NCG in 4 pages

Thijs van den Broek

A very brief introduction into noncommutative geometry

The aim of this small document is to provide the reader (assumed to be somebody with a background in physics) a bird's eye view of noncommutative geometry (NCG) and in particular its application in particle physics. I by far have the intention to give even a remotely complete exhibition of the subject, rather I will touch upon some of its main ideas and concepts.

NCG is a branch of mathematics that has become of vast size; it incorporates ideas and techniques from various other field of mathematics. Although it is a mathematicians playground, there are a number of applications in physics, one of these I will be focussing on. This is the possibility to give a derivation of the Standard Model in geometrical nature. Physics and geometry have in fact already had a rather long and fruitful joint history, think of the geometrical interpretation of gauge theories and the General theory of relativity. As I will explain, NCG may be considered to be a generalization of the latter.

The field is rooted in an idea that dates back to the 1940s stating that any compact space X and the commutative algebra of continous functions on that space,

$$C(X, \mathbb{C}) = \{f : X \to \mathbb{C}, f \text{ is continuous}\}$$

contain the same information (strictly: they are each others dual)[1]. So instead of talking about spaces (which are a topological concept) you might equally well talk about commutative algebras (that are an algebraic concept). Building upon the above correspondence, various geometric properties of the space X can be translated into properties of the corresponding algebra $C(X)$: a link between two completely different fields of mathematics is established! The essential idea behind NCG is to generalize this correspondence to noncommmutative algebras[2] and to provide mathematical techniques in order to handle these noncommutative algebras.

Now for the physically interesting cases, we must enrich the space X with extra structures: we promote it to a *Riemannian manifold*, a space that *locally* looks like the Euclidean space $\mathbb{R}^n$ (for some n) on which we define a Riemannian metric g.[3] We're just referring to curved spacetimes here, but you must keep in mind that your standard Minkowski space is problematic, since it features a metric with minus signs... From here on we write M instead of X.

The aim of the above was to give an intuitive feeling of the basic idea of NCG, what will follow now is considered to be the most prominent object in NCG, the *spectral triple* $(\mathcal{A}, \mathcal{H}, D)$. Here $\mathcal{A}$ is a (non)commutative *unital involutive*[4] algebra (= space!), $\mathcal{H}$ is a Hilbert space —so automatically equipped with an inner product, denoted by $\langle ., . \rangle : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$— on which $\mathcal{A}$ is represented as bounded operators, and a self-adjoint operator $D : \mathcal{H} \to \mathcal{H}$. There are a number of compatibility conditions between these three objects (that I will take for granted now) that in fact make it a rather stringent definition.

[1] In the case that you're not familiar with the concept of an algebra: it is a vector space on which a product —satisfying certain rules— is defined. In this case we define for each f and g in $C(X, \mathbb{C})$ that $(fg)(x) := f(x)g(x) \in \mathbb{C}$ i.e. fg is indeed again an element of $C(X, \mathbb{C})$.

[2] There seems to be some sort of "don't ask, don't tell"-policy towards the question whether or not such noncommutative algebras again correspond to something one might call a "noncommutative space".

[3] To connect with the notation that you're probably more familiar to: this means that for each $x \in X$ we have a map $g(x)$ whose components are denoted by $g_{\mu\nu}$.

[4] The first of these means that there is an element that plays the role of a multiplicative unit, the second means that for each $m \in \mathcal{A}$, $m^* \in \mathcal{A}$ as well.

Okay, you might want an example of a spectral triple now. The following may said to have served as the motivating example of the field; NCG is more or less modelled to be a generalization of it:

$$(\mathcal{A}, \mathcal{H}, D) = (C^\infty(M, \mathbb{C}), L^2(M, S), \not{\partial} = i\gamma^\mu(\partial_\mu + \omega_\mu)), . \tag{1}$$

I will comment on the three different values one by one

- $C^\infty(M, \mathbb{C})$ is the subset of $C(M, \mathbb{C})$ containing only *smooth* (i.e. infinitely differentiable) functions. It can be made involutive (just as $C(M)$ itself) by just defining f^* through $(f^*)(x) := \overline{f(x)}$ for any $x \in M$.
- The space $L^2(M, S)$ consists of smooth, spinor-valued functions[5] ψ (i.e. $\psi(x)$ is a spinor for each $x \in M$). The number of components of that spinor depends on the dimension of the manifold M. In addition for each of these spinors

  $$\langle \psi, \psi \rangle \equiv \int_M \psi^*(x)\psi(x)\sqrt{g}\mathrm{d}^n x < \infty$$

  has to hold. Note that for a given manifold M, $L^2(M, S)$ need not even exist; its existence heavily relies on the properties of M.
- $\not{\partial}$ is the operator that is familiar from the Dirac equation, save for the term ω_μ which is a term that accounts for the manifold M being curved. The Dirac operator $\not{\partial}$ is in fact defined using the Levi-Civita connection, the unique connection that is compatible with the metric on M. Picking a metric thus gives you a unique Dirac operator, and the same holds the other way around: the Dirac operator can therefore be seen *as* a metric.

Given a spectral triple one can enrich it with two operators, J and γ on $\mathcal{H}$, the first of which can be seen as some sort of charge conjugation, whereas the latter allows you to make a distinction into left- and right-handed elements of $\mathcal{H}$ (it is called a *grading*). We denote such an enriched spectral triple by $(\mathcal{A}, \mathcal{H}, D; J, \gamma)$ and call it a *real, even spectral triple*. These operators J and γ satisfy

$$D\gamma = -\gamma D \qquad DJ = \pm JD \qquad J^2 = \pm 1 \qquad J\gamma = \pm\gamma J$$

and the three different signs giving rise to eight possible combinations determine the so called *KO-dimension* ($\in \mathbb{N} \bmod 8$) of the spectral triple.
Two more ingredients are needed. The first stems from the mathematically natural question "to what extent is a given spectral triple unique?". To answer that question one can —given a spectral triple $(\mathcal{A}, \mathcal{H}, D)$— consider a second algebra $\mathcal{B}$ that is *Morita equivalent*[6] to $\mathcal{A}$. The task at hand is then to find a spectral triple $(\mathcal{B}, \mathcal{H}', D')$ of which $\mathcal{B}$ serves as the algebra: this can be called a Morita equivalent spectral triple —though strictly it isn't an equivalence relation. There is a definite answer to this problem, but I will not mention it here, since we'll only be needing a special case here. Since $\mathcal{A}$ is Morita equivalent to itself, one can consider the answer to the problem, but taking $\mathcal{B} = \mathcal{A}$. It turns out that there is a whole family of spectral triples that are Morita equivalent to the original one and they are given by $(\mathcal{A}, \mathcal{H}, D_A = D + A)$, where for each different A we have another member of this family. In the case of a real spectral triple we have a slightly different solution: the members are given by $D_A = D + A \pm JAJ^*$, where A must be selfadjoint.[7] The D_A are called the *inner fluctuations* of D. These inner fluctuations turn out to have the right form in order to be interpreted as *gauge fields*.

[5] Strictly speaking they are square integrable sections of a spinor bundle.

[6] Morita equivalence is a bit weaker than unitary equivalence. Will you settle with the example that each algebra $\mathcal{A}$ is Morita equivalent with the algebra of $\mathcal{A}$-valued $N \times N$-matrices?

[7] Where the sign is the one coming from $DJ = \pm JD$.

The second and last ingredient that we'll need here is a natural functional that can serve as the equivalent of the action we know from high energy physics. For that we want something which only depends on the data that is present in the spectral triple. The most (?) simple choice that meets these requirements would be to count how many of the eigenvalues of D_A are smaller than some mass scale Λ. Now for technical reasons it turns out that taking this isn't allowed and we'll have to settle with something similar:

$$\mathrm{Tr}\left[f(D_A^2/\Lambda^2)\right], \tag{2}$$

where again the mass-scale Λ appears, as does some (a priori arbitrary) function f. The thing is called the *spectral action* postulate. We're considering the spectrum of D_A here, hence the name. Despite its deceivingly simple form it is a highly complicated object and in the cases that are of interest to us, we can only handle it by means of an *asymptotic expansion* in Λ.[8] In the case that M is a four-dimensional manifold[9] and D meets the right conditions[10] this expansion reads:

$$\mathrm{Tr}\left[f(D_A^2/\Lambda^2)\right] \sim \Lambda^4 f_4 a_0(D_A^2) + \Lambda^2 f_2 a_2(D_A^2) + f(0) a_4(D_A^2) + \mathcal{O}(\Lambda^{-2}),$$

where $f_{2,4}$ are the second and fourth *moments* of f:

$$f_p \equiv \int_0^\infty f(x) x^{p-1} \mathrm{d}x$$

and the (*Seeley-DeWitt*) coefficients $a_{0,2,4}(D_A^2)$ only depend on the square of the Dirac operator. Now a second way to get an action (one that features spinors) is to use the inner product $\langle .,. \rangle$ on the Hilbert space $\mathcal{H}$:

$$\tfrac{1}{2}\langle J\psi, D_A\psi\rangle \qquad \psi \in \mathcal{H}. \tag{3}$$

Note the resemblance of this term with the expression

$$\overline{\psi}(x)\gamma^\mu(\partial_\mu + A_\mu)\psi(x)$$

that you might know from quantum field theory. Now the *spectral action principle* states that together (2) and (3) constitute the total action corresponding to a spectral triple. Note that —in contrast to 'normal' high energy physics— there's (for example) no question of adding some terms to the action in order to make something work: a particular term is simply in the action or it's not.

In the beginning I made a comment on NCG being a generalization of the theory of General Relativity. With this I mean that if you would compute the spectral action (2) in the example that I gave before, you exactly get the Einstein-Hilbert action which, in turn, gives you the field equations of General Relativity, including a cosmologial constant!

The spectral triple for the Standard Model

Now what does this mean for particle physics? We can use the fact that the tensor product of two spectral triples is again a spectral triple: given $(\mathcal{A}_{1,2}, \mathcal{H}_{1,2}, D_{1,2}; J_{1,2}, \gamma_{1,2})$ we can form

$$(\mathcal{A}_1 \otimes \mathcal{A}_2, \mathcal{H}_1 \otimes \mathcal{H}_2, D_1 \otimes 1 + \gamma_1 \otimes D_2, J_1 \otimes J_2, \gamma_1 \otimes \gamma_2),$$

where $J_1\gamma_1 = \gamma_1 J_1$ is assumed. It follows that the KO-dimension of the tensor product of two spectral triples is the sum of the KO-dimensions of the separate spectral triples. Now for that first

[8] So it might only be a good approximation for particular (in this case: large enough) values of Λ.

[9] Actually there is a wealth of theory about when such an expansion exists at all and if so, which terms it comprises of.

[10] I will suffice by saying that in the cases under consideration, it does.

spectral triple we can take the motivating example (1) and as for the second we can start with the algebra

$$\mathcal{A}_2 = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C}),$$

where with $\mathbb{H}$ we mean the quaternions. The diligent reader has already noted the resemblance between this algebra and the gauge group of the Standard Model. Note that it is this second algebra that makes the resulting spectral triple a noncommutative geometry[11]. A representation of that algebra (i.e. the Hilbert space) is exactly the particle content of the Standard Model. We can construct a grading γ_2 that distuinguishes between left- and right-handed particles. This Hilbert space describes only one generation of particles so we need to take three copies of it. So the first Hilbert space $L^2(M, S)$ takes into account that fermions are spinors and the second Hilbert space describes the internal structure (e.g. color) of the particles.

Then there is the Dirac operator D_2 for the second spectral triple. Employing all the demands on the Dirac operator, this actually leaves not that much freedom for it. The only thing we need to put in by hand is the fact that we need a massless photon. The result is a Dirac operator that maps between the left- and right-handed particles. Having fully specified the spectral triple, we can check that not only $SU(\mathcal{A}_2)$ equals the gauge group of the Standard Model $SU(3) \times SU(2) \times U(1)$ but also that the resulting hypercharges of the representations match those of the particles of the Standard Model. We're on the right track!

A second step is to calculate the inner fluctuations of the two Dirac operators that we have. For the first one $\partial\!\!\!/$ these turn out to describe exactly the gauge bosons of the Standard Model (the photon, W, Z and gluons) and the inner fluctuations of the second are seen to describe a scalar field that maps between left- and right-handed particles: this is the infamous Higgs field giving mass to the various particles. This still looks very promising.

We are now ready to determine the spectral action for this spectral triple and what we find is rather amazing: not only do we find the action of the full Standard Model but again the Einstein-Hilbert action of General Relativity pops out too. Now the absolute bonus is that all these different terms are accompanied by specific coefficients that are characteristic for NCG (e.g. the moments f_n). Now if you normalize these results such that the coefficients match those of the Standard Model, you automatically find the relation

$$g_3^2 = g_2^2 = \tfrac{5}{3} g_1^2$$

between the three coupling constants. This relation says that at the energy our theory lives on all the three forces (electromagnetic, weak and strong) are of the same strength. This means that the value of Λ must be the GUT-energy. Given this fact, the coefficients in front of the various terms of the Lagrangian are not independent and we can find relations between SM-parameters that aren't present in the 'normal' Standard Model itself. Rather miraculously this allows you to (more or less) determine the masses of the top-quark and Higgs boson. The values that are found are $\lesssim 180$ GeV and 168 GeV respectively.

This would be a perfect end to the story, if it wasn't for two things: first of all, though we pretended that the three forces are of equal strength at one specific energy-scale Λ, we know from experiment that they in fact aren't completely. Secondly, this specific value for the Higss mass is more or less (more more than less) ruled out by experiment. So apparently the story isn't complete yet ...

Are you the one to write the final chapter?

[11]Hence the name 'almost commutative geometry' is sometimes used.