The Spectral Model

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Space-Time

Our knowledge of spacetime is described by two existing theories :

– General Relativity

– The Standard Model of particle physics

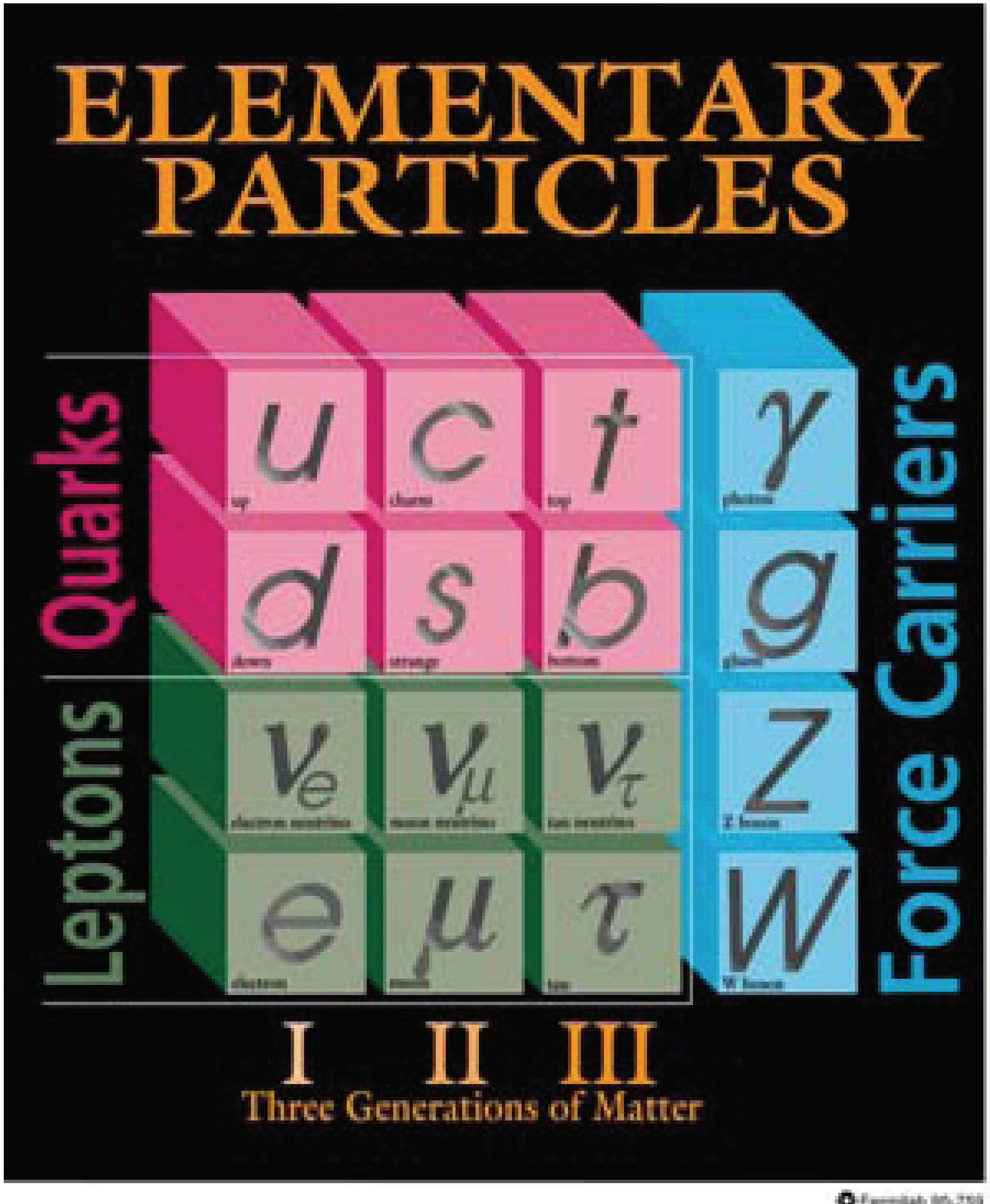
Curved Space, gravitational potential $g_{\mu\nu}$

$$ds^2 = g_{\mu\nu} dx^\mu \, dx^\nu$$

Action principle

$$S_E[\, g_{\mu\nu}] = \frac{1}{G} \int_M r \sqrt{g} \; d^4x$$

$$S = S_E + S_{SM}$$


ELEMENTARY PARTICLES
Quarks
Leptons
Force Carriers
u
c
t
d
s
b
γ
g
Z
W
νe
νμ
ντ
e
μ
τ
I
II
III
Three Generations of Matter

Standard Model

$$\mathcal{L}_{SM} = -\frac{1}{2}\partial_\nu g^a_\mu \partial_\nu g^a_\mu - g_s f^{abc}\partial_\mu g^a_\nu g^b_\mu g^c_\nu - \frac{1}{4}g_s^2 f^{abc} f^{ade} g^b_\mu g^c_\nu g^d_\mu g^e_\nu$$

$$-\partial_\nu W^+_\mu \partial_\nu W^-_\mu - M^2 W^+_\mu W^-_\mu - \frac{1}{2}\partial_\nu Z^0_\mu \partial_\nu Z^0_\mu - \frac{1}{2c_w^2} M^2 Z^0_\mu Z^0_\mu - \frac{1}{2}\partial_\mu A_\nu \partial_\mu A_\nu$$

$$-igc_w(\partial_\nu Z^0_\mu (W^+_\mu W^-_\nu - W^+_\nu W^-_\mu) - Z^0_\nu (W^+_\mu \partial_\nu W^-_\mu - W^-_\mu \partial_\nu W^+_\mu)$$

$$+Z^0_\mu (W^+_\nu \partial_\nu W^-_\mu - W^-_\nu \partial_\nu W^+_\mu)) - igs_w(\partial_\nu A_\mu (W^+_\mu W^-_\nu - W^+_\nu W^-_\mu)$$

$$-A_\nu (W^+_\mu \partial_\nu W^-_\mu - W^-_\mu \partial_\nu W^+_\mu) + A_\mu (W^+_\nu \partial_\nu W^-_\mu - W^-_\nu \partial_\nu W^+_\mu))$$

$$-\frac{1}{2}g^2 W^+_\mu W^-_\mu W^+_\nu W^-_\nu + \frac{1}{2}g^2 W^+_\mu W^-_\nu W^+_\mu W^-_\nu$$

$$+g^2 c_w^2 (Z^0_\mu W^+_\mu Z^0_\nu W^-_\nu - Z^0_\mu Z^0_\mu W^+_\nu W^-_\nu) + g^2 s_w^2 (A_\mu W^+_\mu A_\nu W^-_\nu - A_\mu A_\mu W^+_\nu W^-_\nu)$$

$$+g^2 s_w c_w (A_\mu Z^0_\nu (W^+_\mu W^-_\nu - W^+_\nu W^-_\mu) - 2A_\mu Z^0_\mu W^+_\nu W^-_\nu) - \frac{1}{2}\partial_\mu H \partial_\mu H - \frac{1}{2}m_h^2 H^2$$

$$-\partial_\mu \phi^+ \partial_\mu \phi^- - M^2 \phi^+ \phi^- - \frac{1}{2}\partial_\mu \phi^0 \partial_\mu \phi^0 - \frac{1}{2c_w^2} M^2 \phi^0 \phi^0$$

$$-\beta_h \left(\frac{2M^2}{g^2} + \frac{2M}{g} H + \frac{1}{2}(H^2 + \phi^0 \phi^0 + 2\phi^+ \phi^-) \right) + \frac{2M^4}{g^2}\alpha_h$$

$$-g\alpha_h M \left(H^3 + H\phi^0 \phi^0 + 2H\phi^+ \phi^- \right)$$

$$-\frac{1}{8}g^2\alpha_h\left(H^4+(\phi^0)^4+4(\phi^+\phi^-)^2+4(\phi^0)^2\phi^+\phi^-+4H^2\phi^+\phi^-+2(\phi^0)^2H^2\right)$$

$$-gMW_\mu^+W_\mu^-H-\frac{1}{2}g\frac{M}{c_w^2}Z_\mu^0Z_\mu^0H$$

$$-\frac{1}{2}ig\left(W_\mu^+(\phi^0\partial_\mu\phi^- -\phi^-\partial_\mu\phi^0)-W_\mu^-(\phi^0\partial_\mu\phi^+ -\phi^+\partial_\mu\phi^0)\right)$$

$$+\frac{1}{2}g\left(W_\mu^+(H\partial_\mu\phi^- -\phi^-\partial_\mu H)+W_\mu^-(H\partial_\mu\phi^+ -\phi^+\partial_\mu H)\right)$$

$$+\frac{1}{2}g\frac{1}{c_w}(Z_\mu^0(H\partial_\mu\phi^0-\phi^0\partial_\mu H)-ig\frac{s_w^2}{c_w}MZ_\mu^0(W_\mu^+\phi^- -W_\mu^-\phi^+)$$

$$+igs_wMA_\mu(W_\mu^+\phi^- -W_\mu^-\phi^+)-ig\frac{1-2c_w^2}{2c_w}Z_\mu^0(\phi^+\partial_\mu\phi^- -\phi^-\partial_\mu\phi^+)$$

$$+igs_wA_\mu(\phi^+\partial_\mu\phi^- -\phi^-\partial_\mu\phi^+)-\frac{1}{4}g^2W_\mu^+W_\mu^-\left(H^2+(\phi^0)^2+2\phi^+\phi^-\right)$$

$$-\frac{1}{8}g^2\frac{1}{c_w^2}Z_\mu^0Z_\mu^0\left(H^2+(\phi^0)^2+2(2s_w^2-1)^2\phi^+\phi^-\right)$$

$$-\frac{1}{2}g^2\frac{s_w^2}{c_w}Z_\mu^0\phi^0(W_\mu^+\phi^- +W_\mu^-\phi^+)-\frac{1}{2}ig^2\frac{s_w^2}{c_w}Z_\mu^0H(W_\mu^+\phi^- -W_\mu^-\phi^+)$$

$$+\frac{1}{2}g^2s_wA_\mu\phi^0(W_\mu^+\phi^- +W_\mu^-\phi^+)+\frac{1}{2}ig^2s_wA_\mu H(W_\mu^+\phi^- -W_\mu^-\phi^+)$$

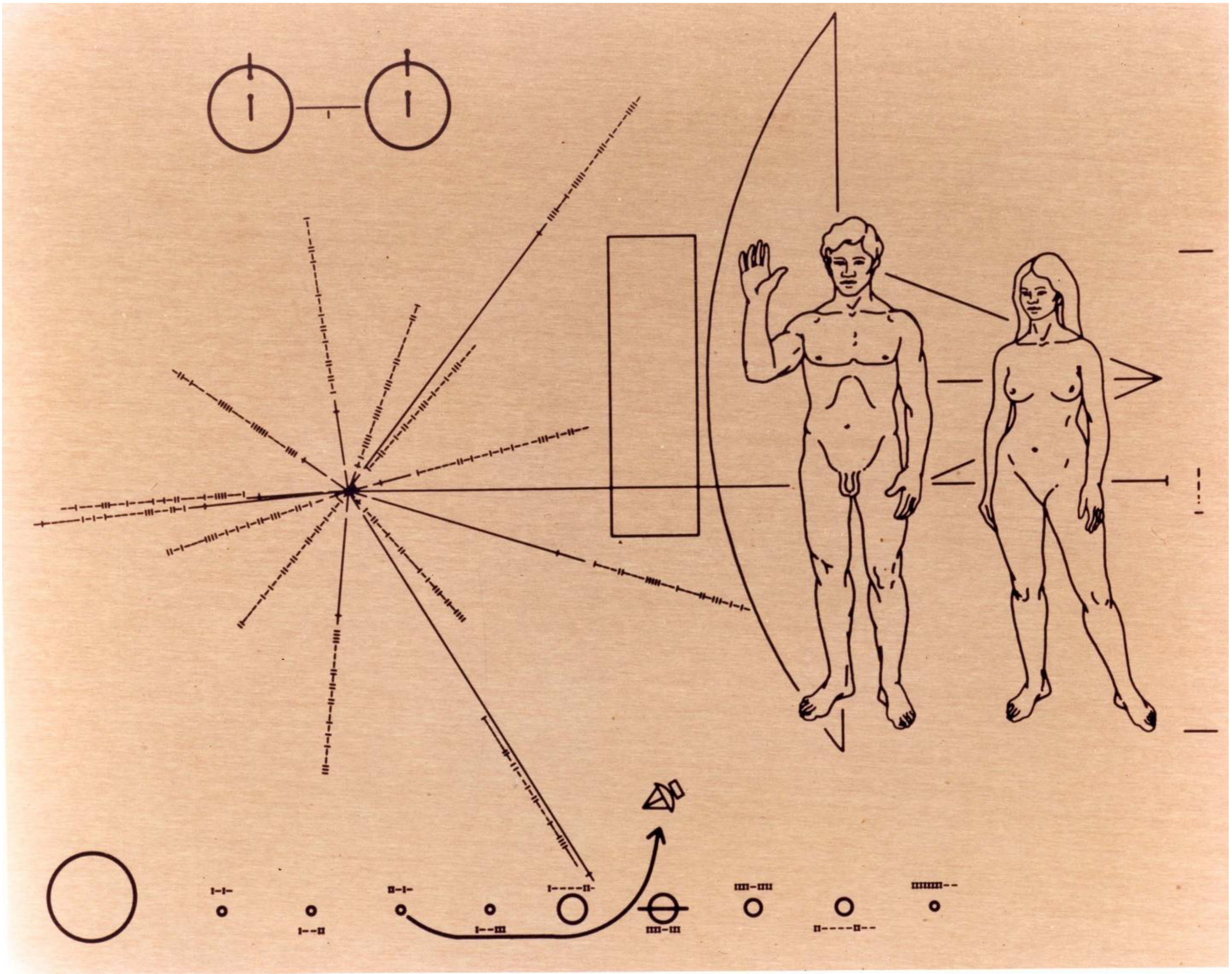
$$-g^2\frac{s_w}{c_w}(2c_w^2-1)Z_\mu^0A_\mu\phi^+\phi^- -g^2s_w^2A_\mu A_\mu\phi^+\phi^-$$

$$
\begin{gathered}
+\frac{1}{2} i g_s \lambda_{ij}^a (\bar{q}_i^\sigma \gamma^\mu q_j^\sigma) g_\mu^a - \bar{e}^\lambda (\gamma\partial + m_e^\lambda) e^\lambda - \bar{\nu}^\lambda \gamma\partial \nu^\lambda - \bar{u}_j^\lambda (\gamma\partial + m_u^\lambda) u_j^\lambda \\
-\bar{d}_j^\lambda (\gamma\partial + m_d^\lambda) d_j^\lambda + i g s_w A_\mu \left(-(\bar{e}^\lambda \gamma^\mu e^\lambda) + \frac{2}{3} (\bar{u}_j^\lambda \gamma^\mu u_j^\lambda) - \frac{1}{3} (\bar{d}_j^\lambda \gamma^\mu d_j^\lambda) \right) \\
+\frac{ig}{4c_w} Z_\mu^0 \{ (\bar{\nu}^\lambda \gamma^\mu (1+\gamma^5) \nu^\lambda) + (\bar{e}^\lambda \gamma^\mu (4 s_w^2 - 1 - \gamma^5) e^\lambda) \\
+(\bar{d}_j^\lambda \gamma^\mu (\frac{4}{3} s_w^2 - 1 - \gamma^5) d_j^\lambda) + (\bar{u}_j^\lambda \gamma^\mu (1 - \frac{8}{3} s_w^2 + \gamma^5) u_j^\lambda) \} \\
+\frac{ig}{2\sqrt{2}} W_\mu^+ \left((\bar{\nu}^\lambda \gamma^\mu (1+\gamma^5) e^\lambda) + (\bar{u}_j^\lambda \gamma^\mu (1+\gamma^5) C_{\lambda\kappa} d_j^\kappa) \right) \\
+\frac{ig}{2\sqrt{2}} W_\mu^- \left((\bar{e}^\lambda \gamma^\mu (1+\gamma^5) \nu^\lambda) + (\bar{d}_j^\kappa C_{\kappa\lambda}^\dagger \gamma^\mu (1+\gamma^5) u_j^\lambda) \right) \\
+\frac{ig}{2\sqrt{2}} \frac{m_e^\lambda}{M} \left(-\phi^+ (\bar{\nu}^\lambda (1-\gamma^5) e^\lambda) + \phi^- (\bar{e}^\lambda (1+\gamma^5) \nu^\lambda) \right) \\
-\frac{g}{2} \frac{m_e^\lambda}{M} \left(H (\bar{e}^\lambda e^\lambda) + i \phi^0 (\bar{e}^\lambda \gamma^5 e^\lambda) \right) \\
+\frac{ig}{2M\sqrt{2}} \phi^+ \left(-m_d^\kappa (\bar{u}_j^\lambda C_{\lambda\kappa} (1-\gamma^5) d_j^\kappa) + m_u^\lambda (\bar{u}_j^\lambda C_{\lambda\kappa} (1+\gamma^5) d_j^\kappa \right) \\
+\frac{ig}{2M\sqrt{2}} \phi^- \left(m_d^\lambda (\bar{d}_j^\lambda C_{\lambda\kappa}^\dagger (1+\gamma^5) u_j^\kappa) - m_u^\kappa (\bar{d}_j^\lambda C_{\lambda\kappa}^\dagger (1-\gamma^5) u_j^\kappa \right) \\
-\frac{g}{2} \frac{m_u^\lambda}{M} H (\bar{u}_j^\lambda u_j^\lambda) - \frac{g}{2} \frac{m_d^\lambda}{M} H (\bar{d}_j^\lambda d_j^\lambda) + \frac{ig}{2} \frac{m_u^\lambda}{M} \phi^0 (\bar{u}_j^\lambda \gamma^5 u_j^\lambda) - \frac{ig}{2} \frac{m_d^\lambda}{M} \phi^0 (\bar{d}_j^\lambda \gamma^5 d_j^\lambda)
\end{gathered}
$$

Our goal is to express the very elaborate Lagrangian given by gravity coupled with the Standard Model, with all its subtleties (V-A, BEH, seesaw, etc etc...) as pure gravity on a geometric space-time whose texture is slightly more elaborate than the 4-dimensional continuum.

This requires rethinking completely what Geometry is, and the simplest manner is to start with the simplest question :

"Where are we ?"



Two questions arise :

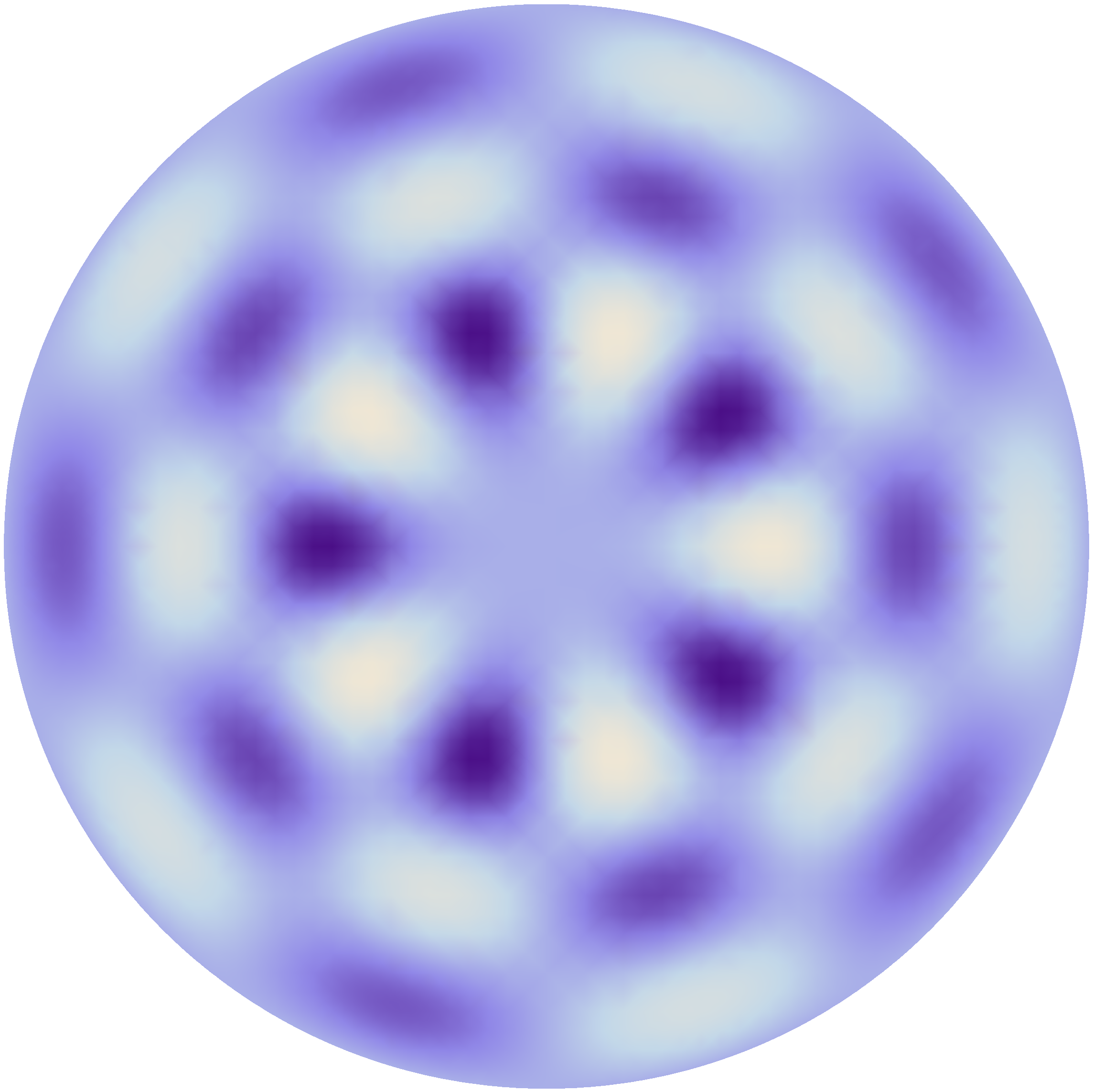
Find complete invariants of
geometric spaces, of "shapes"

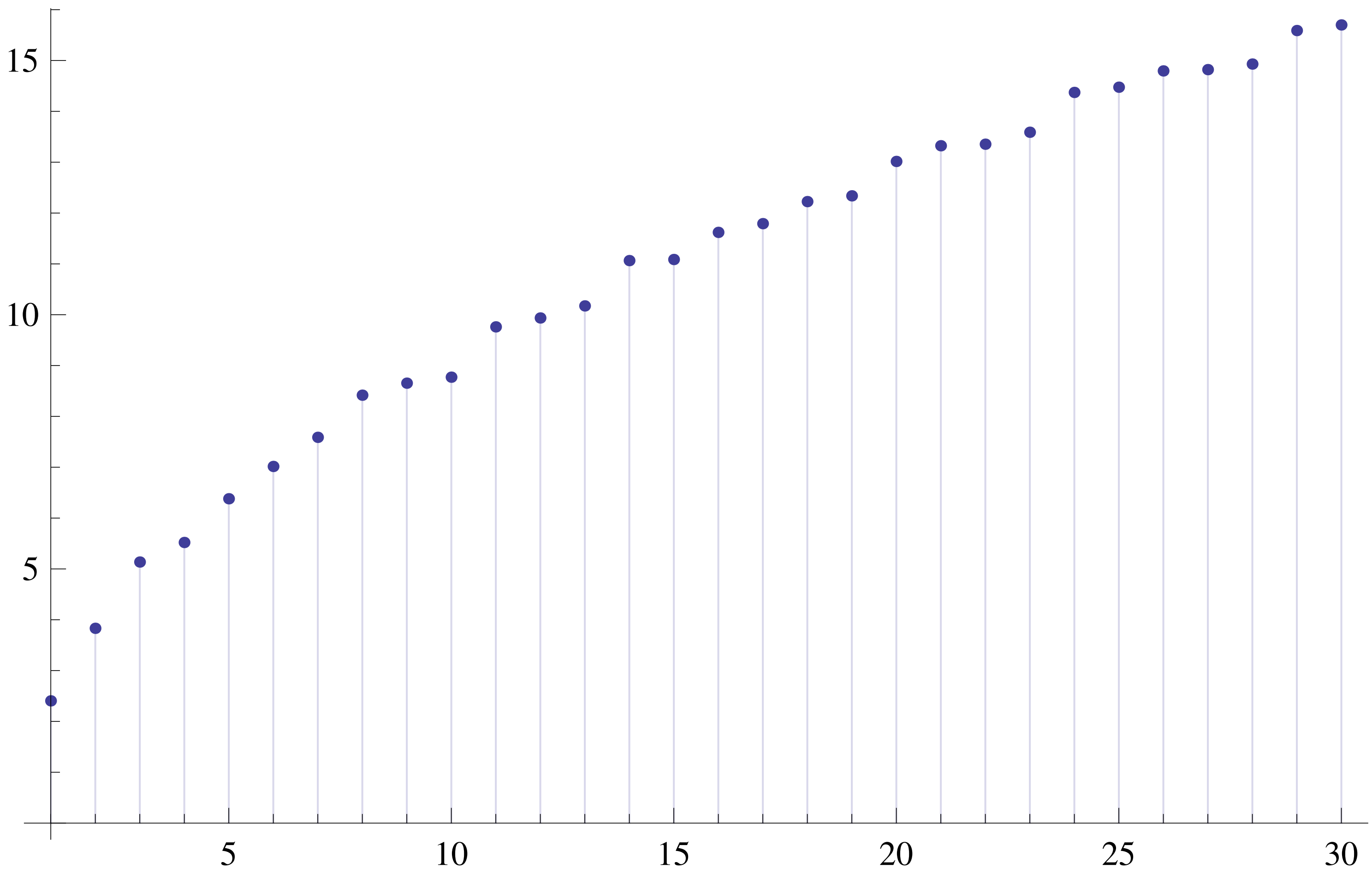
How can we invariantly specify
a point in a geometric space ?

The music of shapes

Milnor, John (1964), "Eigenvalues of the Laplace operator on certain manifolds", Proceedings of the National Academy of Sciences of the United States of America 51

Kac, Mark (1966), "Can one hear the shape of a drum ?", American Mathematical Monthly 73 (4, part 2) : 1–23





Eigenfrequencies of the disk

Spectrum of disk

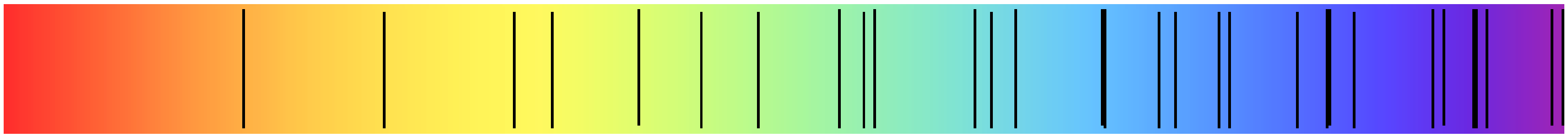
2.40483, 3.83171, 5.13562, 5.52008, 6.38016, 7.01559,

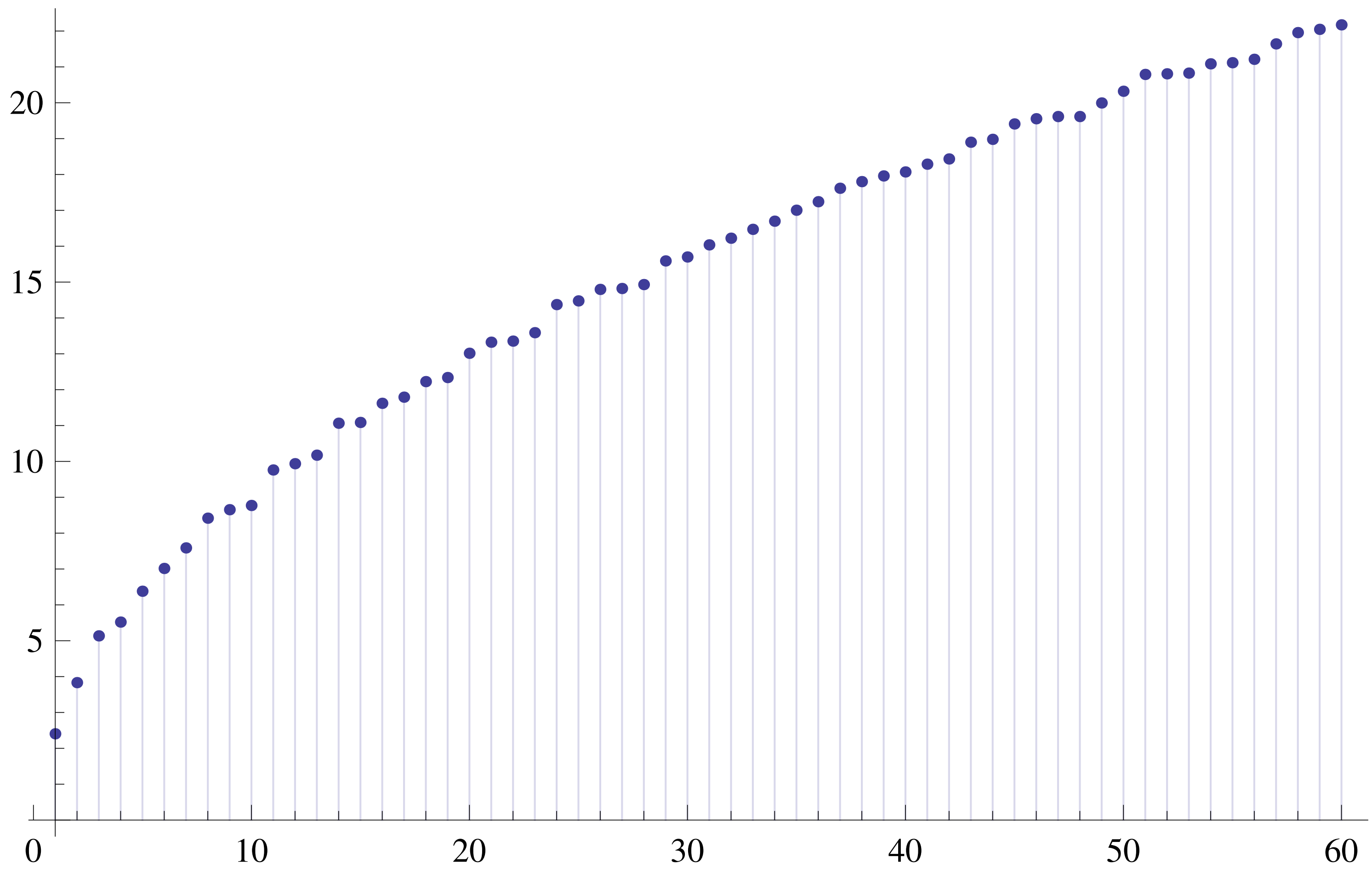
7.58834, 8.41724, 8.65373, 8.77148, 9.76102, 9.93611,

10.1735, 11.0647, 11.0864, 11.6198, 11.7915, 12.2251,

12.3386, 13.0152, 13.3237, 13.3543, 13.5893, 14.3725,

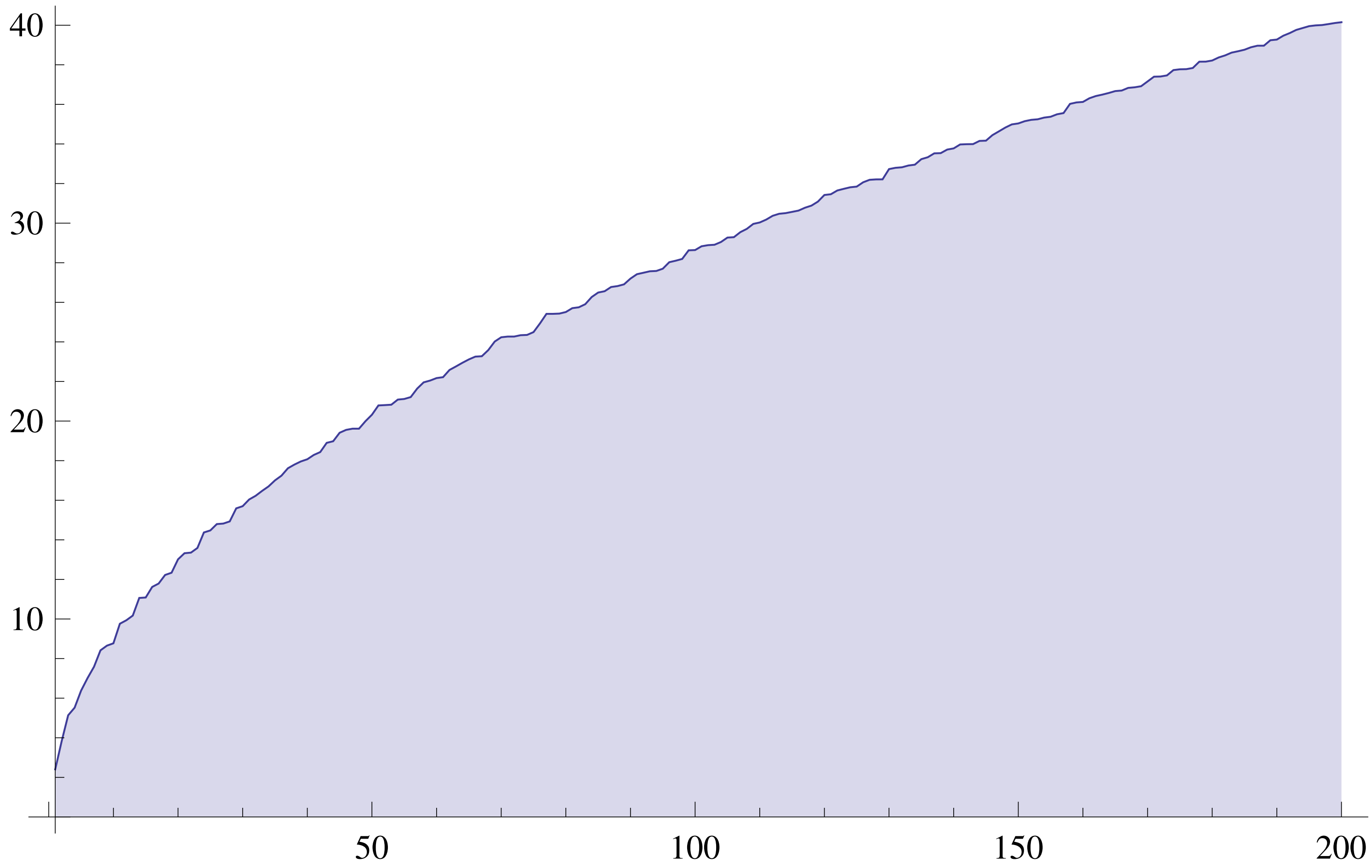
14.4755, 14.796, 14.8213, 14.9309, 15.5898, 15.7002 ...



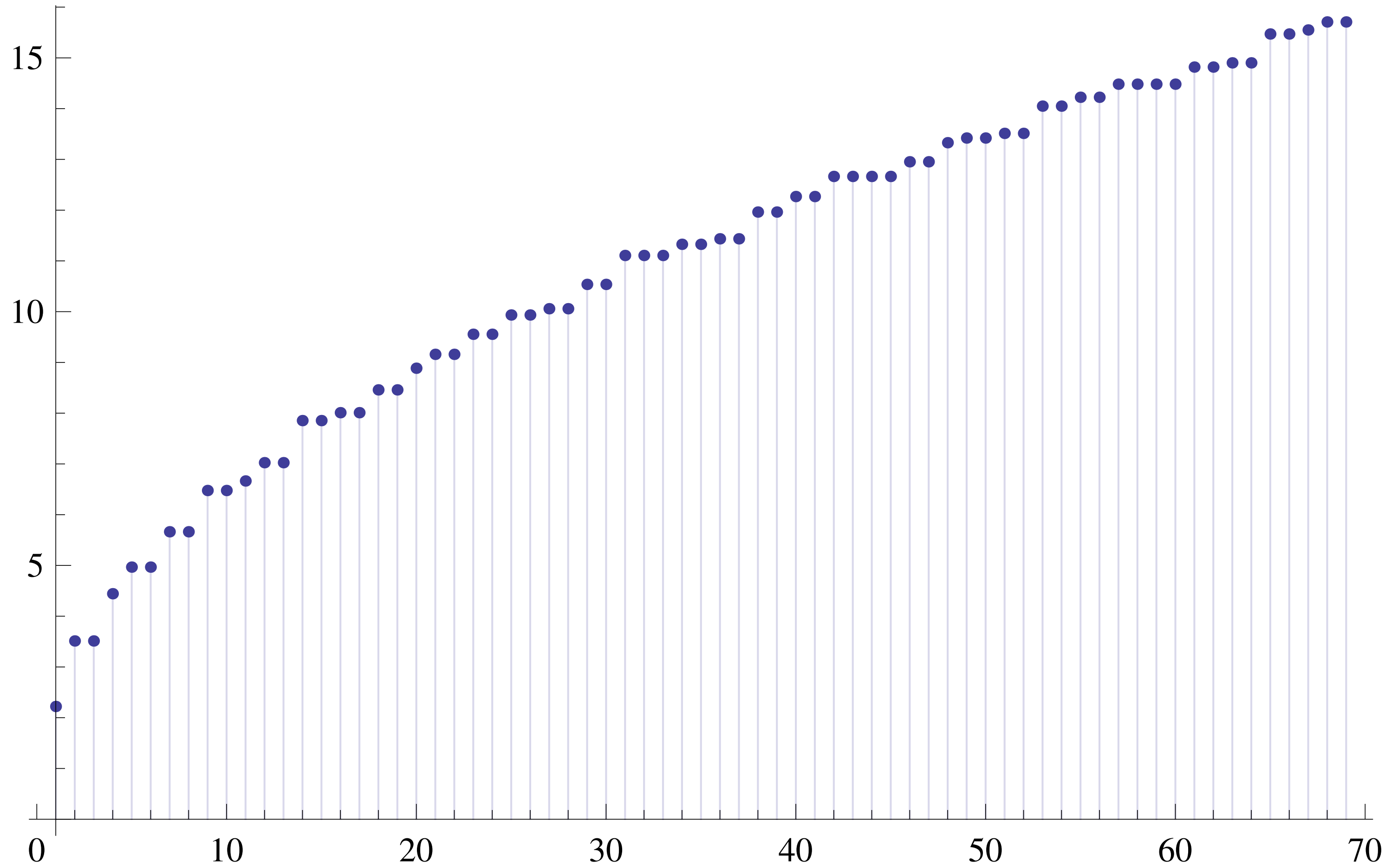


Eigenfrequencies of the disk

High frequencies

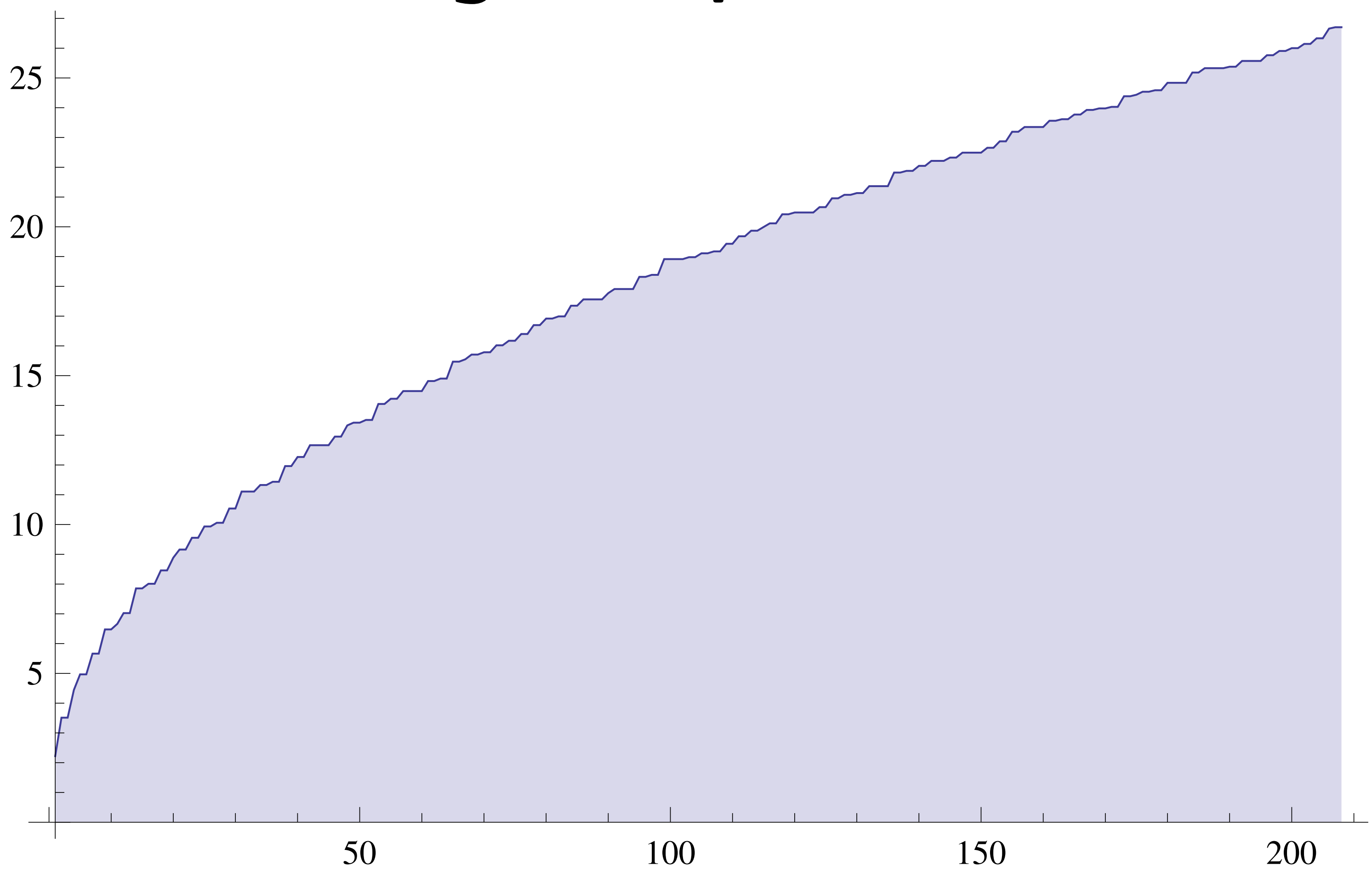


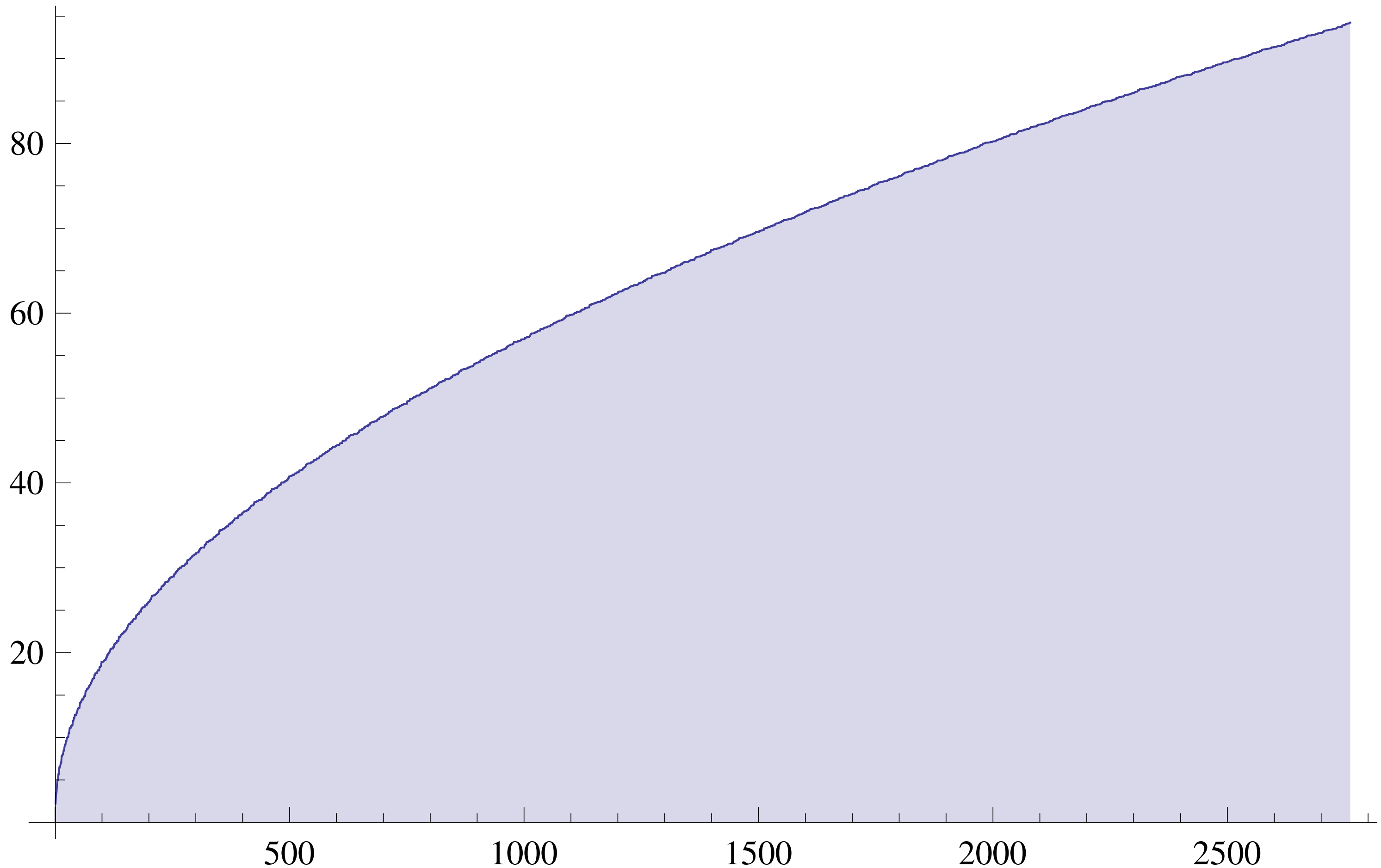
Vibrations of the square



Eigenfrequencies of square

High frequencies

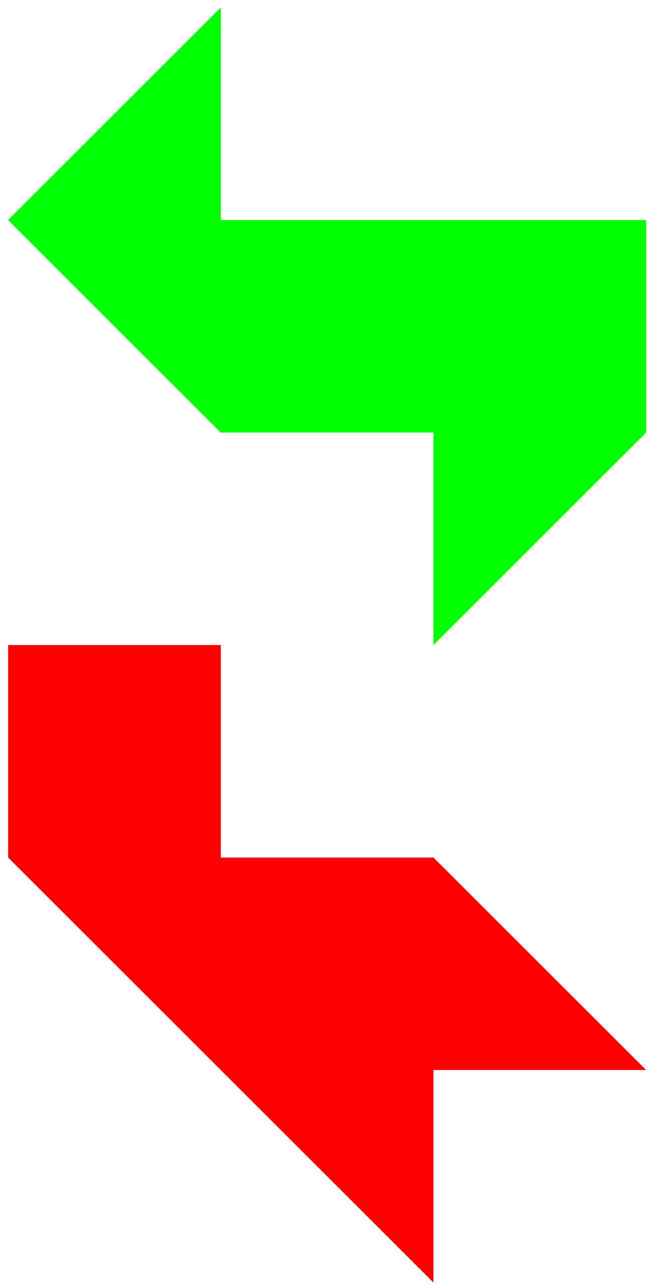


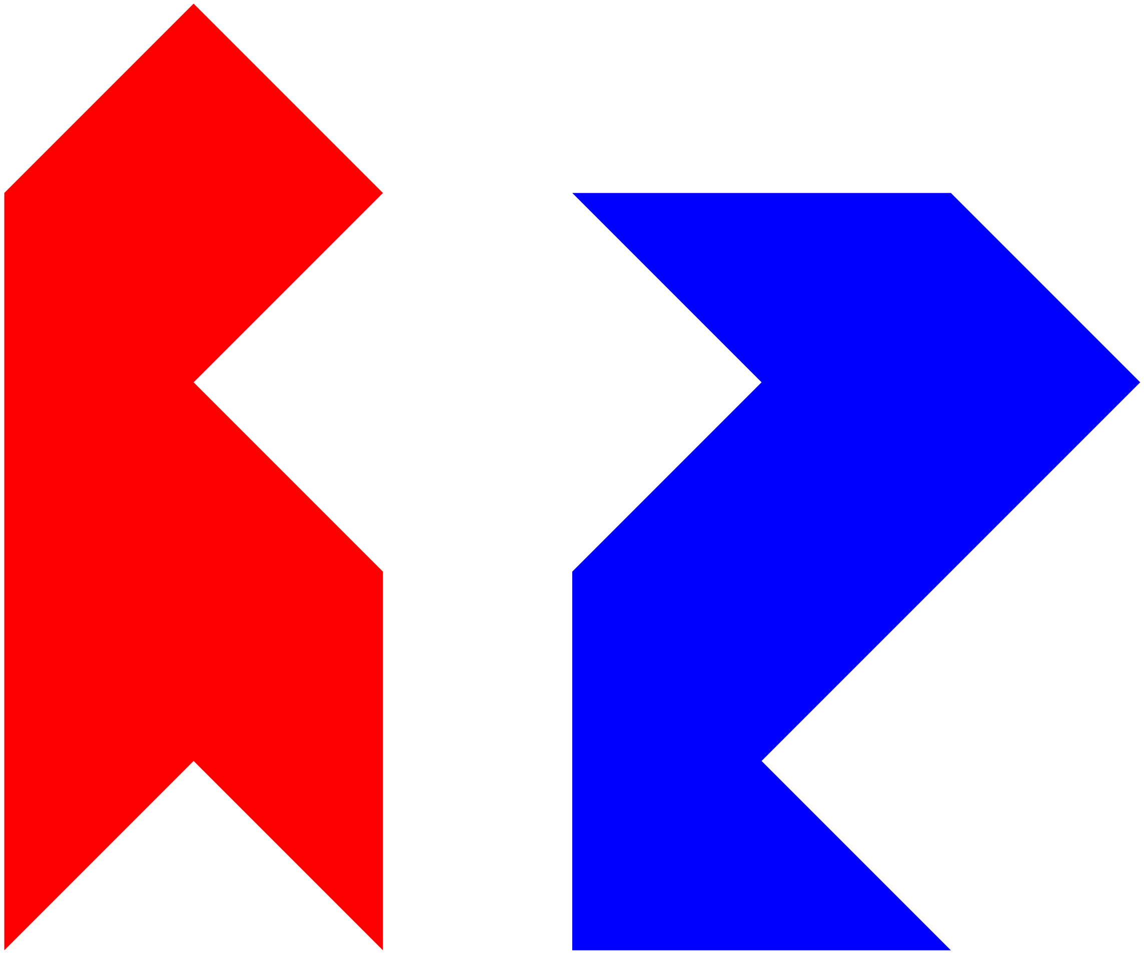


80
60
40
20
500
1000
1500
2000
2500

Gordon, Web, Wolpert

Gordon, C. ; Webb, D. ; Wolpert, S. (1992), "Isospectral plane domains and surfaces via Riemannian orbifolds", Inventiones mathematicae 110 (1) : 1–2

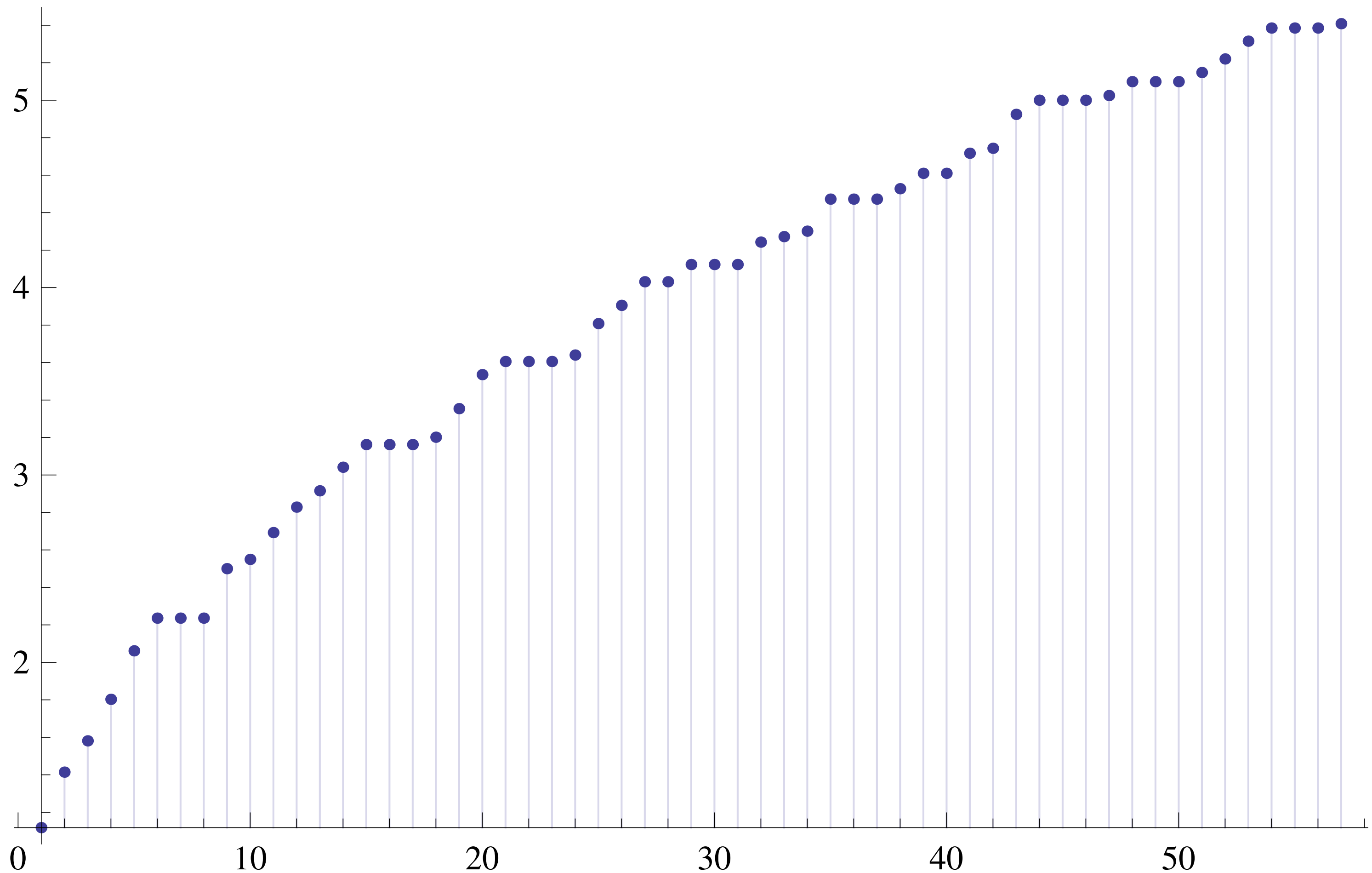




Two shapes with same spectrum (Chapman).

Shape I

Shape II


5
4
3
2
0
10
20
30
40
50

Spectrum $= \{\sqrt{x} \mid x \in S\}$,

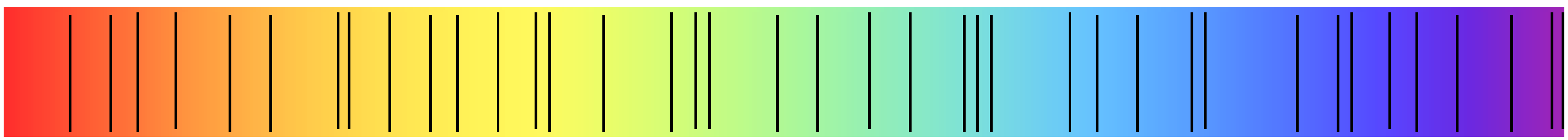
$$S = \{\frac{5}{4}, 2, \frac{5}{2}, \frac{13}{4}, \frac{17}{4}, 5, 5, 5, \frac{25}{4}, \frac{13}{2}, \frac{29}{4}, 8, \frac{17}{2}, \frac{37}{4}, 10, 10, 10, \frac{41}{4}, \frac{45}{4}, \frac{25}{2},$$
$$13, 13, 13, \frac{53}{4}, \frac{29}{2}, \frac{61}{4}, \frac{65}{4}, \frac{65}{4}, 17, 17, 17, 18, \frac{73}{4}, \frac{37}{2}, 20, 20, 20,$$
$$\frac{41}{2}, \frac{85}{4}, \frac{85}{4}, \frac{89}{4}, \frac{45}{2}, \frac{97}{4}, 25, 25, 25, \frac{101}{4}, 26, 26, 26, \frac{53}{2},$$
$$\frac{109}{4}, \frac{113}{4}, 29, 29, 29, \frac{117}{4}, \ldots$$

Same spectrum

$$\{a^2 + b^2 \mid a, b > 0\} \cup \{c^2/4 + d^2/4 \mid 0 < c < d\}$$

$$=$$

$$\{e^2/4 + f^2 \mid e, f > 0\} \cup \{g^2/2 + h^2/2 \mid 0 < g < h\}$$



Three classes of notes

One looks at the fractional part

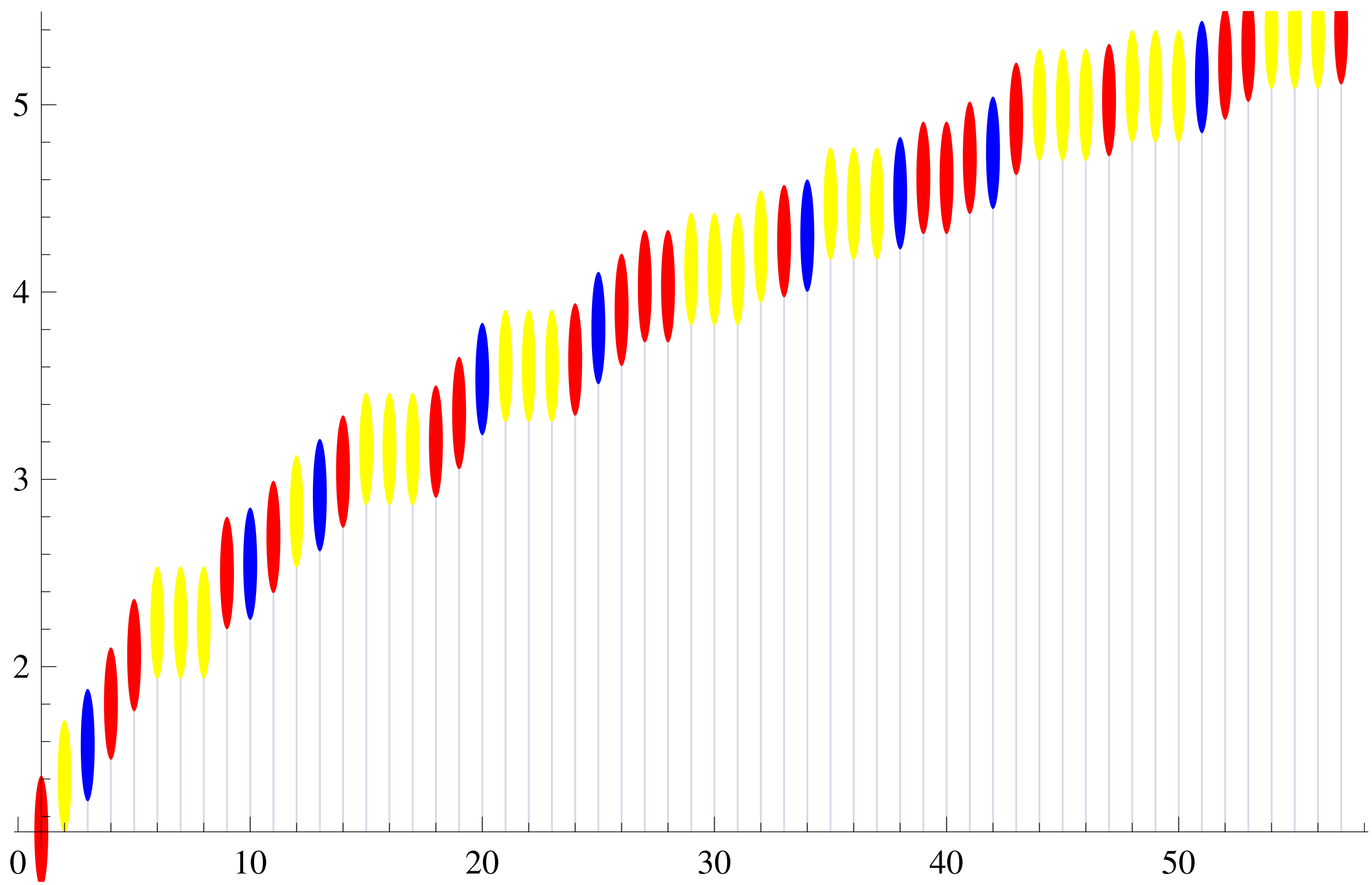
$\frac{1}{4}$: $\{e^2/4+f^2\}$ with $e,f>0 = \{c^2/4+d^2/4\}$ with $c+d$ odd.

$\frac{1}{2}$: The $c^2/4+d^2/4$ with c,d odd and $g^2/2+h^2/2$ with $g+h$ odd.

0 : $\{a^2+b^2 \mid a,b>0\} \cup \{4c^2/4+4d^2/4 \mid 0<c<d\}$ et $\{4e^2/4+f^2 \mid e,f>0\} \cup \{g^2/2+h^2/2 \mid 0<g<h\}$ with $g+h$ even.

Possible chords

The possible chords are not the same. Blue–Red is not possible for shape II the one which contains the rectangle.



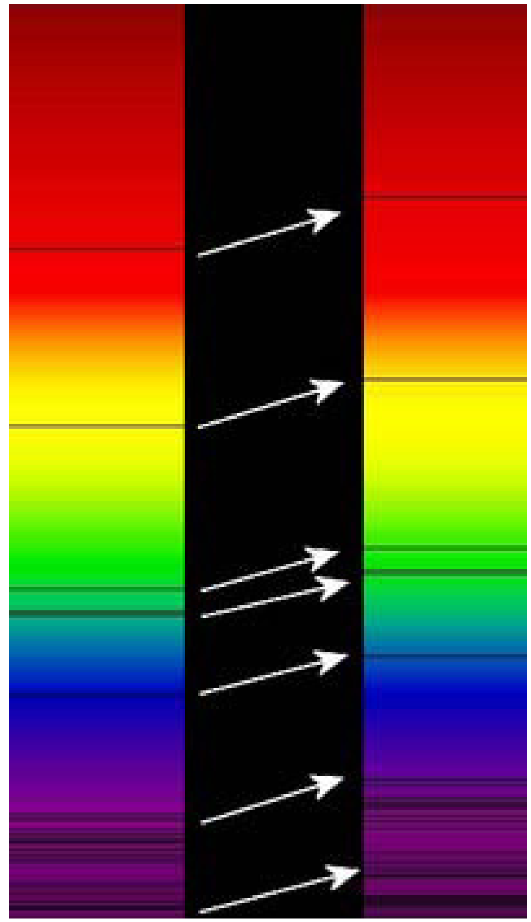
5
4
3
2
0
10
20
30
40
50

Points

The missing invariant should be interpreted as giving the probability for correlations between the possible frequencies, while a “point” of the geometric space X can be thought of as a correlation, *i.e.* a specific positive hermitian matrix $\rho_{\lambda\kappa}$ (up to scale) which encodes the scalar product at the point between the eigenfunctions of the Dirac operator associated to various frequencies *i.e.* eigenvalues of the Dirac operator.

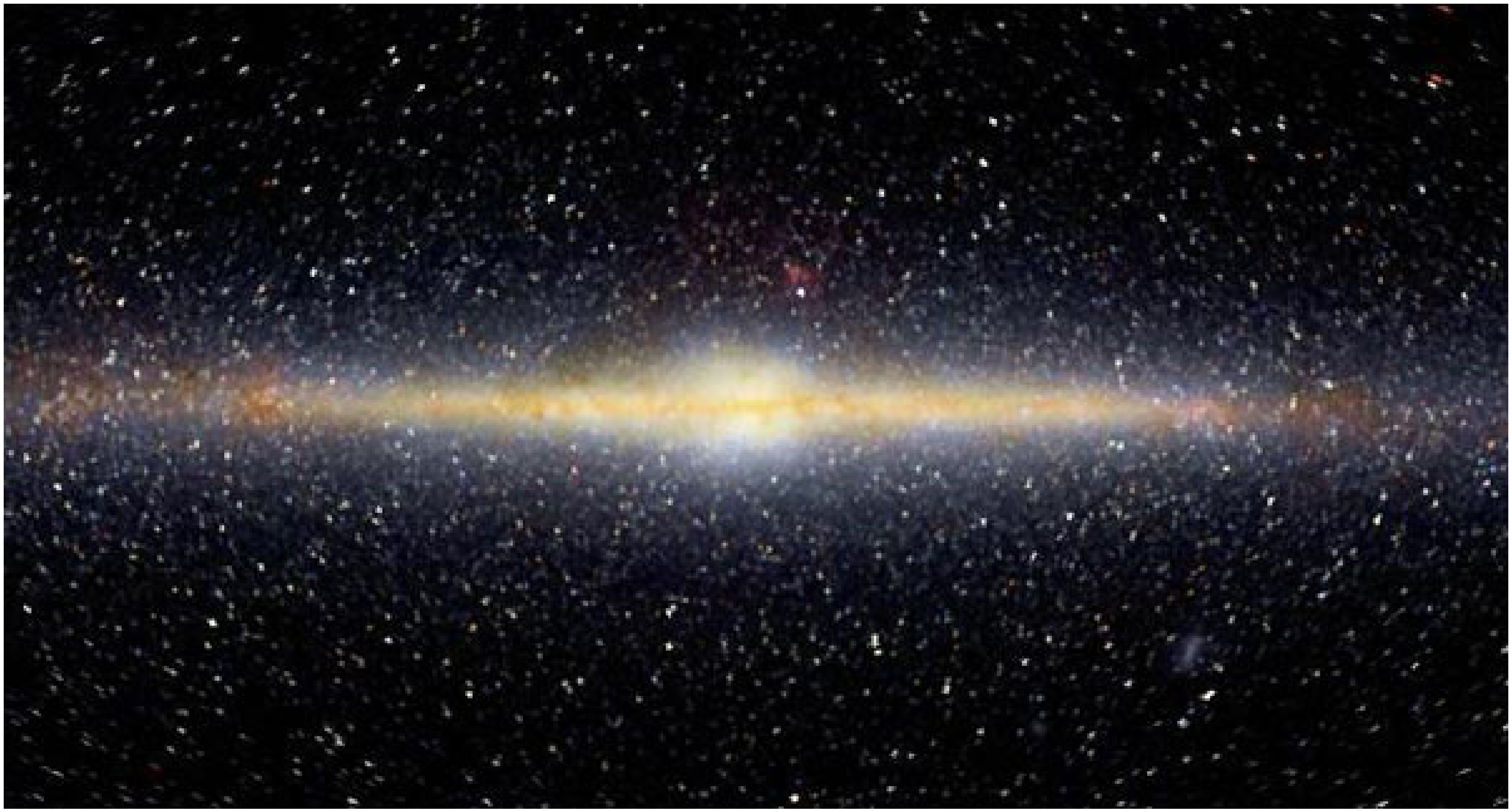
Redshift

Thanks to great telescopes like Hubble we now have an "eye" that allows us to look out in space, but what is striking is that all the information we get is of spectral nature. It is for instance thanks to the red-shift (which is a rescaling factor of spectra and can be as large as 10 and possibly a thousand) that we can estimate cosmic distances.



Radiations emitted in ultraviolet (10^{14} cycles per second) are observed in infrared (10^{12} cycles per second)

It is rather convincing also that our faith in outer space is based on the strong correlations that exist between different frequencies, as encoded by the matrix $g_{\lambda\mu}$, so that the picture in infrared of the milky way is not that different from its visible light counterpart, which can be seen with a bare eye on a clear night.



Musical shape ?

The ear is sensitive to *ratios* of frequencies.

The two sequences

$$\{440, 440, 440, 493, 552, 493, 440, 552, 493, 493, 440\}$$

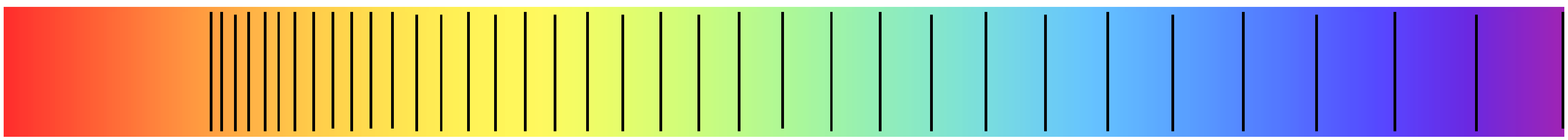
$$\{622, 622, 622, 697, 780, 697, 622, 780, 697, 697, 622\}$$

are in the ratio $\sim \sqrt{2}$.

$$\frac{\log 3}{\log 2} \sim 1 + \frac{1}{1 + \frac{1}{1+\frac{1}{2+\frac{1}{2}}}} = \frac{19}{12}$$

Towards a musical shape

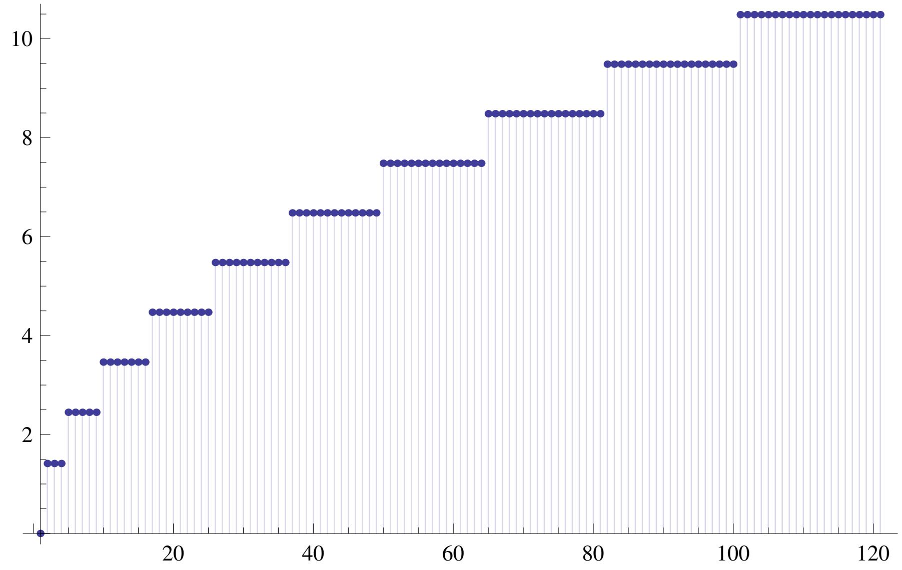
$$\{q^n \mid n \in \mathbb{N}\}\,, \quad q = 2^{\frac{1}{12}}$$

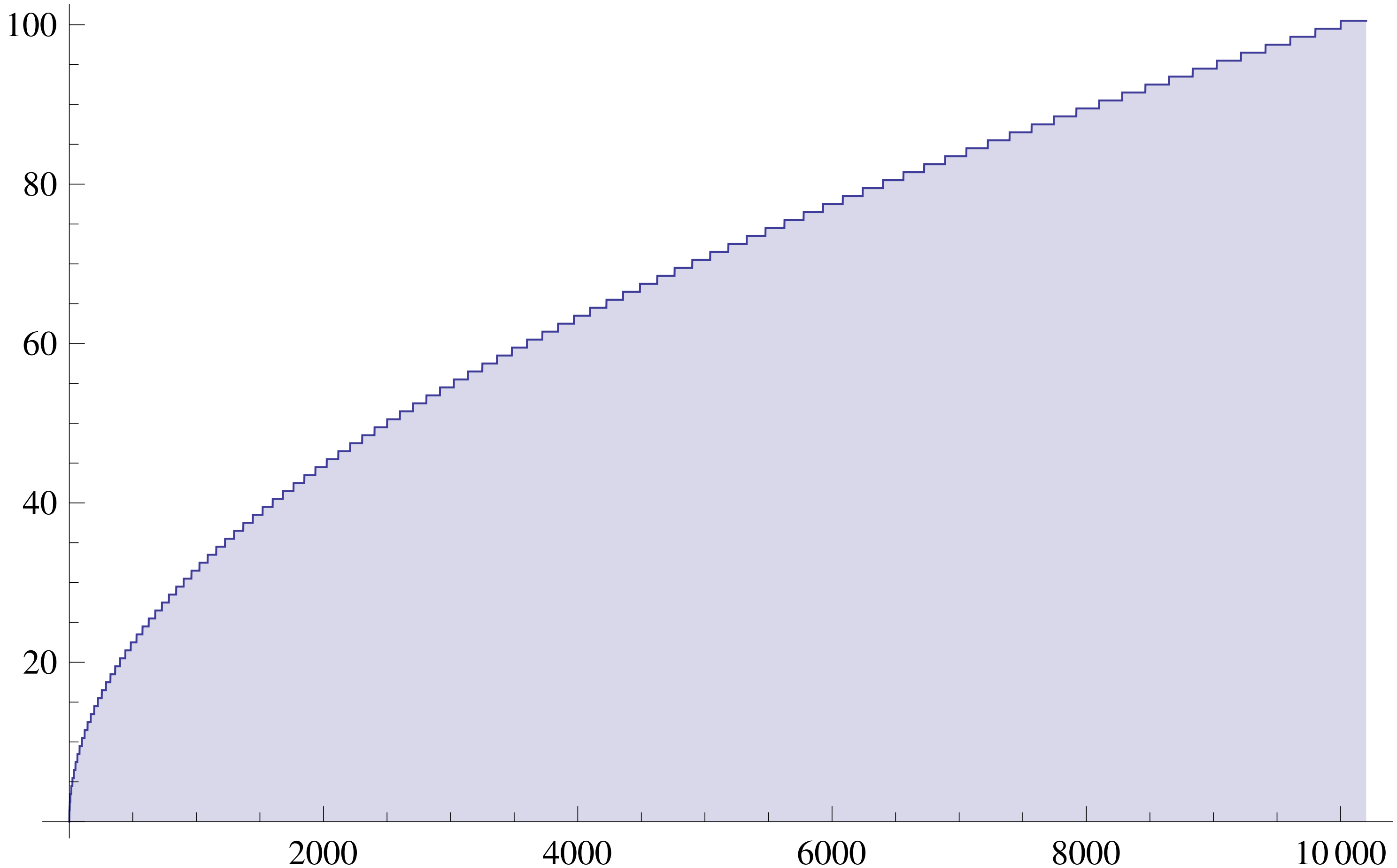


$$2^{1/12} = 1.05946...\,, \quad 3^{1/19} = 1.05953...$$

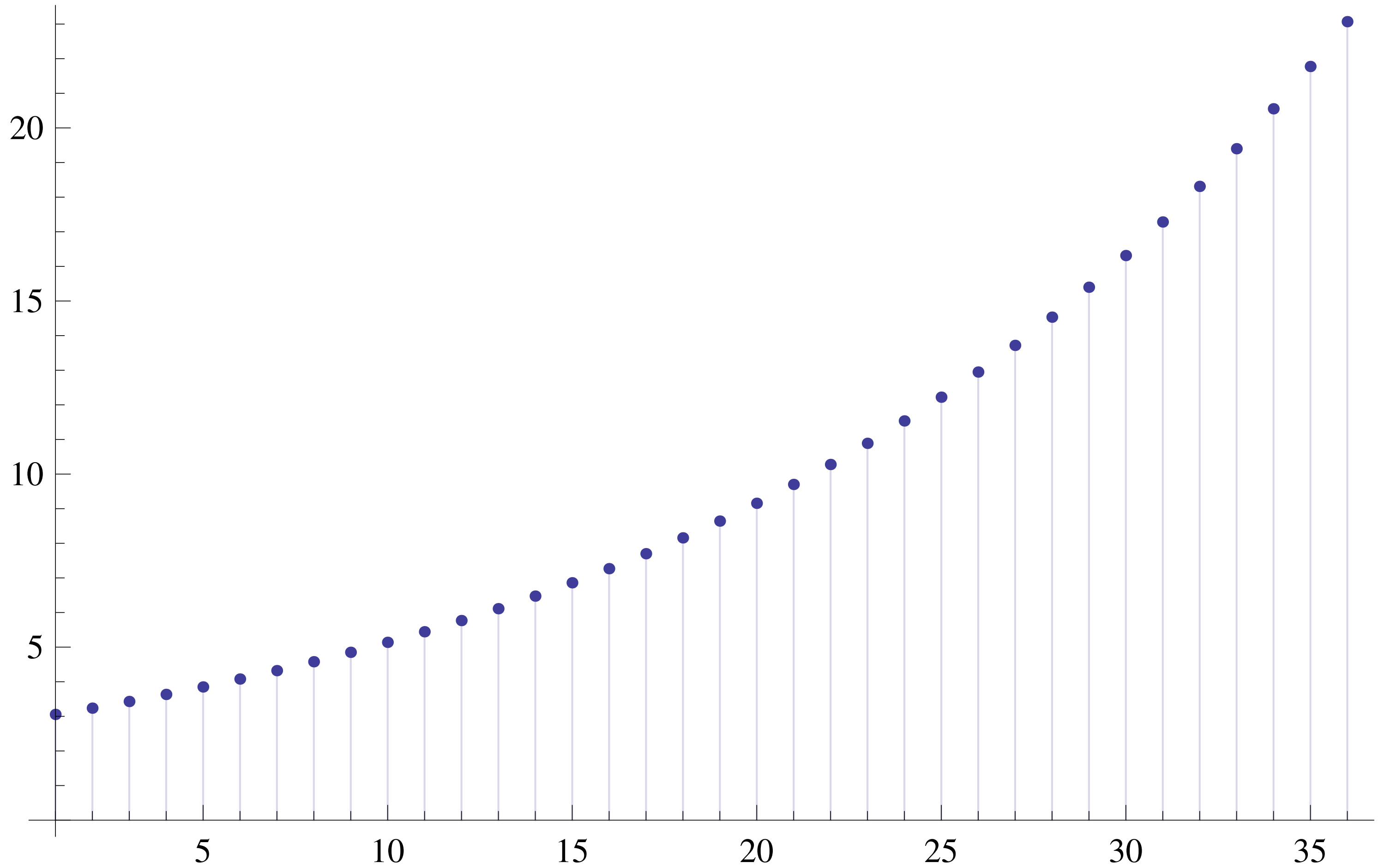
The sphere ?

Spectrum of sphere $= \{\sqrt{j(j+1)} \mid j \geq 0\}$ looks like this for small values





High frequencies of sphere

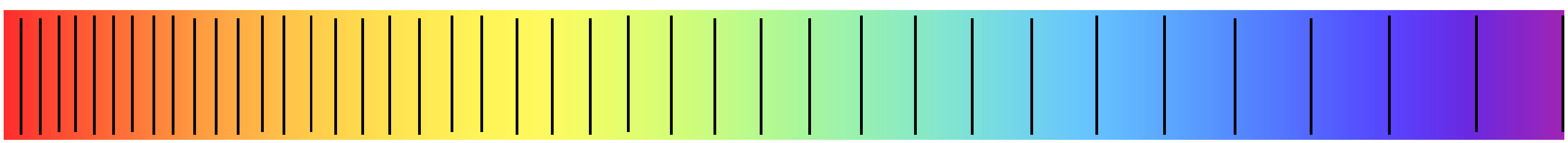


Musical Shape has Dimension $= 0$

The quantum sphere S_q^2

Poddles, Dabrowski, Sitarz, Landi, Wagner, Brain...

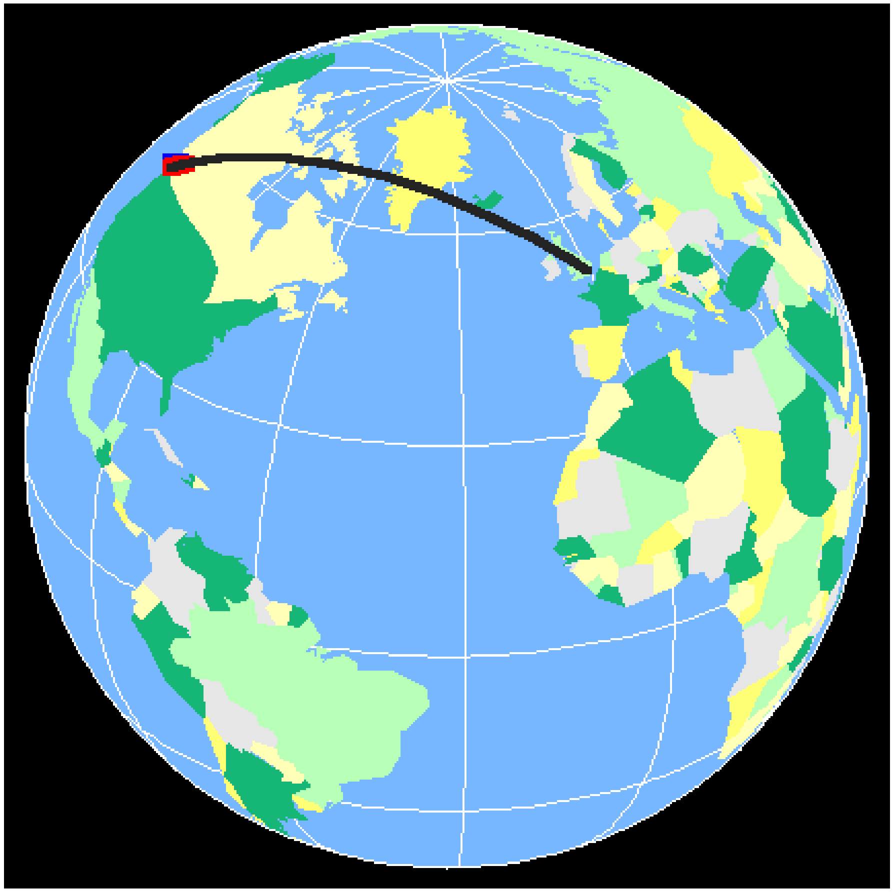
$$\{\frac{q^j - q^{-j}}{q - q^{-1}} \mid j \in \mathbb{N}\} \text{ with multiplicity } O(j)$$



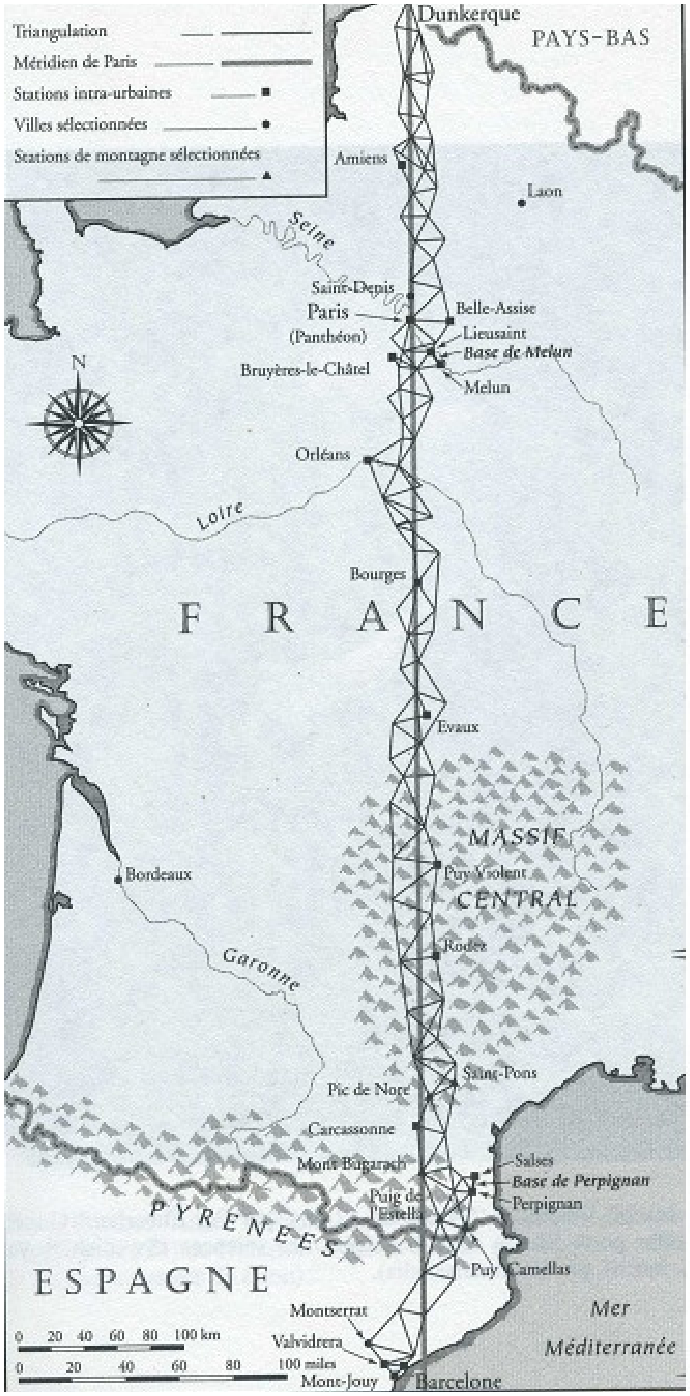
L. Dabrowski, A. Sitarz, *D*irac operator on the standard Podles' quantum sphere. Noncommutative geometry and quantum groups (Warsaw, 2001), 49–58, Banach Center Publ., 61, Polish Acad. Sci., Warsaw, 2003.

L. Dabrowski, F. D'Andrea, G. Landi, E.Wagner, *D*irac operators on all Podles quantum spheres J. Noncomm. Geom. 1 (2007) 213–239 arXiv :math/0606480

S. Brain, G. Landi, *T*he 3D Spin geometry of the quantum 2-sphere Rev. Math. Phys. 22 (2010) 963–993 arXiv :1003.2150



$$d(A,B) = \text{Inf} \int_\gamma \sqrt{g_{\mu\,\nu}\, dx^\mu\, dx^\nu}$$



J-B. J. DELAMBRE

P. F. A. MECHAIN

1792--1799

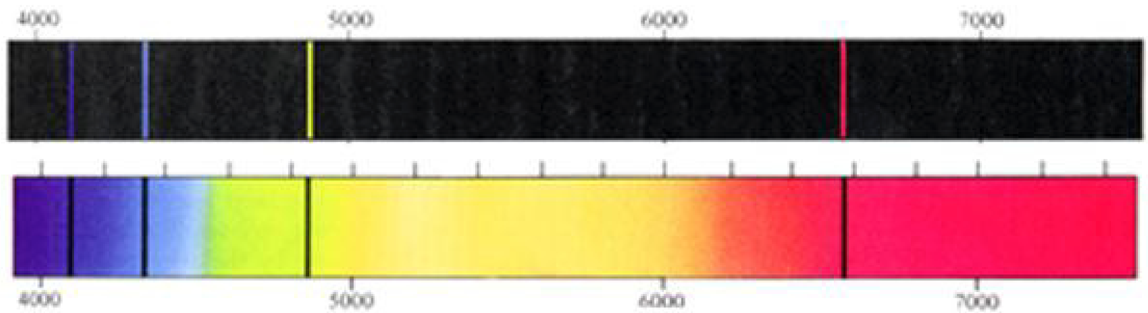
DUNKERQUE--BARCELONE

Dirac's square root of the Laplacian

Spectral triples

$$(\mathcal{A}, \mathcal{H}, D)\,, \quad ds = D^{-1}\,,$$

$$d(A,B) = \text{Sup}\,\{|f(A) - f(B)|\,;\, f \in \mathcal{A}\,,\ \|[D,f]\| \leq 1\,\}$$



Meter $\rightarrow$ Wave length (Krypton (1967) spectrum of 86Kr then Caesium (1984) hyperfine levels of C133)

Gauge transfos = Inn($\mathcal{A}$)

Let us consider the simplest example

$$\mathcal{A} = C^\infty(M, M_n(\mathbb{C})) = C^\infty(M) \otimes M_n(\mathbb{C})$$

Algebra of $n \times n$ matrices of smooth functions on manifold M.

The group Inn($\mathcal{A}$) of inner automorphisms is locally isomorphic to the group $\mathcal{G}$ of smooth maps from M to the small gauge group $SU(n)$

$$1 \to \mathrm{Inn}(\mathcal{A}) \to \mathrm{Aut}(\mathcal{A}) \to \mathrm{Out}(\mathcal{A}) \to 1$$

becomes identical to

$$1 \to \mathrm{Map}(M, G) \to \mathcal{G} \to \mathrm{Diff}(M) \to 1.$$

Einstein–Yang-Mills

We have shown that the study of pure gravity on this space yields Einstein gravity on M minimally coupled with Yang-Mills theory for the gauge group $SU(n)$. The Yang-Mills gauge potential appears as the inner part of the metric, in the same way as the group of gauge transformations (for the gauge group $SU(n)$) appears as the group of inner diffeomorphisms.

Reconstruction Theorem

The restriction to spin manifolds is obtained by requiring a *real structure i.e.* an antilinear unitary operator J acting in $\mathcal{H}$ which plays the same role and has the same algebraic properties as the charge conjugation operator in physics.

In the even case the chirality operator γ plays an important role, both γ and J are decorations of the spectral triple.

The following further relations hold for D, J and γ

$$J^2 = \varepsilon\,,\ DJ = \varepsilon' JD, \quad J\,\gamma = \varepsilon''\gamma J, \quad D\gamma = -\gamma D$$

The values of the three signs $\varepsilon, \varepsilon', \varepsilon''$ depend only, in the classical case of spin manifolds, upon the value of the dimension n modulo 8 and are given in the following table :

n	0	1	2	3	4	5	6	7
ε	1	1	-1	-1	-1	-1	1	1
ε'	1	-1	1	1	1	-1	1	1
ε''	1		-1		1		-1	

Metric dimension and KO-dimension

In the classical case of spin manifolds there is thus a relation between the metric (or spectral) dimension given by the rate of growth of the spectrum of D and the integer modulo 8 which appears in the above table. For more general spaces however the two notions of dimension (the dimension modulo 8 is called the KO-dimension because of its origin in K-theory) become independent since there are spaces F of metric dimension 0 but of arbitrary KO-dimension.

Fine Structure

Starting with an ordinary spin geometry M of dimension n and taking the product $M \times F$, one obtains a space whose metric dimension is still n but whose KO-dimension is the sum of n with the KO-dimension of F.

As it turns out the Standard Model with neutrino mixing favors the shift of dimension from the 4 of our familiar space-time picture to $10 = 4 + 6 = 2$ modulo 8.

Finite spaces

In order to learn how to perform the above shift of dimension using a 0-dimensional space F, it is important to classify such spaces. This was done in joint work with A. Chamseddine. We classified there the *finite* spaces F of given KO-dimension. A space F is finite when the algebra $\mathcal{A}_F$ of coordinates on F is finite dimensional. We no longer require that this algebra is commutative.

Classification

We classified the irreducible $(\mathcal{A}, \mathcal{H}, J)$ and found out that the solutions fall into two classes. Let $\mathcal{A}_{\mathbb{C}}$ be the complex linear space generated by $\mathcal{A}$ in $\mathcal{L}(\mathcal{H})$, the algebra of operators in $\mathcal{H}$. By construction $\mathcal{A}_{\mathbb{C}}$ is a complex algebra and one only has two cases :

1. The center $Z(\mathcal{A}_{\mathbb{C}})$ is $\mathbb{C}$, in which case $\mathcal{A}_{\mathbb{C}} = M_k(\mathbb{C})$ for some k.

2. The center $Z(\mathcal{A}_{\mathbb{C}})$ is $\mathbb{C} \oplus \mathbb{C}$ and $\mathcal{A}_{\mathbb{C}} = M_k(\mathbb{C}) \oplus M_k(\mathbb{C})$ for some k.

Moreover the knowledge of $\mathcal{A}_{\mathbb{C}} = M_k(\mathbb{C})$ shows that $\mathcal{A}$ is either $M_k(\mathbb{C})$ (unitary case), $M_k(\mathbb{R})$ (real case) or, when $k = 2\ell$ is even, $M_\ell(\mathbb{H})$, where $\mathbb{H}$ is the field of quaternions (symplectic case). This first case is a minor variant of the Einstein-Yang-Mills case described above.

It turns out by studying their $\mathbb{Z}/2$ gradings γ, that these cases are incompatible with KO-dimension 6 which is only possible in case (2).

KO-dimension 6

If one assumes that one is in the "symplectic–unitary" case and that the grading is given by a grading of the vector space over $\mathbb{H}$, one can show that the dimension of $\mathcal{H}$ which is $2k^2$ in case (2) is at least 2×16 while the simplest solution is given by the algebra $\mathcal{A} = M_2(\mathbb{H}) \oplus M_4(\mathbb{C})$. This is an important variant of the Einstein-Yang-Mills case because, as the center $Z(\mathcal{A}_{\mathbb{C}})$ is $\mathbb{C} \oplus \mathbb{C}$, the product of this finite geometry F by a manifold M appears, from the commutative standpoint, as two distinct copies of M.

Reduction to SM gauge group

We showed that requiring that these two copies of M stay a finite distance apart reduces the symmetries from the group SU(2) $\times$ SU(2) $\times$ SU(4) of inner automorphisms of the even part of the algebra to the symmetries $U(1) \times$ SU(2) $\times$ SU(3) of the Standard Model. This reduction of the gauge symmetry occurs because of the order one condition

$$[[D,a],b^0] = 0\,, \quad \forall\, a,b \in \mathcal{A}$$

Spectral Model

Let M be a Riemannian spin 4-manifold and F the finite noncommutative geometry of KO-dimension 6 described above. Let $M \times F$ be endowed with the product metric.

1. The unimodular subgroup of the unitary group acting by the adjoint representation Ad(u) in $\mathcal{H}$ is the group of gauge transformations of SM.

2. The unimodular inner fluctuations of the metric give the gauge bosons of SM.

3. The full standard model (with neutrino mixing and seesaw mechanism) minimally coupled to Einstein

gravity is given in Euclidean form by the action functional

$$S = \mathrm{Tr}(f(D_A/\Lambda)) + \frac{1}{2}\langle J\tilde{\xi}, D_A\tilde{\xi}\rangle\,, \quad \tilde{\xi} \in \mathcal{H}^{+}_{cl}\,,$$

where D_A is the Dirac operator with the unimodular inner fluctuations.

Standard Model	Spectral Action
Higgs Boson	Inner metric$^{(0,1)}$
Gauge bosons	Inner metric$^{(1,0)}$
Fermion masses u, ν	Dirac$^{(0,1)}$ in $\uparrow$
CKM matrix Masses down	Dirac$^{(0,1)}$ in $(\downarrow 3)$
Lepton mixing Masses leptons e	Dirac$^{(0,1)}$ in $(\downarrow 1)$

Standard Model	Spectral Action
Majorana mass matrix	Dirac$^{(0,1)}$ on $E_R \oplus J_F E_R$
Gauge couplings	Fixed at unification
Higgs scattering parameter	Fixed at unification
Tadpole constant	$-\mu_0^2 \, \lvert\mathbf{H}\rvert^2$

First interplay with experiment

Historically, the search to identify the structure of the noncommutative space followed the bottom-up approach where the known spectrum of the fermionic particles was used to determine the geometric data that defines the space.

This bottom-up approach involved an interesting interplay with experiments. While at first the experimental evidence of neutrino oscillations contradicted the first attempt, it was realized several years later in 2006 that the obstruction to get neutrino oscillations was naturally eliminated by dropping the equality between the metric dimension of space-time (which is equal to 4 as far as we know) and its KO-dimension which is only defined modulo 8. When the latter is set equal to 2 modulo 8 (using the freedom to adjust the geometry of the finite space encoding the fine structure of space-time) everything works fine, the neutrino oscillations are there as well as the see-saw mechanism which appears for free as an unexpected bonus. Incidentally, this also solved the fermionic doubling problem by allowing a simultaneous Weyl-Majorana condition on the fermions to halve the degrees of freedom.

Second interplay with experiment

The second interplay with experiments occurred a bit later when it became clear that the mass of the Brout-Englert-Higgs boson would not comply with the restriction (that $m_H \succeq 170$ Gev) imposed by the validity of the Standard Model up to the unification scale.

We showed that the inconsistency between the spectral Standard Model and the experimental value of the Higgs mass is resolved by the presence of a real scalar field strongly coupled to the Higgs field. This scalar field was already present in the spectral model and we wrongly neglected it in our previous computations.

It was shown recently by several authors, independently of the spectral approach, that such a strongly coupled scalar field stabilizes the Standard Model up to unification scale in spite of the low value of the Higgs mass. In our recent work, we show that the noncommutative neutral singlet modifies substantially the RG analysis, invalidates our previous prediction of Higgs mass in the range 160–180 Gev, and restores the consistency of the noncommutative geometric model with the low Higgs mass.

Lesson

One lesson which we learned on that occasion is that we have to take all the fields of the noncommutative spectral model seriously, without making assumptions not backed up by valid analysis, especially because of the almost uniqueness of the Standard Model (SM) in the noncommutative setting.

New developments

$(AC)^2$ + Walter van Suijlekom

The SM continues to conform to all experimental data. The question remains whether this model will continue to hold at much higher energies, or whether there is a unified theory whose low-energy limit is the SM. One indication that there must be a new higher scale that effects the low energy sector is the small mass of the neutrinos which is explained through the see-saw mechanism with a Majorana mass of at least of the order of 10^{11}Gev. In addition and as noted above, a scalar field which acquires a vev generating that mass scale can stabilize the Higgs coupling and prevent it from becoming negative at higher energies and thus make it consistent with the low Higgs mass of 126 Gev. Another indication of the need to modify the SM at high energies is the failure (by few percent) of the three gauge couplings to be unified at some high scale which indicates that it may be necessary to add other matter couplings to change the slopes of the running of the RG equations.

This leads us to address the issue of the breaking from the natural algebra $\mathcal{A}$ which results from the classification of irreducible finite geometries of KO-dimension 6 (modulo 8), to the algebra corresponding to the SM. This breaking was effected using the requirement of the first order condition on the Dirac operator. The first order condition is the requirement that the Dirac operator is a derivation of the algebra $\mathcal{A}$ into the commutant of $\widehat{\mathcal{A}} = J\mathcal{A}J^{-1}$ where J is the charge conjugation operator. This in turn guarantees the gauge invariance and linearity of the inner fluctuations under the action of the gauge group given by the unitaries $U = uJuJ^{-1}$ for any unitary $u \in \mathcal{A}$. This condition was used as a mathematical requirement to select the maximal subalgebra

$$\mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C}) \subset \mathbb{H}_R \oplus \mathbb{H}_L \oplus M_4(\mathbb{C})$$

which is compatible with the first order condition and is the main reason behind the unique selection of the SM.

The existence of examples of noncommutative spaces where the first order condition is not satisfied such as quantum groups and quantum spheres provides a motive to remove this condition from the classification of noncommutative spaces compatible with unification. This study was undertaken in a companion paper where it was shown that in the general case the inner fluctuations of D with respect to inner automorphisms of the form $U = u\, Ju\, J^{-1}$ are given by

$$D_A = D + A_{(1)} + \widetilde{A}_{(1)} + A_{(2)}$$

where

$$A_{(1)} = \sum_i a_i \, [D, b_i]$$

$$\widetilde{A}_{(1)} = \sum_i \widehat{a}_i \left[D, \widehat{b}_i\right], \qquad \widehat{a}_i = J a_i J^{-1}, \quad \widehat{b}_i = J b_i J^{-1}$$

$$A_{(2)} = \sum_{i,j} \widehat{a}_i a_j \left[\left[D, b_j\right], \widehat{b}_i\right] = \sum_{i,j} \widehat{a}_i \left[A_{(1)}, \widehat{b}_i\right].$$

Clearly $A_{(2)}$ which depends quadratically on the fields in $A_{(1)}$ vanishes when the first order condition is satisfied.

Our point of departure is that one can extend inner fluctuations to the general case, *i.e.* without assuming the order one condition. It suffices to add a quadratic term which only depends upon the universal 1-form $\omega \in \Omega^1(\mathbb{A})$ to the formula and one restores in this way,

– The gauge invariance under the unitaries $U = uJuJ^{-1}$
– The fact that inner fluctuations are transitive, *i.e.* that inner fluctuations of inner fluctuations are themselves inner fluctuations.

We show moreover that the resulting inner fluctuations come from the action on operators in Hilbert space of a semi-group Pert($\mathbb{A}$) of *inner perturbations* which only depends on the involutive algebra $\mathbb{A}$ and extends the unitary group of $\mathbb{A}$. This opens up two areas of investigation, the first is mathematical and the second is directly related to particle physics and model building :

1. Investigate the inner fluctuations for noncommutative spaces such as quantum groups and quantum spheres.
2. Compute the spectral action and inner fluctuations for the model involving the full symmetry algebra $\mathbb{H} \oplus \mathbb{H} \oplus M_4(\mathbb{C})$ before the breaking to the Standard Model algebra.

(i) The following map η is a surjection

$$\eta : \{\sum a_j \otimes b_j^{\mathrm{op}} \in \mathbb{A} \otimes \mathbb{A}^{\mathrm{op}} \mid \sum a_j b_j = 1\} \to \Omega^1(\mathbb{A}),$$

$$\eta(\sum a_j \otimes b_j^{\mathrm{op}}) = \sum a_j \delta(b_j).$$

(ii) One has

$$\eta\left(\sum b_j^* \otimes a_j^{*\mathrm{op}}\right) = \left(\eta\left(\sum a_j \otimes b_j^{\mathrm{op}}\right)\right)^*$$

(iii) One has, for any unitary $u \in \mathbb{A}$,

$$\eta\left(\sum u a_j \otimes (b_j u^*)^{\mathrm{op}}\right) = \gamma_u\left(\eta\left(\sum a_j \otimes b_j^{\mathrm{op}}\right)\right)$$

where γ_u is the gauge transformation of potentials.

(i) Let $A = \sum a_j \otimes b_j^{\text{op}} \in \mathbb{A} \otimes \mathbb{A}^{\text{op}}$ normalized by the condition $\sum a_j b_j = 1$. Then the operator $D' = D(\eta(A))$ is equal to the inner fluctuation of D with respect to the algebra $\mathbb{A} \otimes \hat{\mathbb{A}}$ and the 1-form $\eta(A \otimes \hat{A})$, that is

$$D' = D + \sum a_i \hat{a}_j [D, b_i \hat{b}_j]$$

(ii) An inner fluctuation of an inner fluctuation of D is still an inner fluctuation of D, and more precisely one has, with A and A' normalized elements of $\mathbb{A} \otimes \mathbb{A}^{\text{op}}$ as above,

$$(D(\eta(A)))\,(\eta(A')) = D(\eta(A'A))$$

where the product $A'A$ is taken in the tensor product algebra $\mathbb{A} \otimes \mathbb{A}^{\text{op}}$.

(i) The self-adjoint normalized elements of $\mathbb{A}\otimes\mathbb{A}^{\mathrm{op}}$ form a semi-group Pert($\mathbb{A}$) under multiplication.

(ii) The transitivity of inner fluctuations (*i.e.* the fact that inner fluctuations of inner fluctuations are inner fluctuations) corresponds to the semi-group law in the semi-group Pert($\mathbb{A}$).

(iii) The semi-group Pert($\mathbb{A}$) acts on real spectral triples through the homomorphism

$$\mu : \mathrm{Pert}(\mathbb{A}) \to \mathrm{Pert}(\mathbb{A} \otimes \hat{\mathbb{A}})$$

given by

$$A \in \mathbb{A} \otimes \mathbb{A}^{\mathrm{op}} \mapsto \mu(A) = A \otimes \hat{A} \in \left(\mathbb{A} \otimes \hat{\mathbb{A}}\right) \otimes \left(\mathbb{A} \otimes \hat{\mathbb{A}}\right)^{\mathrm{op}}$$

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