The Podleś spheres converge to the sphere

Global Noncommutative Geometry Seminar
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Based on joint work with Konrad Aguilar, Thomas Gotfredsen and Jens Kaad

PRELUDE

- The talk revolves around non-commutative analogues of compact metric spaces.

Classical		Quantum
compact Hausdorff space	$\longleftrightarrow$	unital C^*-algebra
compact group	$\longleftrightarrow$	compact quantum group
compact (spin) manifold	$\longleftrightarrow$	spectral triple
compact metric space	$\longleftrightarrow$	??

- This question was first studied by Rieffel in the late 1990's, and drawing inspiration from Connes' NCG, he arrived at the theory of compact quantum metric spaces.

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Compact Quantum Metric Spaces

Definition (Rieffel, Li)

Let A be a unital C^*-algebra equipped with a seminorm $L\colon A \to [0,\infty]$ satisfying that $L(x^*) = L(x)$ for all $x \in A$. Then (A, L) is called a compact quantum metric space (CQMS) if

(i) $L(a) = 0$ *iff* $a \in \mathbb{C}.1$

(ii) *The set* $V := \{a \in A \mid L(a) < \infty\}$ *is dense in* A.

(iii) $\rho_L(\mu,\nu) := \sup\{|\mu(a) - \nu(a)| : L(a) \leqslant 1\}$ *metrises the weak*-topology on* $\mathcal{S}(A)$.

In this case L is called a Lip-norm.

- If (X, d) is a compact metric space then $C(X)$ becomes a CQMS by setting $L_d(f) := \sup\left\{\frac{|f(x)-f(y)|}{d(x,y)} : x \neq y\right\}$.
- L_d is finite on $C^{\mathrm{Lip}}(X) := \{f \in C(X) : f \text{ is Lipschitz cont.}\}$
- The metric ρ_{L_d} on $\mathcal{S}(C(X)) = \mathrm{Prop}(X)$ is the so-called *Monge-Kantorovich metric* and $\rho_{L_d}\restriction_X = d$.

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Examples from NCG

- If $(\mathcal{A}, \mathcal{H}, D)$ is a spectral triple, then sometimes – but not always – one obtains a CQMS by defining

$$L_D(a) := \|[D, a]\| \qquad a \in \mathcal{A}.$$

- This is the case when $\mathcal{A} = C^\infty(S^1)$ and $D = \frac{-i}{2\pi}\frac{d}{d\theta}$, in which case the metric ρ_{L_D} on $\mathcal{S}(C(S^1))$ restricts to the usual arc-length on $S^1 \subset \mathcal{S}(C(S^1))$. True, more generally, for compact, connected, spin manifolds [Connes].
- And also if $A = C^*_{\mathrm{red}}(\Gamma)$ for a word hyperbolic group Γ (eg. $\mathbb{F}_n$) equipped with a length function ℓ and D_ℓ in $\ell^2(\Gamma)$ is defined as $D_\ell(\delta_\gamma) := \ell(\gamma)\delta_\gamma$ [Ozawa-Rieffel].

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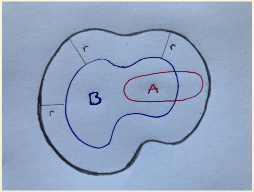
- Then their *Hausdorff distance* is defined by

$$\mathrm{dist}_{\mathrm{H}}^{d}(A, B) := \inf\{r > 0 \mid A \subset \mathbb{B}_r(B) \text{ and } B \subset \mathbb{B}_r(A)\}$$



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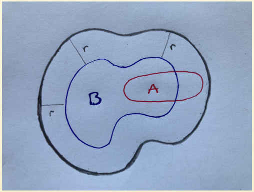

- And for two compact metric spaces (X_1, d_1) and (X_2, d_2) their *Gromov-Hausdorff distance* is defined as

$$\mathrm{dist}_{\mathrm{GH}}(X_1, X_2) := \inf_{d} \left\{ \mathrm{dist}_{\mathrm{H}}^{d}(X_1, X_2) \right\}$$

where the infimum runs over all metrics on $X_1 \sqcup X_2$ restricting to d_1 and d_2 respectively.

$\rightsquigarrow$ quantum-GH-distance

Quantum Gromov-Hausdorff distance

- If (A_1, L_1) and (A_2, L_2) are CQMS then a Lip-norm $L\colon A_1 \oplus A_2 \to [0,\infty]$ is called *admissible* if the induced quotient semi-norms on A_1 and A_2 agree with L_1 and L_2.
- The two projections $A_1 \oplus A_2 \to A_i$ dualise to injections

$$\mathcal{S}(A_i) \hookrightarrow \mathcal{S}(A_1 \oplus A_2)$$

- And Rieffel then defines

$$\mathrm{dist}_{\mathrm{GH}}^{\mathrm{Q}}(A_1, A_2) := \inf\left\{ \mathrm{dist}_{\mathrm{H}}^{\rho_L}(\mathcal{S}(A_1), \mathcal{S}(A_2)) : L \text{ admissible} \right\}$$

- This is symmetric and satisfies the triangle-inequality.
- But distance zero only means Lip-isometric order unit isomorphism $(V_1)_{\mathrm{sa}} \simeq (V_2)_{\mathrm{sa}}$ (at the level of completions). Or affine isometric isomorphism between state spaces.
- $C\colon (X, d) \mapsto (C(X), L_d)$ is a contraction, but *not* an isometry.
- However, it is a homeomorphism onto its image.

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QUANTUM GROMOV-HAUSDORFF DISTANCE

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Convergence and continuity results

- Non-commutative tori vary continuously [Rieffel].
- Fuzzy spheres (i.e. matrix algebras) converge to the classical 2-sphere [Rieffel, van Suijlekom].
- Crossed products with certain one-parameter families of automorphisms vary continuously [K-Kaad].
- AF-algebras are approximated by matrix algebras [Aguilar-Latrémolière].
- But, q-deformations are not yet well understood from the QMS point of view — at least not in a way that reflects their non-commutative geometry well.
- Actually, only in 2018, Aguilar and Kaad showed that the Dąbrowski-Sitarz Dirac operator turns the Podleś sphere S_q^2 into a CQMS.

⇝ q-deformations

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THEOREM (AGUILAR-KAAD-K, 2021)
The Podleś spheres S_q^2 vary continuously in the deformation parameter q and converge to the classical 2-sphere as q tends to 1.

- This is really a theorem about the coordinate algebras $\mathcal{O}(S_q^2) := \mathrm{Alg}(A_q, B_q, B_q^*)$, as these are the domains of the Lip-norm:

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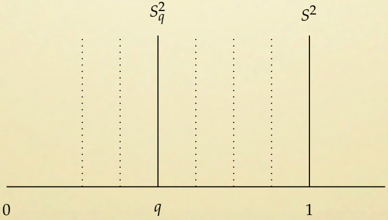

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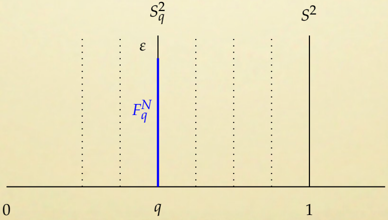

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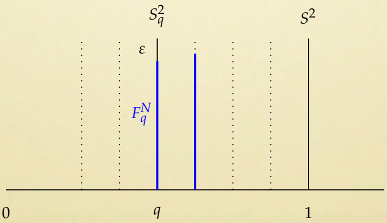

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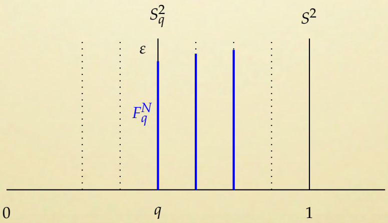

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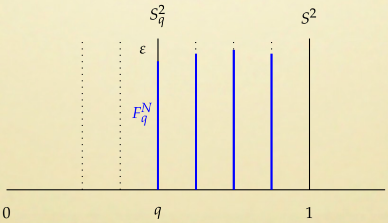

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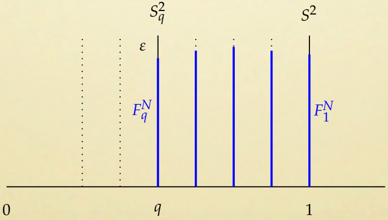

THANK YOU!

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