{D}} \psi$, where $\overline{\psi} = \psi^\dagger \gamma^0$; $\widetilde{\mathcal{D}} = \gamma^0 \mathcal{D}$ - the Krein shift of $\mathcal{D}$.
- $\widetilde{\mathcal{D}}$ - symmetric $\Leftrightarrow$ $\mathcal{D}$ - Krein-self-adjoint: $\mathcal{D} = \gamma^0 \mathcal{D} \gamma^0$

  [Paschke–Sitarz, 2006] [Franco, 2014] [B.–Sitarz, 2018]...
- $\mathcal{D}\gamma = -\gamma \mathcal{D}$, $\mathcal{D}\mathcal{J} = \mathcal{J}\mathcal{D}$.
- $\widetilde{\mathcal{D}}\gamma = \gamma \widetilde{\mathcal{D}}$, $\widetilde{\mathcal{D}}\mathcal{J} = -\mathcal{J}\widetilde{\mathcal{D}}$.

Finite Riemannian spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D}, \pi_L, \pi_R)$

- $\mathcal{A}$ - finite dimensional algebra
- π_L - representation of $\mathcal{A}$ on $\mathcal{H}$
- π_R - representation of $\mathcal{A}^{\mathrm{op}}$ on $\mathcal{H}$
- $[\pi_L(a), \pi_R(b)] = 0$ - (0th order condition)
- $[[\mathcal{D}, \pi_L(a)], \pi_R(b)] = 0$ - (1st order condition)

Additional conditions:

Look for the spin-c or Hodge type condition [Dąbrowski– D'Andrea, 2016], [Dąbrowski– D'Andrea– Sitarz, 2018], [Dąbrowski– Sitarz, 2019]

- spin$_c$ type geometry: $(Cl_{\mathcal{D}}(\pi_L(\mathcal{A})))' = \pi_R(\mathcal{A})$.
  Hodge condition: $(Cl_{\mathcal{D}}(\pi_L(\mathcal{A})))' = Cl_{\mathcal{D}}(\pi_R(\mathcal{A}))$,

where

$CL_{\mathcal{D}}(\pi_L(\mathcal{A}))$ is the algebra generated by $\pi_L(\mathcal{A})$ and $[\mathcal{D}, \pi_L(\mathcal{A})]$.

The idea is to rephrase the universally accepted form of the Standard Model Lagrangian in the language of spectral triples being as close as possible to the pseudo-Riemannian structure and physics.

How does it work for the Standard Model?

- We start with the particle content (one generation):

$$\Psi = \begin{pmatrix} \nu_R & u_R^1 & u_R^2 & u_R^3 \\ e_R & d_R^1 & d_R^2 & d_R^3 \\ \nu_L & u_L^1 & u_L^2 & u_L^3 \\ e_L & d_L^1 & d_L^2 & d_L^3 \end{pmatrix} \in M_4(H_W)$$

- Algebra $\mathcal{A}$ of $\mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$-valued smooth functions on the spacetime, with representations:

$$\pi_L(\lambda, q, m)\Psi = \begin{pmatrix} \lambda & & \\ & \overline{\lambda} & \\ & & q \end{pmatrix} \Psi, \quad \pi_R(\lambda, q, m)\Psi = \Psi \begin{pmatrix} \overline{\lambda} & \\ & m^\dagger \end{pmatrix}.$$

Dirac operator for the Standard Model

- At every point of the Minkowski space, linear operators on the space of particles can be encoded as a matrix from $M_4(\mathbb{C}) \otimes M_2(\mathbb{C}) \otimes M_4(\mathbb{C})$.
- Dirac operator: $\mathcal{D}_{SM}\Psi = \mathcal{D}\Psi + \mathcal{D}_F\Psi$, where

$$\mathcal{D} = \begin{pmatrix} & & i\widetilde{\sigma}^\mu\partial_\mu & \\ & & & i\widetilde{\sigma}^\mu\partial_\mu \\ i\sigma^\mu\partial_\mu & & & \\ & i\sigma^\mu\partial_\mu & & \end{pmatrix},$$

and $\mathcal{D}_F$ is a finite endomorphism of $M_4(H_W)$. Here $\widetilde{\sigma}^0 = \sigma^0 = 1_2$ and $\widetilde{\sigma}^j = -\sigma^j$ for $j = 1, 2, 3$.

Dirac operator for the Standard Model

- $\mathcal{D}_F$ has to be in $M_4(\mathbb{C}) \otimes 1_2 \otimes M_4(\mathbb{C})$ in order to have the Lorentz invariance of the full Dirac operator.
- Hence $\mathcal{D}_F$ commutes with the chirality $\Gamma = \pi_L(1, -1, 1)$.
- Therefore, $\mathcal{D}_{SM} = \mathcal{D} + \mathcal{D}_F$ with $\{\mathcal{D}, \Gamma\} = 0$ and $[\mathcal{D}_F, \Gamma] = 0$.
- Krein-shifted operators behave in the opposite way.

The result

Theorem

With the above assumptions,

- *Requiring $\widetilde{\mathcal{D}_{SM}}$ to satisfy the first order condition implies*

$$\widetilde{\mathcal{D}_F} = \begin{pmatrix} & M_l \\ M_l^\dagger & \end{pmatrix} \otimes 1_2 \otimes e_{11} + \begin{pmatrix} & M_q \\ M_q^\dagger & \end{pmatrix} \otimes 1_2 \otimes (1_4 - e_{11}),$$

  where $M_l, M_q \in M_2(\mathbb{C})$.

- *if M_l, M_q nondegenerate then $\widetilde{\mathcal{D}_{SM}}$ satisfies the spin$_c$ condition.*

The Standard Model with three generations

- Hilbert space: $M_4(H_W) \otimes \mathbb{C}^3$.
- Representation enlarged diagonally.
- $M_l, M_q \in M_2(\mathbb{C}) \otimes M_3(\mathbb{C})$:

$$M_l = \begin{pmatrix} \Upsilon_\nu & \\ & \Upsilon_e \end{pmatrix}, \qquad M_q = \begin{pmatrix} \Upsilon_u & \\ & \Upsilon_d \end{pmatrix},$$

with Υ_e, Υ_u - diagonal, $\Upsilon_\nu = U\widetilde{\Upsilon_\nu}U^\dagger$, $\Upsilon_d = V\widetilde{\Upsilon_d}V^\dagger$,

U– Pontecorvo–Maki–Nakagawa–Sakata matrix,

V– Cabibbo–Kobayashi–Maskawa matrix.

The Standard Model with three generations of particles

Theorem

The spin-c condition holds provided that for both pairs of matrices $(\Upsilon_\nu, \Upsilon_e)$ and (Υ_u, Υ_d) their eigenvalues are pairwise different.

- This is the same condition as for Hodge duality [Dąbrowski–Sitarz, 2019]
- This condition is satisfied for physical Standard Model provided that there is no massless neutrino [Dąbrowski–Sitarz, 2019]
- The model can be doubled: the resulting spectral triple satisfies the Hodge duality and is the finite part of the one studied in the almost-commutative framework.

CP violation and reality of the spectral triple

- The usual 0th order condition is not implemented by $\mathcal{J}$, but its milder version is: $\pi_R(\mathcal{A}) \subseteq \mathcal{J}\pi_L(\mathcal{A})\mathcal{J}^{-1}$.
- A real Dirac operator implies the reality of M_l and M_q.
- One generation: fermion masses are real.
- Three generations: both Wolfenstein parameter $\overline{\eta}$ and CP-violating phase δ^{ν}_{CP} have to vanish.
- **CP-violation** $\Leftrightarrow$ shadow of the $\mathcal{J}$-symmetry violation in the nondoubled spectral triple.
- $\widetilde{\mathcal{D}_{SM}}$ satisfies the order one condition, while $\mathcal{D}_{SM}$ satisfies its twisted version: $[[\mathcal{D}_{SM}, \pi_L(a)]_\beta, \pi_R(b)]_\beta = 0$, where $[x, y]_\beta = xy - \beta y \beta^{-1} x$.

Gauge transformations

- $U_{LR} := \pi_L(U)\pi_R(U)$ for $U = (u_1, u_2, u_3) \in \mathcal{U}(\mathcal{A})$. They form a group $(U(1) \times SU(2) \times U(3))/(\mathbb{Z}/2\mathbb{Z})$.
- To have $SU(3)$ rather than $U(3)$ one could impose unimodularity condition.
- The left action is already unimodular, while for the right one it could be imposed either on each fundamental component or in the full representation.
- In the first case: $u_1 \det u_3 = 1$ and the gauge group of the Standard Model $(U(1) \times SU(2) \times SU(3))/(\mathbb{Z}/6\mathbb{Z})$, while in the second one: $(u_1 \det u_3)^{12} = 1$ and the group differs by a finite factor.

Fluctuated Dirac operator

- $\widetilde{\mathcal{D}_{SM}}^{\omega} = \widetilde{\mathcal{D}_{SM}} + \omega$ with

$$\omega = A_\mu e_{11} \otimes \sigma^\mu \otimes (1_4 - e_{11}) - 2A_\mu e_{22} \otimes \sigma^\mu \otimes e_{11}$$
$$- A_\mu e_{22} \otimes \sigma^\mu \otimes (1_4 - e_{11}) - A_\mu (e_{33} + e_{44}) \otimes \tilde{\sigma}^\mu \otimes e_{11}$$
$$+ \begin{pmatrix} 0_2 & \\ & W_\mu \end{pmatrix} \otimes \widetilde{\sigma}^\mu \otimes 1_4 + \begin{pmatrix} 1_2 & \\ & 0_2 \end{pmatrix} \otimes \sigma^\mu \otimes \begin{pmatrix} 0_1 & \\ & G_\mu \end{pmatrix}$$
$$+ \begin{pmatrix} 0_2 & \\ & 1_2 \end{pmatrix} \otimes \widetilde{\sigma}^\mu \otimes \begin{pmatrix} 0_1 & \\ & G_\mu \end{pmatrix} + \begin{pmatrix} & M_l \Phi \\ \Phi^\dagger M_l^\dagger & \end{pmatrix} \otimes 1_2 \otimes e_{11}$$
$$+ \begin{pmatrix} & M_q \Phi \\ \Phi^\dagger M_q^\dagger & \end{pmatrix} \otimes 1_2 \otimes (1_4 - e_{11}).$$

Physical parametrization (for one generation)

- Since $\Phi \in \mathbb{H}$ we can write $\Phi = \begin{pmatrix} \phi_1 & \phi_2 \\ -\overline{\phi_2} & \overline{\phi_1} \end{pmatrix}$.
- Define $\Phi_x := M_x(1_2 + \Phi)$, for $x = l, q$.
- Define the Higgs dublet $H = \begin{pmatrix} 1 + \phi_1 \\ \phi_2 \end{pmatrix}$

Static and spatial model

We consider time-independent and spatial part of the Dirac operator.

$$\widetilde{\mathcal{D}_L} = i\begin{pmatrix} 1_2 & \\ & -1_2 \end{pmatrix} \otimes \sigma^j \partial_j + \begin{pmatrix} & \Phi_l \\ \Phi_l^\dagger & \end{pmatrix} \otimes 1_2$$
$$+ A_j \begin{pmatrix} \sigma^3 - 1_2 & \\ & 1_2 \end{pmatrix} \otimes \sigma^j - \begin{pmatrix} 0_2 & \\ & W_j \end{pmatrix} \otimes \sigma^j.$$

$$\widetilde{\mathcal{D}_Q} = i\begin{pmatrix} 1_2 & \\ & -1_2 \end{pmatrix} \otimes \sigma^j \partial_j \otimes 1_3 + \begin{pmatrix} & \Phi_q \\ \Phi_q^\dagger & \end{pmatrix} \otimes 1_2 \otimes 1_3$$
$$+ A_j \begin{pmatrix} \sigma^3 + \frac{1}{3}1_2 & \\ & -\frac{1}{3}1_2 \end{pmatrix} \otimes \sigma^j \otimes 1_3$$
$$- \begin{pmatrix} 0_2 & \\ & W_j \end{pmatrix} \otimes \sigma^j \otimes 1_3 + \begin{pmatrix} 1_2 & \\ & -1_2 \end{pmatrix} \otimes \sigma^j \otimes G_j.$$

Static and spatial model

- The physical values of hypercharges are reproduced (in quark sector: because of the unimodularity condition).

Static and spatial model

- The physical values of hypercharges are reproduced (in quark sector: because of the unimodularity condition).
- Gilkey-Seeley-DeWitt coefficients for three generations:

$$a_2 = -\frac{1}{4\pi^2} a \int d^4x \, |H|^2,$$

$$a_4 = \frac{1}{8\pi^2} \int d^4x \left[b|H|^4 + a\mathrm{Tr}|D_j H|^2 + \frac{20}{3} F^2 + 2\mathrm{Tr} W^2 + 2\mathrm{Tr} G^2 \right],$$

where

$$a = \mathrm{Tr}(\Upsilon_\nu^\dagger \Upsilon_\nu) + \mathrm{Tr}(\Upsilon_e^\dagger \Upsilon_e) + 3\mathrm{Tr}(\Upsilon_u^\dagger \Upsilon_u) + 3\mathrm{Tr}(\Upsilon_d^\dagger \Upsilon_d),$$

$$b = \mathrm{Tr}(\Upsilon_\nu^\dagger \Upsilon_\nu)^2 + \mathrm{Tr}(\Upsilon_e^\dagger \Upsilon_e)^2 + 3\mathrm{Tr}(\Upsilon_u^\dagger \Upsilon_u)^2 + 3\mathrm{Tr}(\Upsilon_d^\dagger \Upsilon_d)^2.$$

Static and spatial model

- Effective Lagrangian: $\mathcal{L} = \mathcal{L}_{\text{Higgs}} + \mathcal{L}_{\text{gauge}}$, where

$$\mathcal{L}_{\text{Higgs}} = \frac{bf(0)}{2\pi^2}|H|^4 - \frac{2f_2\Lambda^2 a}{\pi^2}|H|^2 + \frac{af(0)}{2\pi^2}\mathrm{Tr}|D_j H|^2,$$

$$\mathcal{L}_{\text{gauge}} = \frac{f(0)}{\pi^2}\left(\frac{10}{3}F^2 + \mathrm{Tr}\,W^2 + \mathrm{Tr}\,G^2\right).$$

- This result is consistent with taking the static part of the Lorentzian Lagrangian for the Standard Model.
- All the relations between the coefficients of the model are exactly the same as in the usual almost-commutative Euclidean formulation. Therefore, the measurable quantities will have the same values.

Wick rotated model - leptonic sector

We take the Lorentzian Dirac operator:

$$\mathcal{D}_L = i \begin{pmatrix} & 1_2 \otimes \widetilde{\sigma}^\mu \\ 1_2 \otimes \sigma^\mu & \end{pmatrix} \partial_\mu + A_\mu \begin{pmatrix} & -1_2 \otimes \widetilde{\sigma}^\mu \\ (\sigma^3 - 1_2) \otimes \sigma^\mu & \end{pmatrix}$$

$$+ \begin{pmatrix} & W_\mu \otimes \widetilde{\sigma}^\mu \\ 0_4 & \end{pmatrix} + \begin{pmatrix} \Phi_l^\dagger & \\ & \Phi_l \end{pmatrix} \otimes 1_2.$$

and Wick rotate it: $\sigma^j \to i\sigma^j$:

$$\begin{aligned} \mathcal{D}_{L,w} =& i \begin{pmatrix} & 1_2 \\ 1_2 & \end{pmatrix} \otimes 1_2 \partial_0 + i \begin{pmatrix} & -i1_2 \\ i1_2 & \end{pmatrix} \otimes \sigma^j \partial_j \\ & + A_0 \begin{pmatrix} & -1_2 \\ (\sigma^3 - 1_2) & \end{pmatrix} \otimes 1_2 + A_j \begin{pmatrix} & i1_2 \\ i(\sigma^3 - 1_2) & \end{pmatrix} \otimes \sigma^j \\ & + \begin{pmatrix} & W_0 \\ 0_2 & \end{pmatrix} \otimes 1_2 - \begin{pmatrix} & iW_j \\ 0_2 & \end{pmatrix} \otimes \sigma^j + \begin{pmatrix} \Phi_l^\dagger & \\ & \Phi_l \end{pmatrix} \otimes 1_2 \end{aligned}$$

Wick rotated model:results

Doing the same for the quark sector we compute $\mathcal{D}^\dagger\mathcal{D}$ and the Gilkey-Seeley-DeWitt coefficients:

$$a_2 = \frac{3}{4\pi^2} a \int d^4x |H|^2,$$

$$a_4 = \frac{1}{8\pi^2} \int d^4x \Big[b|H|^4 - a\mathrm{Tr}|D_\mu|^2 + \frac{20}{3}F^2 + 2\mathrm{Tr}(W^2) + 2\mathrm{Tr}(G^2) + 12\varepsilon^{jkl}F_{jk}F_{0l} - 6\varepsilon^{jkl}\mathrm{Tr}(W_{jk}W_{0l}) \Big]$$

which leads to the Euclidean action:

$$\mathcal{L}_{\mathrm{gauge}} = \frac{f(0)}{\pi^2}\left(\frac{10}{3}F^2 + \mathrm{Tr}(W^2) + \mathrm{Tr}(G^2) + 6\varepsilon^{jkl}F_{jk}F_{0l} - 3\varepsilon^{jkl}\mathrm{Tr}(W_{jk}W_{0l}) \right),$$

$$\mathcal{L}_H = \frac{bf(0)}{2\pi^2}|H|^4 + \frac{6f_2\Lambda^2}{\pi^2}a|H|^2 - \frac{af(0)}{2\pi^2}\mathrm{Tr}|D_\mu H|^2$$

Wick rotated model: summary

- **No fermion doubling**.
- **No $SU(3)$ breaking**.
- Order-one condition holds.
- Lack of real structure $\rightarrow$ **CP violation**.
- Spectral triple obeys the Morita condition of spin$_c$ geometry.
- **Spectral action** reproduces the action of the Lorentzian Standard Model with an additional electroweak "θ-term".
- Potentially different coefficients do not finally affect the numerical values of the measurable parameters.

Nonproduct geometries for *bimetric* gravity

A different possibility to modify *almost-commutative geometries* is to assume dependence of the Dirac operator on the discrete coordinate. Assume two-point (Connes-Lott) discrete geometry over a manifold and a Dirac operator:

$$\mathcal{D} = \begin{pmatrix} D_1 & \gamma\Phi \\ \gamma\Phi^* & D_2 \end{pmatrix}$$

where D_1 and D_2 are independent Dirac operators.

Problems:

- compute the spectral action (at least two leading parts) ?
- is the model physically viable (link to bimetric gravity) ?
- is there a mechanism leading to $D_1 = D_2$?

Nonproduct geometries for „bimetric" gravity

Assume FLRW type geometry (Euclidean):

$$ds^2 = b(t)^2 dt^2 + a(t)^2 \left(d\chi^2 + S_k^2(\chi)\left(d\theta^2 + \sin^2(\theta) d\phi^2\right)\right),$$

where $S_k(\chi) = \chi \,\mathrm{sinc}(\chi\sqrt{k})$ for $k = 0, \pm 1$, so

$$D = \gamma^a dx^\mu(\theta_a)\frac{\partial}{\partial x^\mu} + \frac{1}{4}\gamma^c \omega_{cab}\gamma^a\gamma^b,$$

with $\{\gamma^a, \gamma^b\} = -2\delta^{ab} I$. Apply conformal rescalling: $D_h = h^{-1} D h$ with $h(t) = a(t)^{-3/2} b(t)^{-1/2}$ and take the doubled geometry with D_1, D_2 as earlier, with Φ some function, $\gamma^2 = \kappa = \pm 1$ and with $a_i(t), b_i(t)$ positive smooth functions ($i = 1, 2$).

Nonproduct geometries for „bimetric” gravity

The method: we compute the terms of the spectral action using the Wodzicki residue:

$$\begin{aligned}\mathcal{S}(\mathcal{D}) &= \Lambda^4 \,\mathrm{Wres}(\mathcal{D}^{-4}) + c\Lambda^2 \,\mathrm{Wres}(\mathcal{D}^{-2}) \\ &= \int_M \int_{\|\xi\|=1} \left(\Lambda^4 \,\mathrm{Tr}\,\mathrm{Tr}_{Cl}\, \mathfrak{b}_0^2 + c\Lambda^2 \,\mathrm{Tr}\,\mathrm{Tr}_{Cl} \mathfrak{b}_2\right)\end{aligned}$$

where $\sigma_{\mathcal{D}^{-2}}(\xi) = \mathfrak{b}_0 + \mathfrak{b}_1 + \mathfrak{b}_2 + \ldots$, and $\mathfrak{b}_k(x, \xi)$ is homogeneous of order $-2-k$ in ξs

The action

$$\mathcal{S}(\mathcal{D}) \sim \int dt \, \Big\{ \left(\Lambda^4 - c\kappa\Lambda^2|\Phi|^2\right)\left(a_1^3 b_1 + a_2^3 b_2\right)$$
$$-\frac{c\Lambda^2}{12}\left(a_1^3 b_1 R(a_1, b_1) + a_2^3 b_2 R(a_2, b_2)\right)$$
$$+c\kappa\Lambda^2|\Phi|^2 b_1 b_2 \frac{(a_1 - a_2)^2}{(a_1 b_2 + a_2 b_1)^2}\left[a_1^2(2a_2 b_1 + a_1 b_2)\right.$$
$$\left. + \; a_2^2(2a_1 b_2 + a_2 b_1)\right] +$$
$$+c\kappa\Lambda^2|\Phi|^2 \frac{(b_1 - b_2)^2}{(a_2 b_1 + a_1 b_2)^2} a_1^2 a_2^2 (a_1 b_1 + a_2 b_2) \Big\}$$

where $R(a, b)$ is the usual scalar curvature density.

The potential and bimetric gravity models

- Introduce $x = \frac{b_1}{b_2}$, $y = \frac{a_1}{a_2}$, which depend only on the entries of $X^a{}_c = g_2^{ab} g_{1bc}$.
- The interaction potential between the metrics satisfies:

$$\mathbb{V}\left(\sqrt{g_2^{-1} g_1}\right)\sqrt{g_2} = \mathbb{V}\left(\sqrt{g_1^{-1} g_2}\right)\sqrt{g_1}, \qquad (1)$$

$$\mathbb{V}\left(\sqrt{g_2^{-1} g_1}\right) = \frac{1}{(x+y)^2}\left(x^2 + 2xy - 2x^2y + y^2 - 6xy^2\right.$$
$$\left. + 4x^2y^2 + 4xy^3 - 6x^2y^3 + x^3y^3 - 2xy^4 + 2x^2y^4 + xy^5\right)$$

- $\mathbb{V}$ is a rational function of symmetric polynomials in $\sqrt{X}$.
- For bimetric gravity models [Hassan– Rosen, 2012]: polynomial function of symmetric polynomials in $\sqrt{X}$ satisfying (1).

Equations of motion

- Wick rotate into $(-,+,+,+)$
- Introduce effective parameters:

$$\Lambda_e = \frac{12}{c}(\Lambda^2 - c\kappa|\Phi|^2), \quad \alpha = 12|\Phi|^2\kappa.$$

- $\Lambda_e > 0$, $\Lambda_e < 0$ or $\Lambda_e = 0$.
- Introduce interactions with matter and/or radiation.
- We are free to fix one of b_1, b_2. We will later use the choice $b_{1,2}(t) = 1 \pm b(t)$.

Equations of motion

$$6H_{b,i}^2 + \frac{6k}{a_i^2} - \Lambda_e - \frac{\alpha}{a_i} V(a_i, a_{i'}, b_i, b_{i'}) = -2T^0{}_0(a_i, b_i),$$

$$12\frac{\partial_t^2 a_i}{a_i b_i^2} + 6H_{b,i}^2 - 3\Lambda_e + \frac{6k}{a_i^2} - 12\frac{\partial_t a_i\, \partial_t b_i}{a_i b_i^3} - \alpha W(a_i, a_{i'}, b_i, b_{i'})$$
$$= -6T^1{}_1(a_i, b_i),$$

with $H_{b,j} = \frac{\partial_t a_j}{a_j b_j}$ and

$$V(a_1, a_2, b_1, b_2) = a_1 + \frac{8a_1 a_2(a_1^2 - a_2^2)b_2^3}{(a_2 b_1 + a_1 b_2)^3} + \frac{2a_2(a_2^2 + 2a_1 a_2 - 5a_1^2)b_2^2}{(a_2 b_1 + a_1 b_2)^2},$$

$$W(a_1, a_2, b_1, b_2) = 3 - 2\frac{a_2 b_2(a_2^2 - 4a_1 a_2 + 9a_1^2)}{a_1^2(a_2 b_1 + a_1 b_2)} + 2\frac{a_2 b_2^2(11a_1^2 - 2a_1 a_2 - 3a_2^2)}{a_1(a_2 b_1 + a_1 b_2)^2} - 8\frac{a_2 b_2^3(a_1^2 - a_2^2)}{(a_2 b_1 + a_1 b_2)^3}.$$

Perfect fluid models

- $T^{g}_{\mu\nu} = (\rho + P)u_\mu u_\nu + P g_{\mu\nu}$,
- $P(t) = w\rho(t)$,
- continuity equation $\Rightarrow \rho(t) = \eta\, a(t)^{-3(1+w)}$.

Perturbative solutions and stability

- $a_1(t) = a(t) + \epsilon r_1(t), \quad a_2(t) = a(t) + \epsilon r_2(t), \quad b(t) = \epsilon s(t)$
- Linearized EOMs:

$$\dot{r}(t) = \frac{3\lambda^2 a(t)^2(1+w) - (\dot{a}(t)^2 + k)(1+3w)}{2a(t)\dot{a}(t)} r(t) + \left(\dot{a}(t) + \alpha\frac{a(t)^2}{6\dot{a}(t)}\right) s(t),$$

$$\dot{s}(t) = 3\frac{\dot{a}(t)}{a(t)}\left(\frac{r(t)}{a(t)} - s(t)\right),$$

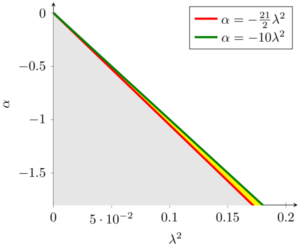
where $\Lambda_e = 6\lambda^2$.

Empty de Sitter universe $(\eta = 0, k = 0)$; $a(t) = a_0 e^{\lambda t}$

$$s(t) = C_1 e^{-\frac{3}{2}\lambda t + \frac{1}{2}\sqrt{21\lambda^2 + 2\alpha}\, t} + C_2 e^{-\frac{3}{2}\lambda t - \frac{1}{2}\sqrt{21\lambda^2 + 2\alpha}\, t},$$

$$r(t) = C_3 e^{-\frac{1}{2}\lambda t + \frac{1}{2}\sqrt{21\lambda^2 + 2\alpha}\, t} + C_4 e^{-\frac{1}{2}\lambda t - \frac{1}{2}\sqrt{21\lambda^2 + 2\alpha}\, t},$$

$$C_3 = C_1 \frac{a_0}{6\lambda} \left(3\lambda + \sqrt{21\lambda^2 + 2\alpha}\right), \qquad C_4 = C_2 \frac{a_0}{6\lambda} \left(3\lambda - \sqrt{21\lambda^2 + 2\alpha}\right).$$



- yellow region: exponential damping,
- gray region: exponentially growth
- red line: degenerate solution

Empty universe with $k = \mp 1$

- $k = -1$

$$s(t) = c_1\, {}_2F_1\left(\frac{3}{4} - \zeta, \frac{3}{4} + \zeta; 3; -\tau^2\right) + c_2\, G^{2,0}_{2,2}\left(-\tau^2 \middle| \begin{array}{c} \frac{1}{4} - \zeta, \frac{1}{4} + \zeta \\ -2, 0 \end{array}\right)$$

- $k = 1$

$$s(t) = c_1\, {}_2F_1\left(\frac{3}{4} - \zeta, \frac{3}{4} + \zeta; -\frac{1}{2}; -\tau^2\right) + c_2\, t^3\, {}_2F_1\left(\frac{9}{4} - \zeta, \frac{9}{4} + \zeta; \frac{5}{2}; -\tau^2\right)$$

where

$$\zeta = \frac{\sqrt{21\lambda^2 + 2\alpha}}{4\lambda}, \qquad \tau = \sinh(\lambda t).$$

Matter ($w = 0$) and radiation ($w = \frac{1}{3}$) dominated universes

- matter dominated with $k = 0$:

$$s(t) = c_1 t^{-\frac{3}{2}} J_{\sqrt{\frac{19}{12}}}\left(\sqrt{-\frac{\alpha}{2}}\, t\right) + c_2 t^{-\frac{3}{2}} Y_{\sqrt{\frac{19}{12}}}\left(\sqrt{-\frac{\alpha}{2}}\, t\right)$$

- radiation domination with $k = 0$:

$$s(t) = c_1 t^{-\frac{5}{4}} J_{\sqrt{\frac{13}{16}}}\left(\sqrt{-\frac{\alpha}{2}}\, t\right) + c_2 t^{-\frac{5}{4}} Y_{\sqrt{\frac{13}{16}}}\left(\sqrt{-\frac{\alpha}{2}}\, t\right)$$

- radiation dominated with $k = -1$: confluent Heun functions
- radiation dominated with $k = 1$: cyclic solutions

Summary

- Allowing for nonproduct geometries leads to models resembling bimetric gravity theories.
- For the considered range of models there exist a range of parameters so that the solution is dynamically stable.
- Bimetric gravity: huge freedom in the choice of potential – spectral action fixes the action.
- Modified gravity models are of interests in cosmology and nonproduct geometries may shed some light on them.

Thank you for your attention!