From almost-commutative to almost-associative geometries: Model Building.

Shane Farnsworth (with Latham Boyle)

October 17, 2013

Overview

1. Motivation revisted: Non-associative algebras
2. Framework revisted: The Basic tools
3. A family of non-associative Geometries
4. Where to go from here (crazy ideas section).

Motivation revisted: Non-associative algebras

General Alternative algebras

$$\begin{aligned} [a,b,c] &= -[a,c,b] = [c,a,b] \\ \rightarrow [a,a,b] &= [b,a,a] = [a,b,a] = 0 \\ \delta_{a,b} &= [L_a, L_b] + [L_a, R_b] + [R_a, R_b] \in \mathcal{D}(\mathcal{A}) \end{aligned} \quad (1)$$

Example: The Octonions with automorphism group G_2

$$\swarrow \qquad \searrow$$

$$SU_3 \qquad SU_2 \times SU_2$$

The octonions have 8 basis elements, could we fit $\frac{1}{2}$ a generation of particles (6 quarks, 2 leptions)???

Motivation revisted: Non-associative algebras

Jordan Algebras

$$[a^2, b, a] = [a, b, a^2] = 0$$
$$\delta_{a,b} = [L_a, L_b] = [L_a, R_b] = [R_a, R_b] \in \mathcal{D}(\mathcal{A}) \qquad (2)$$

Example 1: The Jordan Algebra of 10×10 real matrices with automorphism group SO_{10}.
Example 2: The Exceptional Jordan Algebra of 3×3 hermitian octonion matrices with automorphism group F_4.

Motivation revisted: Non-associative algebras

Brown Algebra Example:

$$\begin{pmatrix} a & j \\ j' & a' \end{pmatrix} \begin{pmatrix} b & s \\ s' & b' \end{pmatrix} = \begin{pmatrix} ab + Tr(j, s') & as + b'j + (j' \times s') \\ bj' + a's' + (j \times s) & a'b' + Tr(j', s) \end{pmatrix}$$

where $a \times b = ab - \frac{1}{2}Tr(a)b - \frac{1}{2}Tr(b)a + \frac{1}{2}[Tr(a)Tr(b) - Tr(ab)]1.$

A very complicated class of algebras. **DON'T WORRY**, just remember that they have automorphisms of type E_6

$$\begin{array}{c} E_6 \\ \downarrow \\ SO_{10} \\ \downarrow \\ SU_2 \times SU_2 \times SU_4 \\ \downarrow \\ U_1 \times SU_2 \times SU_3 \end{array}$$

Motivation revisted: Non-associative algebras

LIE ALGEBRAS,ETC...

Framework revisted: The Basic tools

Keep most of the known and loved structure of NCG!

Just a few small changes

- The Hilbert space H now acts as a (non-associative) bimodule over the (non-associative) input algebra.
  - New Order Zero Condition(s)
  - New Order One Condition(s)
- New Fluctuations!
  - Inhomoheneous Fluctuation given by $[D, \delta]$
- New compatibility checks!
  - $[\gamma, \delta] = 0$
  - $[J, \delta] = 0$

Introducing a Family of Non-associative Geometries!

A simple first example

Take the yang mills Finite spectral triple as inspiration:

$$\{A, H, D, J, \gamma\} = \{M_n(\mathbb{C}), M_n(\mathbb{C}), 0, (.)^{\dagger}, \mathbb{I}\}. \quad (3)$$

Let's make the simplest model we can imagine:

$$\{A, H, D, J, \gamma\} = \{\mathbb{O}, \mathbb{O}, 0, (.)^{*}, \mathbb{I}\}. \quad (4)$$

Does it work?

$$\begin{aligned} [\gamma, \delta] = [J, \delta] &= 0\checkmark \\ [L_a, JL_{a^*}J^*] &= 0\checkmark \\ \text{Order one}&\checkmark \end{aligned} \quad (5)$$

Introducing a Family of Non-associative Geometries!

A simple first example

The almost-assocaitive space:

$$A = C^\infty(M, \mathbb{O}), \qquad H = \mathcal{L}^2(M, S) \otimes \mathbb{O}$$
$$D = -i\gamma_c^\mu \nabla_\mu^S \otimes \mathbb{I}, \qquad J = J_c \otimes (.)^*$$
$$\gamma = \gamma_5 \otimes \mathbb{I}_F$$

Fluctuations:

$$\begin{aligned} B_0 &= [D, \delta_{a,b}] \\ &= -i\gamma_c^\mu([L_{\partial_\mu a}, L_b] + [L_{\partial_\mu a}, R_b] + [R_{\partial_\mu a}, R_b] - (a \leftrightarrow b)) \\ &= -i\gamma_c^\mu[(\partial_\mu a)^i b^j - (\partial_\mu b)^i a^j]\delta_{e_i, e_j} \end{aligned} \tag{6}$$

By inspection:

$$\begin{aligned} B = -i\gamma_c^\mu A_\mu^\alpha \delta_\alpha \rightarrow D_A = -i\gamma_c^\mu(\nabla_\mu^S + A_\mu^\alpha \delta_\alpha) \\ A_\mu^\alpha = \sum_{a,b} (a[\partial_\mu, b])^\alpha \end{aligned} \tag{7}$$

Introducing a Family of Non-associative Geometries!

Generalizing the Yang-mills example.

Problem! We want $\{D, \gamma\} = 0$, and $D_F \neq 0$.

Simplest solution! increase the size of the hilbert space.

Operator	Constraints
$\pi(a) = pa$	$\pi(ab) = \pi(a)\pi(b) \to p = p^2$ $\pi(a^*) = \pi(a)^* \to p = \overline{p} = p^\dagger$
$D = \gamma_F^\alpha \delta_\alpha$	$D = D^\dagger \to \gamma_F^\alpha = -(\gamma_F^\alpha)^T$ $\{D, \gamma\} = 0 \to \{\gamma_F^\alpha, \gamma\} = 0$ $[D, L_a] = D(a)p \to [\gamma_F^\alpha, p] = 0$
$J = j \circ (.)^*$	$< J^\dagger a \vert b >= \overline{< a \vert Jb >}, J^2 = \epsilon \to j^T = \epsilon j$ $DJ = \epsilon' JD \to j\gamma_F^\alpha = \epsilon' \gamma_F^\alpha j$ $J\gamma = \epsilon'' \gamma J \to j\gamma = \epsilon'' \gamma j$ $[L_a, R_b] = [a,, b] \to [p, jpj^\dagger] = 0$
γ	$\gamma = \gamma^{-1} = \gamma^\dagger$ $[\gamma, L_{\pi(a)}] = 0$

Introducing a Family of Non-associative Geometries!

KO	$\pi(a)$	D	J
0	$a\mathbb{I}_2$	$D = \begin{pmatrix} 0 & +\delta \\ -\delta & 0 \end{pmatrix}$	$J = \begin{pmatrix} +1 & 0 \\ 0 & +1 \end{pmatrix} \circ (.)^*$
1	$a\mathbb{I}_2$	$D = \begin{pmatrix} 0 & +\delta \\ -\delta & 0 \end{pmatrix}$	$J = \begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix} \circ (.)^*$
2/3	$a\mathbb{I}_2$	$D = \begin{pmatrix} 0 & +\delta \\ -\delta & 0 \end{pmatrix}$	$J = \begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix} \circ (.)^*$
4	$a\mathbb{I}_4$	$D = \begin{pmatrix} 0 & +\triangle_+ \\ +\triangle_+ & 0 \end{pmatrix}$	$J = \begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix} \circ (.)^*$
5	$a\mathbb{I}_4$	$D = \begin{pmatrix} 0 & +\triangle_- \\ -\triangle_- & 0 \end{pmatrix}$	$J = \begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix} \circ (.)^*$
6	$a\mathbb{I}_4$	$D = \begin{pmatrix} 0 & \sigma\delta \\ \sigma\delta & 0 \end{pmatrix}$	$J = \begin{pmatrix} \mathbb{I}_2 & 0 \\ 0 & \mathbb{I}_2 \end{pmatrix} \circ (.)^*$
7	$a\mathbb{I}_2$	$D = \begin{pmatrix} 0 & +\delta \\ -\delta & 0 \end{pmatrix}$	$J = \begin{pmatrix} +1 & 0 \\ 0 & +1 \end{pmatrix} \circ (.)^*$

$$\sigma = \begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix}, \quad \triangle_\pm = \begin{pmatrix} +\delta_1 & +\delta_2 \\ \mp\delta_2 & \pm\delta_1 \end{pmatrix}, \quad \gamma = \begin{pmatrix} +\mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix}$$

Introducing a Family of Non-associative Geometries!

Fluctuations

We have a total space Dirac operator of the form

$$D = i\gamma_c^\mu \nabla_\mu^S \otimes \mathbb{I}_F + \gamma_c^5 \otimes \gamma_F^\alpha \delta_\alpha$$

The inhomogeneous fluctuations are given by:

$$[D, \hat{\delta}] = \gamma_c^\mu (i\partial_\mu a^\alpha) \otimes \mathbb{I}_F \delta_\alpha + \gamma_c^5 (f_{\beta\alpha}^\kappa a^\alpha) \otimes \gamma_F^\beta \delta_\kappa$$

By inspection the full fluctuations are given by:

$$B = \gamma_c^\mu A_\mu^\alpha \otimes \mathbb{I}_F \delta_\alpha + \gamma_c^5 \phi_\kappa^\alpha \otimes \gamma_F^\kappa \delta_\alpha$$

Where to go from here (crazy ideas section).

Does the Pati-Salam NCG model hint towards a non-associativity Geometry?

The Gauge Fluctuations for the Pati-Salam model are of the form:

$$B = A_{(1)} + A_{(2)} + \epsilon' J A_{(1)} J^*$$
$$\text{with } A_{(2)} = \epsilon' J A_{(2)} J^*$$

The Gauge Fluctuations for a general alternative model are of the form:

$$B = [A_{(1)}, A_{(0)}] + [A_{(1)}, JA^*_{(0)}J^*] + \epsilon' J[A_{(1)}, A_{(0)}]J^*$$
$$\text{with } [A_{(1)}, JA^*_{(0)}J^*] = \epsilon' J[A_{(1)}, JA^*_{(0)}J^*]J^*$$