NONCOMMUTATIVE GEOMETRY OF THE STANDARD MODEL

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Noncommutative geometry proposes an intriguing broadening of the concept of *geometry*. As a key application to physics, it allows for a geometric description of the Standard Model of particle physics, thus putting the Standard Model on the same footing as Einstein's general theory of Relativity.

This short paper gives an introduction to the use of noncommutative geometry in particle physics and describes how it unifies the four fundamental forces in nature.

1. Geometry in mathematics and physics

Geometry first started to play a serious role in physics through the work of Albert Einstein. His general theory of relativity is based on a field in mathematics developed in the 19th century, essentially by Bernhard Riemann. This *Riemannian geometry* describes in addition to flat spaces also non-Euclidean spaces, such as spherical and hyperbolic surfaces (see Figure 1).

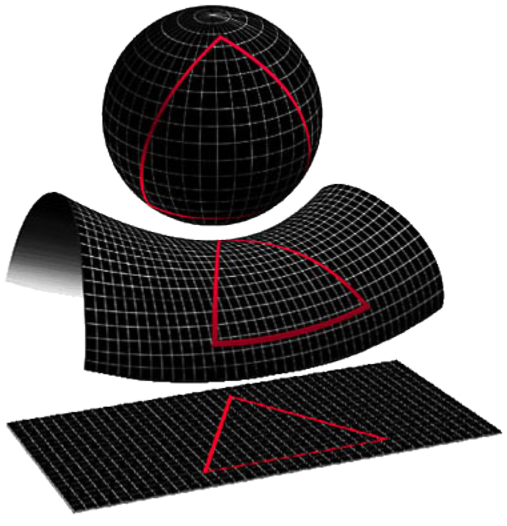


Figure 1. Riemannian geometry in two dimensions: spherical, hyperbolic and flat surfaces

In higher dimensions the idea of a Riemannian manifold is intuitively described by it being locally flat (=Euclidean) at first order. This mathematical framework is tailor-made for an accurate description of the physical principle of general covariance in relativity.

In this way Riemannan geometry describes gravity and the question that immediately comes to mind is whether there is a geometrical theory for the other three fundamental forces. In this article we will show that *noncommutative geometry* provides such a generalization of Riemannian geometry. Indeed, the full Standard Model of particle physics can be described geometrically, albeit by a noncommutative space. Before we try to explain what this means, we consider the physical input needed to describe a particle propagating in curved spacetime.

First of all, curved spacetime is described by a four-dimensional (pseudo) Riemannian manifold M. This means that there are local coordinates x_0, x_1, x_2 and x_3, comparable with the four vectors in special relativity. Then, if we want to describe the propagation in curved spacetime of a fermion of mass m, we should solve the Dirac equation. In compact form this equation can be written as $(\partial\!\!\!/_M - m)\psi = 0$, where $\partial\!\!\!/_M$ is the *Dirac operator*. It is the *general* relativistic analogue to the operator that Dirac found in his search for a *special* relativistic version of Schrödinger's equation. The wave function ψ thus describes the fermion propagating through curved spacetime.

A question which Einstein already played with was whether it is possible to geometrically describe also the other fundamental forces (at his time: the electromagnetic force). In this way one would be let to a unified theory of gravity and electromagnetism, and possibly even the nuclear forces. An elegant solution could have been Kaluza–Klein theory in which spacetime is replaced by *spacetime times a circle*, but this was rejected for physical reasons.

Our claim here —following Alain Connes and coauthors [4]— is that the full Standard Model of particle physics can be unified with general relativity by replacing spacetime by the product of spacetime with a *noncommutative space*. We will thus look at the noncommutative space

$$M \times F$$

where we consider F as an internal, noncommutative space. This is comparable to Kaluza–Klein theory in philosophy, but without its physical objections.

The noncommutative space F is described by giving coordinates, just as we did for M with x_μ.

The difference with spacetime is that the coordinates on F do not commute anymore. In fact, they are typically given by matrices. An example is given by the complex 3×3 matrices, for which indeed $ab \neq ba$ given two of such matrix 'coordinates' a and b.

The 'propagation' of a particle through the internal space F is described by a Dirac-type operator $\partial\!\!\!/_F$, which in this case is nothing but a symmetric matrix.

Example 1 (Electroweak theory)**.** *A 'coordinate' of the space F for the Glashow–Weinberg–Salam electroweak theory consists of*

- *a complex number z;*
- *a quaternion $q = q_0 + \sum_i q_i \sigma_i$, where σ_1, σ_2 and σ_3 are the Pauli matrices.*

Finally, the Dirac operator on F is given by the matrix

$$\partial\!\!\!/_F^+ = \begin{pmatrix} \varphi_1 & \varphi_2 \\ -\bar{\varphi}_2 & \bar{\varphi}_1 \end{pmatrix} .S$$

and the negative chirality part is $\partial\!\!\!/_F^- = (\partial\!\!\!/_F^+)^\dagger$. Notice the Higgs-type form of the matrix $\partial\!\!\!/_F$, which is no coincidence. We will get back to this point later.

But how then does this lead to physics? In general, given a noncommutative spacetime $M \times F$ and Dirac operator $\partial\!\!\!/_M$ and $\partial\!\!\!/_F$, let us search for a Lagrangian describing the dynamics and interactions of the physical model.

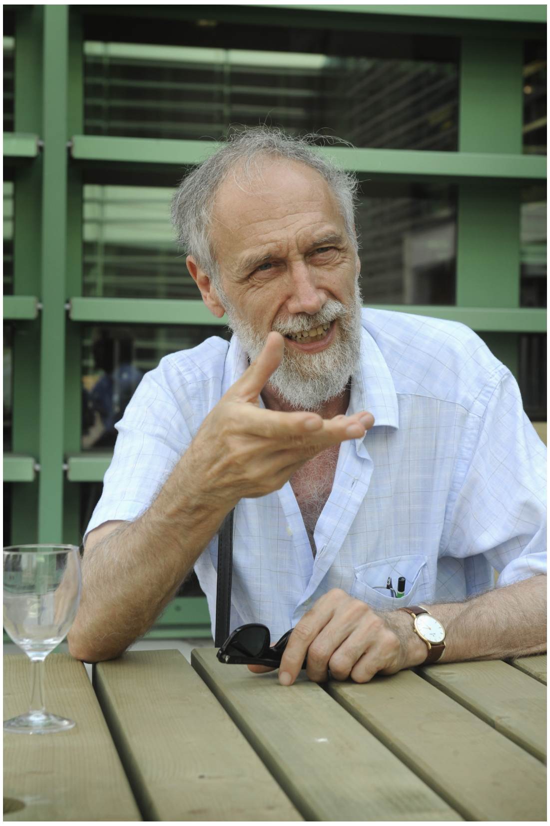
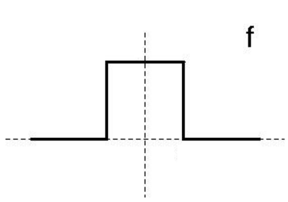


FIGURE 2. French mathematician Alain Connes, the founder of noncommutative geometry [7]

It turns out that there is an extremely simple prescription for such Lagrangians if we take a *spectral* point of view. Namely, if we count the number of eigenvalues of $\partial\!\!\!/_M + \partial\!\!\!/_F$ up to a certain cutoff scale Λ:

$$S_\Lambda = \mathrm{Trace}\left(f\left(\frac{\partial\!\!\!/_M + \partial\!\!\!/_F}{\Lambda} \right) \right) \tag{1}$$

The function f is a (smooth version of) the function that is 1 between -1 and 1 and vanishes elsewhere.



This so-called *spectral action* [1] gives us the Lagrangian of the theory. Only the spectrum of the relevant operator is needed to get to the physical action (or Lagrangian). The coupling constants of the theory are related to the form of the chosen cutoff function. Let us illustrate how this works for Einstein's general theory of relativity.

1.1. **Commutative NCG.** If there is no noncommutative space F then the above recipe applied to $\partial\!\!\!/_M$ gives the Einstein–Hilbert Lagrangian of General Relativity, whose equations of motion are

$$R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R = 0,$$

i.e. the Einstein equations in vacuum. Indeed, a long calculation [2] (see also [10]) based on so-called 'heat-kernel' expansions shows that in this case

$$S_\Lambda = \frac{\Lambda^2 f_2}{48\pi^2} \int_M \sqrt{g} R \, dx + \cdots$$

where f_2 is the area under the graph of f. If we now identify $\frac{\Lambda^2 f_2}{24\pi^2}$ with $\frac{1}{8\pi G}$ where G is Newton's constant, this is precisely the *Einstein–Hilbert Lagrangian* of General Relativity! As said, the equations of motion of this Lagrangian are precisely the Einstein equations. There are some additional terms in the spectral action, which are of interest in themselves; we refer to [9] for more information.

As a conclusion to this paragraph let us add a remark on the transition from metric to spectrum [8]. That this is not such a crazy idea can be seen from the historical transition of the definition of the meter in 1791 by a platinum bar, to the definition in terms of the frequency —a property of the spectrum— of a certain transition radiation in Caesium.

2. Noncommutative geometry of the Standard Model

The success of the above noncommutative approach gets even more appreciated for the Standard Model of particle physics. In fact, it turns out that there exists a noncommutative spacetime F that allows for a derivation of the *full Standard Model*, minimally coupled to gravity and including the Higgs mechanism. For all computation details we refer to [4] and the book [9] (see also the review [11] that also includes a noncommutative description of electrodynamics and the electroweak theory).

The internal space F is described by the following non-commuting 'coordinates'

- a complex number z;
- a quaternion $q = q_0 + \sum_i q_i \sigma_i$;
- a complex 3×3 matrix a.

Compare this to Example 1 for the electroweak model.

Next, $\not{\partial}_F$ is a 96×96 dimensional matrix. The number 96 is the number of fermionic degrees of freedom in the Standard Model: 3 families $\times$ (2 leptonen + 2 quarks, each in 3 colors), of which left and right-handed components and to the total we also add the anti-particles. The mathematical conditions that noncommutative geometry puts on the matrix $\not{\partial}_F$ assures that it, fortunately, contains mostly zeroes, and for the remaining part is parametrized by all bosons in the Standard Model, including Higgs boson.

Now, if we apply the spectral action principle to the operator $\not{\partial}_M + \not{\partial}_F$ a long calculation yields the full Lagrangian of the Standard Model, including the Higgs mechanism and minimally coupled to gravity.

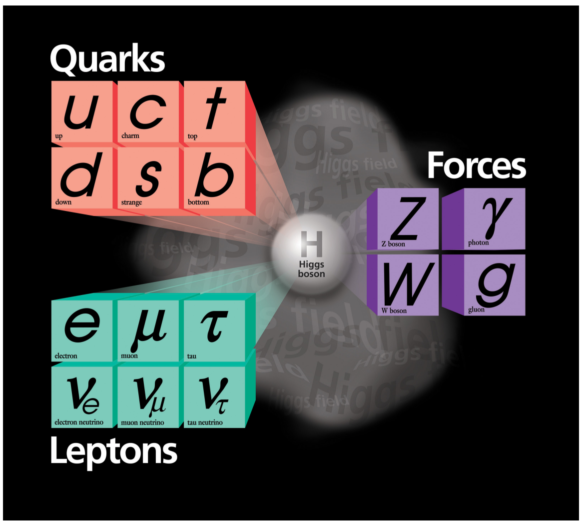


Figure 3. The elementary particles in the Standard Model

As said, the (single) cutoff function f enters in the different coupling constants of the theory, in this case the Standard Model. It follows that the couplings g_1, g_2 and g_3 (electromagnetic, weak and strong interaction) are related via the *GUT-relation*

$$g_3^2 = g_2^2 = \frac{5}{3} g_1^2.$$

We interpret this result as follows. Even though the particle content of the noncommutative model is equal to that of the Standard Model, and includes their mutual couplings, the spectral action implements the GUT-relation for the coupling constants. Just as in eg. $SU(5)$-grand unification, the noncommutative model describes a unified theory with particle content equal to that of the Standard Model. In comparison to the usual $SU(5)$-grand unification this has the advantage that the noncommutative model avoids the notorious leptoquarks.

But, when does such a relation hold? In Figure 4 we show the 'running' of the coupling constants $\alpha_i = g_i^2/4\pi$ in the Standard Model, depending on the energy scale μ. This dependence is dictated by the *renormalization group equations* of the Standard Model. We conclude that the noncommutative model sits at GUT-scale, near the so-called GUT-triangle.

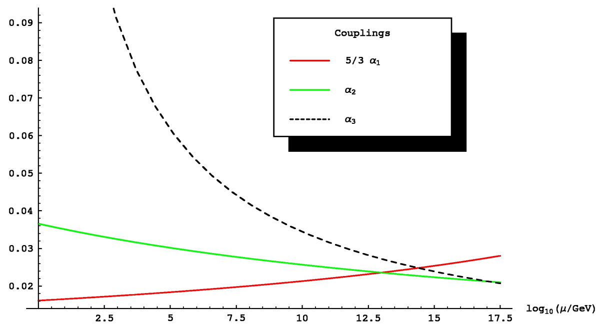


Figure 4. The 'running' of the three coupling constants of the Standard Model.

Another, important relation that our Lagrangian implies is between the Higgs self-coupling and g_3:

$$\lambda \sim \frac{4}{3} g_3^2$$

As our field theory sits at GUT-scale, we interpret this relation as being valid at the same GUT-scale. If we then let the Higgs self-coupling run according to the renormalization group equations of the Standard Model, with boundary condition given by the above relation, we obtain a value for λ at lower energy. Here, we assume the *big desert*: no new particles arise up to GUT-scale besides those present in the Standard Model. At lower energy (at the mass of the Z-boson) λ is related to the mass of the Higgs boson, for which we then determine the value:

$$m_H^2 = 8\lambda \frac{M_W^2}{g_2} \sim 170 \text{ GeV} \qquad (\text{ at } M_Z)$$

If we take into account neutrino mixing (as also described by the noncommutative model) we find a somewhat lower expected value: 168 GeV. A similar computation allows for a postdiction of (an upper limit of) the mass of the top quark, to wit $m_t <$ 180 GeV, compatibly with its measured value.

A few remarks on the predicted Higgs mass are in order, as it is incompatible with the measured value at CERN, which seems to falsify the noncommutative approach. However, it is the big desert hypothesis that is is unlikely to hold, and it is natural to look for noncommutative models that describes theories *beyond* the Standard Model. The converse analysis is also possible: find a noncommutative model that predicts a Higgs mass of eg. 125 GeV. This could then lead to a prediction of new particle content, described by the noncommutative model. This resulted in [3] in an extension of the Standard Model with a real scalar particle, allowing for a lower Higgs mass and at the same time guaranteeing the Higgs vacuum stability. The symmetry of this model is as in Pati–Salam unification, and has been described in detail in [6, 5].

References

[1] A. H. Chamseddine and A. Connes. Universal formula for noncommutative geometry actions: Unifications of gravity and the standard model. *Phys. Rev. Lett.* 77 (1996) 4868–4871.

[2] A. H. Chamseddine and A. Connes. The spectral action principle. *Commun. Math. Phys.* 186 (1997) 731–750.

[3] A. H. Chamseddine and A. Connes. Resilience of the Spectral Standard Model. *JHEP* 1209 (2012) 104.

[4] A. H. Chamseddine, A. Connes, and M. Marcolli. Gravity and the standard model with neutrino mixing. *Adv. Theor. Math. Phys.* 11 (2007) 991–1089.

[5] A. H. Chamseddine, A. Connes, and W. D. van Suijlekom. Beyond the Spectral Standard Model: Emergence of Pati-Salam Unification. (2013).

[6] A. H. Chamseddine, A. Connes, and W. D. Van Suijlekom. Inner fluctuations in noncommutative geometry without the first order condition. *J.Geom.Phys.* 73 (2013) 222–234.

[7] A. Connes. *Noncommutative Geometry.* Academic Press, San Diego, 1994.

[8] A. Connes. On the foundations of noncommutative geometry. In *The unity of mathematics*, volume 244 of *Progr. Math.*, pages 173–204. Birkhäuser Boston, Boston, MA, 2006.

[9] A. Connes and M. Marcolli. *Noncommutative Geometry, Quantum Fields and Motives.* AMS, Providence, 2008.

[10] G. Landi. *An Introduction to Noncommutative Spaces and their Geometry.* Springer-Verlag, 1997.

[11] K. van den Dungen and W. D. van Suijlekom. Particle Physics from Almost Commutative Spacetimes. *Rev.Math.Phys.* 24 (2012) 1230004.

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