Noncommutative geometry and flavour mixing

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1 Introduction: the universality problem

"The origin of the quark and lepton masses is shrouded in mystery" [1]. Some thirty years ago, attempts to solve the enigma based on *textures* of the quark mass matrices, purposedly reflecting mass hierarchies and "nearest-neighbour" interactions, were very popular. Now, in the late eighties, Branco, Lavoura and Mota [2] showed that, within the SM, the zero pattern

$$\begin{pmatrix} a & b & 0 \\ c & 0 & d \\ 0 & e & 0 \end{pmatrix}, \tag{1}$$

a central ingredient of Fritzsch's well-known *Ansatz* for the mass matrices, is devoid of any particular physical meaning. (The top quark is above on top.)

Although perhaps this was not immediately clear at the time, paper [2] marked a watershed in the theory of flavour mixing. In algebraic terms, it establishes that the linear subspace of matrices of the form (1) is *universal* for the group action of unitaries effecting chiral basis transformations, that respect the charged-current term of the Lagrangian. That is, any mass matrix can be transformed to that form without modifying the corresponding CKM matrix.

To put matters in perspective, consider the unitary group acting by similarity on three-by-three matrices. The classical triangularization theorem by Schur ensures that the zero patterns

$$\begin{pmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{pmatrix}, \quad \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix} \tag{2}$$

are universal in this sense. However, proof that the zero pattern

$$\begin{pmatrix} a & b & 0 \\ 0 & c & d \\ e & 0 & f \end{pmatrix}$$

is universal was published [3] just three years ago! (Any off-diagonal $n(n-1)/2$ zero pattern with zeroes at some (ij) and no zeroes at the matching (ji), is universal in this sense, for *complex* $n \times n$ matrices.)

Fast-forwarding to the present time, notwithstanding steady experimental progress [4] and a huge amount of theoretical work by many authors, we cannot be sure of being any closer to solving the "Meroitic" problem [5] of divining the spectrum behind the known data. Disappointingly, textures are still going strong in some quarters [6,7]. However, it seems fair to state that the focus of attention in the search for underlying symmetry and/or dynamics has turned to the *mixing matrix* V_{CKM} itself —or the lepton mixing matrix V_{PMNS}, for that matter.

Models seeking to discern finite groups of "horizontal symmetry" behind the mixing patterns [8–10] and studies such as [11] of empirical mass relations do appear to respond to that type of investigation. There are of course many other ideas hawked in the market.

Still, "the Higgs boson must know something we do not know",[1] and we would dearly like to know it. Perhaps it is time again that we bend the stick again towards the issue of the mass matrices. A perennial question in flavour-mixing theory is the following. Suppose the mass eigenvalues and the empirical mixing matrix, or equivalent data for fermion multiplets, are given: what is the space of mass matrices *compatible* with these data?

A possible path towards the answer to that question involves a detour through the realm of noncommutative geometry (NCG). In [12], when grappling with the classification problem for Riemannian manifolds, Connes dubbed the CKM construction a "toy model" for geometrical placement problems in general. His abstract formulation of the latter helps to inject some fresh thinking into the subject. We make no apology for this relatively high-brow approach to such an apparently simple matter.

2 Relative spectrum in finite dimensions

As is famously known [13], the spectrum of the Laplacian (or Dirac operator) is not enough to solve the classification problem for Riemannian manifolds. On the other hand, it is now a theorem that a (compact, boundaryless) Riemannian manifold is retrievable from a "spectral triple" [14,15]. Such a triple $(M, \mathcal{H}, D)$ consists of an infinite-dimensional Hilbert space $\mathcal{H}$, as a *tabula rasa*; an unbounded self-adjoint operator D on $\mathcal{H}$ with compact resolvent; and a continuous[2] abelian von Neumann subalgebra M of the algebra of bounded operators $\mathcal{L}(\mathcal{H})$. The operator D has a *discrete* spectrum of eigenvalues of finite multiplicity, and thus the von Neumann algebra N generated by D has many minimal projectors, subordinate to the spectral projectors of D. Here again, N is abelian.

Thus the situation one is faced with, when attempting to identify Riemannian manifolds, is roughly as follows. One lays out a complete, commuting set of eigenprojectors on $\mathcal{H}$

[1] *Dixit* M. Veltman, as quoted in [1].

[2] An abelian von Neumann algebra is "continuous" if it contains no minimal projectors.

(including a few for the finite-dimensional kernel of D, if it is nonzero), generating an abelian von Neumann algebra N. The other abelian von Neumann algebra M acts by multiplication operators M_f on $\mathcal{H}$. But now one must "diagonalize" the second algebra properly, so that the commutators like $[D, M_f]$ —which contain the metric information— give the correct results in $\mathcal{L}(\mathcal{H})$. In [12, Sect. 3], a Riemannian invariant $\operatorname{Spec}_N(M)$, called the *relative spectrum*, is described which, in the presence of N and $\mathcal{H}$, allows one to build the algebra M. One can freely allow an overall conjugation by a unitary element of N, since this amounts to adjusting the chosen eigenvectors of D by phase factors. Therefore $\operatorname{Spec}_N(M)$ must be defined modulo an action of the unitary group $\mathcal{U}(N)$.

Still following Connes, we may look at the parallel role of the CKM matrix. Let there be given two sets of n minimal projectors (ketbras of rank one), commuting within each set, but not globally as $n \times n$ matrices. Choosing one set as the diagonal matrices will "undiagonalize" the other. Of course, a change of orthonormal basis is given by a unitary matrix, which is what we must exhibit; but there is the *nuance* that adjusting the phase factors in either basis is deemed irrelevant, so we quotient these adjustments out. The problem is codified as follows.

Lemma. *On $\mathcal{H} \simeq \mathbb{C}^n$, choose two sets of minimal projectors, $\{e_1, \dots, e_n\}$ and $\{f_1, \dots, f_n\}$, orthogonal among themselves ($e_i e_j = 0 = f_i f_j$ for $i \neq j$). Suppose that $e_1 f_j \neq 0$ for each j and $e_i f_1 \neq 0$ for each i. One can find orthonormal bases for $\mathcal{H}$ of eigenvectors $\{\xi_i\}$ with $e_i \xi_i = \xi_i$ and $\{\eta_j\}$ with $f_j \eta_j = \eta_j$; by suitably adjusting their phases, one can also ensure that each $\langle \xi_1 \,|\, \eta_j \rangle > 0$ and each $\langle \xi_i \,|\, \eta_1 \rangle > 0$. This may be done* ***uniquely****, up to an overall phase. Let V denote the unitary change-of-basis matrix for $\xi_i \mapsto \eta_j$. Then $V e_j V^\dagger = f_j$.*

This is Lemma 2.1 of [12]. Here $M = \operatorname{span}\langle e_1, \dots, e_n \rangle$ and $N = \operatorname{span}\langle f_1, \dots, f_n \rangle$ are maximal abelian von Neumann subalgebras of $\mathcal{L}(\mathcal{H}) \simeq \mathbb{C}^{n \times n}$, in "general position" with respect to each other. We note that $|v_{ij}|^2 = \operatorname{tr}(e_i f_j)$.

Evidently V is a CKM matrix by another name. It does not conform to the conventions of the Review of Particle Properties [16]; nevertheless, the condition that the first row and column of V be positive is a well-known device in particle physics to reduce the number of its phase factors. Except for trivial sign changes, it does coincide with the recommended parametrization for CKM matrices in [5], when keeping the top on top. Checking the parameter count is routine: assume that the $2n - 1$ positive numbers are given:

$$\alpha_1 = \beta_1 = \langle \xi_1 \,|\, \eta_1 \rangle; \quad \alpha_j := \langle \xi_j \,|\, \eta_1 \rangle; \quad \beta_k := \langle \xi_1 \,|\, \eta_k \rangle; \quad \text{with} \quad \sum_1^n \alpha_j^2 = \sum_1^n \beta_k^2 = 1.$$

Then η_1 is completely fixed relative to $\{\xi_i\}$, with expenditure of $2n - 3$ degrees of freedom. The constraints on η_2 are: $|\eta_2| = 1$; $\eta_2 \perp \eta_1$; $\langle \xi_1 \,|\, \eta_2 \rangle = \beta_2$. This takes $2n - 5$ degrees of freedom. Successively, the constraints on η_3 are: $|\eta_3| = 1$; $\eta_3 \perp \eta_1, \eta_2$; $\langle \xi_1 \,|\, \eta_3 \rangle = \beta_3$, yielding $2n - 7$ degrees of freedom; and so on. Therefore, we obtain $\sum_{l=1}^{n-1} (2n - 2l - 1) = (n - 1)^2$ as expected.

The next procedure in [12] is to suppress the "row labels" of V while keeping enough information about the "relative position" of the algebras M and N, acting on $\mathcal{H}$. We regard the representation of N as fixed and given, with $f_j = |\eta_j\rangle\langle\eta_j|$. Any row of V determines an element of N, namely $\sum_j v_{ij} f_j \in N$; the corresponding row vector $\sum_j v_{ij}\,\eta_j$ lies in the unit sphere $\mathbb{S}_N \simeq \mathbb{S}^{2n-1}$. Denote by $\widetilde{M} \subset \mathbb{S}_N$ the set of unit vectors coming from M in this way. Suppressing one phase, we land in $\mathbb{P}_N \simeq \mathbb{CP}^{n-1}$, the projective space with $n-1$ complex dimensions. Call $p\colon \mathbb{S}_N \to \mathbb{P}_N$ the usual quotient map. Then we mod out by the "gauge" action of the unitary group $\mathcal{U}(N) \simeq U(1)^n$, which affects the representation of N on $\mathcal{H}$ by adjusting the isomorphism $\mathcal{H} \simeq \mathbb{C}^n$ while preserving its rays.

Definition. The *relative spectrum* $\mathrm{Spec}_N(M)$ is the finite (n-element) subset of projective space $p(\widetilde{M}) \subset \mathbb{P}_N$, modulo the action of $\mathcal{U}(N)$.

This definition makes sense since (a) it is invariant under all permissible phase changes; and (b) it allows one to recover the representation of M from the known representation of N. We can indeed go back to the subset $\widetilde{M}$, up to phases, by selecting arbitrary phases from $\mathcal{U}(N)$. From a vector $\sum_j v_{ij}\,\eta_j$ in $\widetilde{M}$ we can reconstruct an element $e_i := \sum_j v_{ij}\,|\eta_j\rangle\langle\eta_j| \in M$, since the standard basis diagonalizing N is known; and thus we land on a set of minimal projectors which generate M.

3 Quick rewording of the standard procedure

The mass terms in the Yukawa sector of the Lagrangian (say, for quarks) are *a priori* of the form

$$\mathcal{L}_{\mathrm{mass}} = \bar{d}'_L M_d\, d'_R + \bar{u}'_L M_u\, u'_R\,; \tag{3}$$

where the "mass matrices" M_d, M_u are in principle arbitrary (although we always suppose them nonsingular with distinct eigenvalues), reality of the Lagrangian being attained by adding the conjugate terms. In the SM the left-handed and right-handed fermion fields are treated as unrelated to each other. Thus we regard M_d, M_u as $n_L \times n_R$ matrices (with $n_L = n_R = n$) in generation space.

We naturally diagonalize the mass matrices. Common knowledge holds that the best we can do in this respect is the singular value decomposition, called by physicists a biunitary transformation. We formulate this diagonalization in line with the development in the previous section. By definition, a set of operators $\{P_i\}_{i=1}^n$ is a complete set of minimal, pairwise orthogonal *partial isometries* if

$$P_i^\dagger P_j = P_i P_j^\dagger = 0 \quad \text{for} \quad i \neq j,$$

and $P_i^\dagger P_i$, $P_i P_i^\dagger$ are minimal projectors.

We let henceforth $n = 3$ *pour les besoins de la cause*. Generically, the singular value decomposition is of the form

$$M_u = m_u P_u + m_c P_c + m_t P_t; \qquad M_d = m_d P_d + m_s P_s + m_b P_b;$$

where the operators P_u, P_c, etc. may be expressed obviously in the form:

$$P_u = |u_L\rangle\langle u_R|,\ P_c = |c_L\rangle\langle c_R|,\ P_t = |t_L\rangle\langle t_R|; \quad P_d = |d_L\rangle\langle d_R|,\ P_s = |s_L\rangle\langle s_R|,\ P_b = |b_L\rangle\langle b_R|.$$

However, note that the vectors $u_L, u_R, c_L, \dots$ are not uniquely defined; only the partial isometries are. This is the most intrinsic formulation of a biunitary transformation. Note that taking the masses $m_u, m_c, \dots$ to be positive is just a convention; obviously the decomposition works with any sign (or even phase) of the masses.

One then *chooses* pairs $(u_L, c_L, t_L)\,(u_R, c_R, t_R)$ —and similarly for the d-type quarks— and fabricates unitary matrices $U_L^{(u)}$, $U_L^{(d)}$, $U_R^{(u)}$, $U_R^{(d)}$ with these as columns, obtaining

$$U_L^{(u)\dagger} M_u U_R^{(u)} = \operatorname{diag}(m_u, m_c, m_t, \dots) =: D_u\,;$$
$$U_L^{(d)\dagger} M_d U_R^{(d)} = \operatorname{diag}(m_d, m_s, m_b, \dots) =: D_d\,;$$

Thus D_u and D_d yield the quark-mass abelian von Neumann algebra. Clearly the $U_L^{(q)}$, with q standing for either u or d, diagonalize the $|M_q^\dagger|^2$ and the $U_R^{(q)}$ diagonalize $|M_q|^2$. In fact, the columns of $U_L^{(q)}$ and $U_R^{(q)}$ are eigenvectors for $|M_q^\dagger|$ and $|M_q|$, respectively. By writing

$$u'_L = U_L^{(u)} u_L\,; \qquad u'_R = U_R^{(u)} u_R\,, \qquad d'_L = U_L^{(d)} d_L\,; \qquad d'_R = U_R^{(d)} d_R\,,$$

we appreciate that there is every reason to work with the diagonalized matrices when describing the coupling of quarks to gluons.

There are however in the SM Lagrangian, among the weak vertices, the charged-current terms:

$$\mathcal{L}_{qW} = \frac{g}{\sqrt{2}} W_\mu^+ \bar{u}'_L \gamma^\mu\, d'_L + \frac{g}{\sqrt{2}} W_\mu^- \bar{d}'_L \gamma^\mu\, u'_L. \tag{4}$$

On the face of it, this becomes

$$\mathcal{L}_{qW} = \frac{g}{\sqrt{2}} W_\mu^+ \bar{u}_L \gamma^\mu V_{\text{CKM}}\, d_L + \frac{g}{\sqrt{2}} W_\mu^- \bar{d}_L \gamma^\mu V_{\text{CKM}}^\dagger\, u_L,$$

where $V_{\text{CKM}} := U_L^{(u)\dagger} U_L^{(d)}$ is by the definition the CKM matrix, just called V henceforth:

$$V = \begin{pmatrix} v_{tb} & v_{ts} & v_{td} \\ v_{cb} & v_{cs} & v_{cd} \\ v_{ub} & v_{us} & v_{ud} \end{pmatrix}.$$

Note however, that $U_L^{(u)\dagger}$ and $U_L^{(d)}$ belong to different spaces. Only the identification between u-and d-quarks provided by the sesquilinear form (4) allows us to multiply them.

In summary, the CKM matrix describes the *placement problem* for the von Neumann algebras corresponding to the two invoked pieces (3) and (4) of the Lagrangian.

4 A remarkable matrix

Consider the Fourier transform for the group $\mathbb{Z}_3$, given by the symmetric unitary matrix

$$\widetilde{V} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \overline{\omega} \\ 1 & \overline{\omega} & \omega \end{pmatrix}, \quad \text{where} \quad \omega = e^{2\pi i/3}. \tag{5}$$

The pair of maximal abelian subalgebras on $l^2(\mathbb{Z}_3)$ consists of the algebra N of *multiplication* operators and the algebra M of *convolution* operators. The three rows of $\widetilde{V}$, which are unit vectors in $\mathbb{C}^3$, determine a set of three pairwise orthogonal points in the projective space $\mathbb{CP}^2$. This set is the relative spectrum for this particular CKM matrix.

This example is invoked in different roles; at least by:

- Connes in [12]; it is special in that M and N are mutually orthogonal as von Neumann algebras.
- In physics it is well known that the maximum possible absolute value for the Jarlskog invariant —yielding maximal CP violation— is attained with this spectrum.
- As indicated before, finite discrete groups of symmetry are all the rage to explain the patterns (?) in the fermion mass matrices. One of the most popular is the humble alternating group A_4. As it turns out, $\widetilde{V}$ plays a role in its representation theory —see the next section.
- The (in)famous Koide formula [17] for the masses of the charged leptons is given by

  $$\frac{\left(\sqrt{m_e} + \sqrt{m_\mu} + \sqrt{m_\tau}\right)^2}{m_e + m_\mu + m_\tau} = \frac{3}{2}.$$

  At the moment of its inception, the mass of the tau lepton was not yet determined very precisely, so this was a *prediction*, that came out right on the mark, even if his rationale for it then has been deservedly forgotten. Clearly, the formula was equivalent to an exact angle between the vector of square roots of the masses and the permutation invariant tuple:

  $$\left(m_e, m_\mu, m_\tau\right) \angle (1,1,1) = \frac{\pi}{4}.$$

  to allow a phase angle to parametrize the cone around $(1,1,1)$. Note that

  $$\sqrt{M}\left(1 - 1/\sqrt{2}, 1 - 1/\sqrt{2}, 1 + \sqrt{2}\right)$$

  is a Koide tuple. Therefore: for $k = 1,2,3$, with $\omega = e^{2\pi i/3}$ and $(e, \mu, \tau) \equiv (1,2,3)$ —or any permutation thereof:

  $$\sqrt{m_k} = \sqrt{M}\left(1 + \sqrt{2}\Re\left(\omega^k + e^{i\delta}\right)\right) \tag{6}$$

is a Koide tuple. We see the cubic roots of 1 to emerge again! I may quote the following "experimental" values: $M \simeq 313.8$Mev and $\delta \simeq 0.2222324 \simeq 2/9$.

Exercice 1. *Derive* the fact

$$\sqrt{\widetilde{V}} = \frac{1}{4\sqrt{3}} \begin{pmatrix} 2\sqrt{3}+2+2(\sqrt{3}-1)i & 2-2i & 2-2i \\ 2-2i & (\sqrt{3}+\sqrt{6}-1)+(\sqrt{3}+\sqrt{6}+1)i & (\sqrt{3}-\sqrt{6}-1)+(\sqrt{3}-\sqrt{6}+1)i \\ 2-2i & (\sqrt{3}-\sqrt{6}-1)+(\sqrt{3}-\sqrt{6}+1)i & (\sqrt{3}+\sqrt{6}-1)+(\sqrt{3}+\sqrt{6}+1)i \end{pmatrix}.$$

Exercice 2. Prove the assertion on maximal CP violation for the matrix of (5). Or check it in reference [18, Ch. 13].

5 Horizontal symmetry

By definition, a (pure) horizontal symmetry is a unitary matrix $F_{3\times 3}$ commuting with both $|M_u|^2$ and $|M_d|^2$. This implies that these two matrices have the same eigenspaces, which in case that the eigenvalues of F are non-degenerate results in no mixing. If one eigenvalue is doubly degenerate, still one of the quarks does not mix.

So let us consider *two* unitary matrices F, G, respectively commuting with $|M_u|^2$ and $|M_d|^2$, and assume that they are "*residual*" symmetries of a bigger group of matrices $\mathbb{H}$, assumed finite, which must contain the finite group generated by F, G, and of course possess a faithful 3-dimensional representation. We work in the base in which $M_u M_d^\dagger$ is diagonal, equal to D_u^2. Therefore the eigenvalues of G are given by the columns of V. There must be an integer k such that its eigenvalues are k-roots of unity, by the finiteness assumption. Similarly for F, that moreover has to be diagonal. For obvious reasons, F has to be non-degenerate. One can handle either the case in which G has two coincident eigenvalues or the case in which all three are different.

Following Lam [19] and Altarelli and Feruglio [8], we choose to investigate $\mathbb{H} = A_4$, the tetrahedron motion group. It is generated by two basic even permutations of $\{1234\}$,

$$G = \{4321\}; \quad F = \{2314\}, \quad \text{with} \quad G^2 = F^3 = 1 = (GF)^3.$$

There are four conjugacy classes. Besides $C_1 := \{1234\}$:

$$C_2 := \{G, GFG^2, F^2GF\}; \quad C_3 := \{F, GF, FG, GFG\}; \quad C_4 := \{F^2, GF^2, F^2G, GF^2G\}.$$

Thus there are four characters,

$$\chi^0 = (1,1,1,1); \quad \chi^1 = (1,1,\omega,\overline{\omega}); \quad \chi^2 = (1,1,\overline{\omega},\omega); \quad \chi^3 = (3,-1,0,0),$$

with a self-evident notation. Note their orthogonality properties. The group has thus four inequivalent representations, of dimensions $1,1,1,3 : 1^2+1^2+1^2+3^2 = 12$. They correspond to (with some abuse of notation):

$$G=1, F=1;\ G=1, F=\omega;\ G=1, F=\overline{\omega};\ F = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \overline{\omega} \end{pmatrix},\ G = \frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix};$$

choosing for the last a basis in which F is diagonal. Note that G is symmetric. From this one has the twelve matrices of the 3-dimensional irrep of A_4. That old acquaintance of ours, the matrix $\widetilde{V}$, intertwines this representation with an equivalent one in which G is diagonal.

Now, in the lepton sector the following PMNS matrix, called of "tribimaximal mixing", has been popular for over ten years [20]:

$$V = \begin{pmatrix} \sqrt{2}/\sqrt{3} & 1/\sqrt{3} & 0 \\ -1/\sqrt{6} & 1/\sqrt{3} & -1/\sqrt{2} \\ -1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \end{pmatrix};$$

The order here is: $\{e, \mu, \tau\}$. In this example $\sin^2\theta_{12} = 4\sin^2\theta_{12}\cos\theta_{12} = 8/9$. One may have, more generally,

$$V = \begin{pmatrix} \cos\theta_{12} & \sin\theta_{12} & 0 \\ -\sin\theta_{12}/\sqrt{2} & \cos\theta_{12}/\sqrt{2} & -1/\sqrt{2} \\ -\sin\theta_{12}/\sqrt{2} & \cos\theta_{12}/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}.$$

Working in the basis in which the charged leptons "are diagonal", and certainly invariant under conjugation by F, the neutrino mass pattern corresponding to the last matrix is conventionally[3] of the form

$$\begin{pmatrix} \cos\theta_{12} & \sin\theta_{12} & 0 \\ -\sin\theta_{12}/\sqrt{2} & \cos\theta_{12}/\sqrt{2} & -1/\sqrt{2} \\ -\sin\theta_{12}/\sqrt{2} & \cos\theta_{12}/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix} \begin{pmatrix} \cos\theta_{12} & -\sin\theta_{12}/\sqrt{2} & -\sin\theta_{12}/\sqrt{2} \\ \sin\theta_{12} & \cos\theta_{12}/\sqrt{2} & \cos\theta_{12}/\sqrt{2} \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$
$$= \begin{pmatrix} m_1\cos\theta_{12} & m_2\sin\theta_{12} & 0 \\ -m_1\sin\theta_{12}/\sqrt{2} & m_2\cos\theta_{12}/\sqrt{2} & -m_3/\sqrt{2} \\ -m_1\sin\theta_{12}/\sqrt{2} & m_2\cos\theta_{12}/\sqrt{2} & m_3/\sqrt{2} \end{pmatrix} \begin{pmatrix} \cos\theta_{12} & -\sin\theta_{12}/\sqrt{2} & -\sin\theta_{12}/\sqrt{2} \\ \sin\theta_{12} & \cos\theta_{12}/\sqrt{2} & \cos\theta_{12}/\sqrt{2} \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix},$$
$$= \begin{pmatrix} m_1^2\cos^2\theta_{12} + m_2\sin^2\theta_{12} & (m_2 - m_1)\sin\theta_{12}\cos\theta_{12} & (m_2 - m_1)\sin\theta_{12}\cos\theta_{12} \\ (m_2 - m_1)\sin\theta_{12}\cos\theta_{12} & m_1^2\sin^2\theta_{12} + m_2^2\cos^2\theta_{12} + m_3^2 & m_1^2\sin^2\theta_{12} + m_2^2\cos^2\theta_{12} - m_3^2 \\ (m_2 - m_1)\sin\theta_{12}\cos\theta_{12} & m_1^2\sin^2\theta_{12} + m_2^2\cos^2\theta_{12} - m_3^2 & m_1^2\sin^2\theta_{12} + m_2^2\cos^2\theta_{12} + m_3^2 \end{pmatrix}.$$

This is invariant under conjugation by G! Thus we have realized a reflection symmetry in the neutrino sector and a Z_3 symmetry in the charged lepton sector, both allegedly coming from the breaking of a A_4 symmetry.

Now, after the Daya Bay experiment, it is known that $\theta_{13} \neq 0$. Even so, tribimaximal mixing is perhaps not completely ruled out, since a correction for it, of the order of the Cabibblo angle, is expected [8].

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