SM and Beyond in NCG

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1 Motivation

- With the discovery of the Higgs field all the pieces of the SM are now in place.
- Within the SM settings there are many questions begging for an answer such as:
- Why the gauge group is $SU(3) \times SU(2) \times U(1)$?
- Why the Why are there 16 fermions per family in the representations $(1,1,1)$, $\left(1,2,\frac{1}{2}\right)$, $\left(3,1,-\frac{2}{3}\right)$, $\left(3,2,\frac{1}{3}\right)$ representations of $SU(3) \times SU(2) \times U(1)?$
- Why there is one Higgs doublet and how is the spontaneous symmetry breaking occurs?
- Are the three coupling constants related?
- Why the neutrinos are very light?
- Why there are three families?
- Why there is a very small CP violation of order $10^{-9}?$
- Can one predict some of the fermion masses or the Higgs mass?

At present there is no model that can answer few of these questions, and the ones that

do answer some questions are ruled out or suffer from arbitrariness.
Alain Connes in his talk set the stage for a geometrical unification of all fundamental interactions including gravity. He showed that the answer cannot be within Reimannian geometry and one must consider instead noncommutative geometry.
We proceed by assuming that:

1. Space-time is a product of a continuous four-dimensional manifold times a finite space.
2. One of the algebras $M_4(\mathbb{C})$ is subject to symplectic symmetry reducing it to $M_2(\mathbb{H})$.
3. The commutator of the Dirac operator with the center of the algebra is non trivial $[D, Z(\mathcal{A})] \neq 0$.
4. The unitary algebra $U(\mathcal{A})$ is restricted to $SU(\mathcal{A})$.
5. There is no fermion doubling problem which restricts the KO dimension of the NC space to be 2 mod 8.

We have then shown that the first possibility for the NC space is to have the algebra $M_2(\mathbb{H}) \oplus M_4(\mathbb{C})$ which is broken by the grading operator γ to $\mathbb{H} \oplus \mathbb{H} \oplus M_4(\mathbb{C})$.

At this point one imposes the first order condition $\left[[D,a]\,,\widehat{b}\right]=0$ which is necessary to make the inner fluctuations of the Dirac operator linear. As explained in Walter's talk this is a condition that could be relaxed as one can show that the order one condition is not essential to have a consistent model.
Imposing the first order condition reduces the algebra of the finite space to $\mathbb{C}\oplus\mathbb{H}\oplus M_3\left(\mathbb{C}\right)$.
This then gives rise to the following predictions:

1. The number of fundamental fermions is 16.
2. The algebra of the finite space is $\mathbb{C}\oplus\mathbb{H}\oplus M_3\left(\mathbb{C}\right)$.
3. One gets the correct representations of the fermions with respect to $SU(3)\times SU(2)\times U(1)$.
4. Higgs doublet and spontaneous symmetry breaking mechanism. This is highly non-trivial especially that the mass term of the Higgs field comes with the correct negative sign.
5. Mass of the top quark compatible with experiment.

6. See-saw mechanism to give very light left-handed neutrinos.
7. Vanishing of the θ QCD term $\epsilon^{\mu\nu\rho\sigma}V_{\mu\nu}^{m}V_{\rho\sigma}^{m}$at tree level and loop corrections can only change this by orders of less than 10^{-9}.
8. Generating of the Gibbons-Hawking boundary term for gravity with the correct sign and factor which is necessary for the consistency of obtaining a Hamiltonian for gravity.
9. Predicting the existence of a scalar field needed to give Majorana masses to the right-handed neutrinos which in turn solves th problem of the stability of the Higgs potential preventing the Higgs coupling from turning to negative.

We see that many of the questions asked in the beginning are answered in the NCG approach to the SM, but many questions still remain.
We have seen from the experimental talks that at present there are no indications of any new physics beyond the SM, but this does not rule out that some new physics will appear at very high energies. Indications that this is the case can be seen by the fact that the three gauge couplings do not meet at high energies as required by the spectral action. In addition the presence of the sigma field at energies of the order of 10^{11} Gev suggests that new physics would start to play a role at such high energies. Accepting

this lead us to consider relaxing the order one condition and to investigate which model one gets.
The first order condition is what restricted a more general gauge symmetry based on the algebra $\mathbb{H}_R \oplus \mathbb{H}_L \oplus M_4(\mathbb{C})$ to the subalgebra $\mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$. It is thus essential to understand the physical significance of such a requirement. In what follows we shall examine the more general algebra allowed without the first order condition, and shall show that the number of fundamental fermions is still dictated to be 16. We determine the inner automorphisms of the algebra $\mathcal{A}$ and show that the resulting gauge symmetry is a Pati-Salam type left-right model

$$SU(2)_R \times SU(2)_L \times SU(4)$$

where $SU(4)$ is the color group with the lepton number as the fourth color. In addition we observe that the Higgs fields appearing in $A_{(2)}$ are composite and depend quadratically on those appearing in $A_{(1)}$ provided that the initial Dirac operator (without fluctuations) satisfy the order one condition. Otherwise, there will be additional fundamental Higgs fields. In particular, the representations of the fundamental Higgs fields when the initial Dirac operator satisfies the order one condition are $(2_R, 2_L, 1)$, $(2_R, 1_L, 4)$ and $(1_R, 1_L, 1+15)$ with respect to $SU(2)_R \times SU(2)_L \times SU(4)$. When the order one condition is not satisfied for the initial Dirac operator, the representations of the additional Higgs fields are $(3_R, 1_L, 10)$, $(1_R, 1_L, 6)$ and $(2_R, 2_L, 1 + 15)$. There are simplifications if the Yukawa coupling of the up quark is equated with that of the neutrino and of the down quark equated with that of the electron. In addition the $1 + 15$ of $SU(4)$ decouple if we assume that at unification scale there is exact $SU(4)$ symmetry between the quarks and leptons. The resulting model is very similar to the one considered by Marshak and Mohapatra.

2 Doing calculations

Although it is possible to use matrix notation to deal with the physical model, the fact that the matrix representation (which is a product of matrices) is 384×384 dimensional making the task daunting and not very transparent, although only involving products of matrices. We find it much more efficient and practical to use a tensorial notation which simplifies greatly the algebraic operations. This also has the added advantage of allowing to check all the steps using computer programs with algebraic manipulations such as Mathematica and Maple.

We will restrict to the case where $Z(\mathcal{A}_{\mathbb{C}}) = \mathbb{C} \oplus \mathbb{C}$. An element of the Hilbert space

$\Psi \in \mathcal{H}$ is represented by

$$\Psi_M = \begin{pmatrix} \psi_A \\ \psi_{A'} \end{pmatrix}, \quad \psi_{A'} = \psi_A^c \tag{1}$$

where ψ_A^c is the conjugate spinor to ψ_A. Thus all primed indices A' correspond to the Hilbert space of conjugage spinors. It is acted on by both the left algebra $M_2(\mathbb{H})$ and the right algebra $M_4(\mathbb{C})$. Therefore the index A can take 16 values and is represented by

$$A = \alpha I \tag{2}$$

where the index α is acted on by quaternionic matrices and the index I by $M_4(\mathbb{C})$ matrices. Moreover, when grading breaks $M_2(\mathbb{H})$ into $\mathbb{H}_R \oplus \mathbb{H}_L$ the index α is decomposed to $\alpha = \dot{a}, a$ where $\dot{a} = \dot{1}, \dot{2}$ (dotted index) is acted on by the first quaternionic algebra $\mathbb{H}_R$ and $a = 1, 2$ is acted on by the second quaternionic algebra $\mathbb{H}_L$. When $M_4(\mathbb{C})$ breaks into $\mathbb{C} \oplus M_3(\mathbb{C})$ (due to symmetry breaking or through the use of the order one condition) the index I is decomposed into $I = 1, i$ where the 1 is acted on by the $\mathbb{C}$ and the i by $M_3(\mathbb{C})$. Therefore the various components of the spinor ψ_A are

$$\psi_{\alpha I} = \begin{pmatrix} \nu_R & u_{iR} & \nu_L & u_{iL} \\ e_R & d_{iR} & e_L & d_{iL} \end{pmatrix}$$
$$= \left(\psi_{\dot{a}1}, \psi_{\dot{a}i}, \psi_{a1}, \psi_{ai}\right), \qquad a = 1, 2, \qquad a = \dot{1}, \dot{2}, \qquad i = 1, 2, 3.$$

The power of the abstract notation can be seen by noting that the Dirac action takes the very simple form

$$\Psi_M^* D_M^N \Psi_N \tag{3}$$

which could be expanded to give

$$\psi_A^* D_A^B \psi_B + \psi_{A'}^* D_{A'}^B \psi_B + \psi_A^* D_A^{B'} \psi_{B'} + \psi_{A'}^* D_{A'}^{B'} \psi_{B'} \tag{4}$$

The Dirac operator can be written in matrix form

$$D = \begin{pmatrix} D_A^B & D_A^{B'} \\ D_{A'}^B & D_{A'}^{B'} \end{pmatrix}, \tag{5}$$

where

$$A = \alpha I, \quad \alpha = 1, \cdots, 4, \quad I = 1, \cdots, 4 \tag{6}$$

$$A' = \alpha' I', \quad \alpha' = 1', \cdots, 4', \quad I = 1', \cdots, 4' \tag{7}$$

Thus $D_A^B = D_{\alpha I}^{\beta J}$. Elements of the algebra

$$\mathcal{A} = M_4(\mathbb{C}) \oplus M_4(\mathbb{C}) \tag{8}$$

are represented by

$$a = \begin{pmatrix} X_\alpha^\beta \delta_I^J & 0 \\ 0 & \delta_{\alpha'}^{\beta'} Y_{I'}^{J'} \end{pmatrix} \tag{9}$$

where the first block is the tensor product of elements of $M_4(\mathbb{C}) \otimes 1_4$ and the second blcok is the tensor product of elments of $1_4 \otimes M_4(\mathbb{C})$. The reality operator J is anti-linear and interchange the first and second blcoks and satsify $J^2 = 1$. It is represented by

$$J = \begin{pmatrix} 0 & \delta_\alpha^{\beta'} \delta_I^{J'} \\ \delta_{\alpha'}^{\beta} \delta_{I'}^{J} & 0 \end{pmatrix} \times \text{complex conjugation} \tag{10}$$

In this form

$$a^{o}=Ja^{\ast}J^{-1}=\begin{pmatrix}\delta_{\alpha}^{\beta}Y_{I}^{tJ} & 0\\ 0 & X_{\alpha^{\prime}}^{t\beta^{\prime}}\delta_{I^{\prime}\prime}^{J^{\prime}}\end{pmatrix} \tag{11}$$

where the superscript t denotes the transpose matrix. This clearly satisfies the commutation relation

$$[a,b^{o}]=0. \tag{12}$$

Writing

$$b=\begin{pmatrix}Z_{\alpha}^{\beta}\delta_{I}^{J} & 0\\ 0 & \delta_{\alpha^{\prime}}^{\beta^{\prime}}W_{I^{\prime}}^{J^{\prime}}\end{pmatrix} \tag{13}$$

then

$$b^{o}=\begin{pmatrix}\delta_{\alpha}^{\beta}W_{I}^{tJ} & 0\\ 0 & Z_{\alpha^{\prime}}^{t\beta^{\prime}}\delta_{I^{\prime}}^{J^{\prime}}\end{pmatrix} \tag{14}$$

and so $\left[\left[D,a\right] ,b^{o}\right]$ is equal to

$$\begin{pmatrix} \left[\left[D,X\right] ,W^{t}\right]_{A}^{B} & \left(\left(DY-XD\right) Z^{t}-W^{t}\left(DY-XD\right) \right)_{A}^{B^{\prime}} \\ \left(\left(DX-YD\right) W^{t}-Z^{t}\left(DX-YD\right) \right)_{A^{\prime}}^{B} & \left[\left[D,Y\right] ,Z^{t}\right]_{A^{\prime}}^{B^{\prime}} \end{pmatrix} \tag{15}$$

The order one condition is

$$\left[\left[D,a\right] ,b^{o}\right] =0 \tag{16}$$

have only one solution with non-zero mixing between primed and unprimed indices:

$$D_{\alpha I}^{\beta^{\prime}K^{\prime}}=\delta_{\alpha}^{\dot{1}}\delta_{\dot{1}^{\prime}}^{\beta^{\prime}}\delta_{I}^{1}\delta_{1^{\prime}}^{K^{\prime}}k^{\ast\nu_{R}}\sigma \tag{17}$$

where the $k^{\ast\nu_{R}}$ are matrices in generation space which will be assumed to be 3×3. We also note that the property that $DJ=JD$ implies that

$$D_{A^{\prime}}{}^{B^{\prime}}=\overline{D}_{A}^{B}.$$

We further impose the condition of symplectic isometry on the first $M_{4}\left(\mathbb{C}\right)$

$$\left(\sigma_{2}\otimes1\right) \ \left(\overline{a}\right) \left(\sigma_{2}\otimes1\right) \ =a,\quad a\in M_{4}\left(\mathbb{C}\right)$$

which reduces $M_4(\mathbb{C})$ to $M_2(\mathbb{H})$. From the property of commutation of the grading operator G_α^β with $M_2(\mathbb{H})$

$$[G, X] = 0$$

where $G_\alpha^\beta = \begin{pmatrix} 1_2 & 0 \\ 0 & -1_2 \end{pmatrix}$, reduces the algebra $M_2(\mathbb{H})$ to $\mathbb{H}_R \oplus \mathbb{H}_L$. Thus we now have

$$X_\alpha^\beta = \begin{pmatrix} X_{\dot{a}}^{\dot{b}} & 0 \\ 0 & X_a^b \end{pmatrix}, \qquad X_a^b = \begin{pmatrix} X_1^1 & X_1^2 \\ -\overline{X}_1^2 & \overline{X}_1^1 \end{pmatrix} \in \mathbb{H}_L$$

and similarly for $X_{\dot{a}}^{\dot{b}} \in \mathbb{H}_R$. In matrix form the operator D_F has the sub-matrices

$$D_{\alpha 1}{}^{\beta 1} = \begin{pmatrix} 0 & D_{a1}^{\dot{b}1} \\ D_{\dot{a}1}^{b1} & 0 \end{pmatrix}, \qquad D_{a1}^{\dot{b}1} = \left(D_{\dot{a}1}^{b1}\right)^* \equiv D_{a(l)}^{\dot{b}}$$

$$D_{\alpha i}{}^{\beta j} = \begin{pmatrix} 0 & D_{a(q)}^{\dot{b}} \delta_i^j \\ D_{\dot{a}(q)}^{b} \delta_i^j & 0 \end{pmatrix}, \qquad D_{\dot{a}(q)}^{b} = \left(D_{a(q)}^{\dot{b}}\right)^*$$

where

$$D_{a1}^{\dot{b}1} = D_{a(l)}^{\dot{b}} = \begin{pmatrix} k^{*\nu} & 0 \\ 0 & k^{*e} \end{pmatrix}, \qquad a = 1, 2, \quad \dot{b} = \dot{1}, \dot{2}$$

and

$$D_{a(q)}^{\dot{b}} = \begin{pmatrix} k^{*u} & 0 \\ 0 & k^{*d} \end{pmatrix}.$$

The Yukawa couplings k^ν, k^e, k^u, k^d are 3×3 matrices in generation space. Notice that this structure gives Dirac masses to all the fermions, but Majorana masses only for the right-handed neutrinos. This was shown in to be the unique possibility consistent with the first order condition on the subalgebra (**??**). We can summarize all the information about the finite space Dirac operator without fluctuations, in the tensorial equation

$$\begin{aligned}
(D_F)_{\alpha I}{}^{\beta J} &= \left(\delta_\alpha^1 \delta_{\dot{1}}^\beta k^{*\nu} + \delta_\alpha^{\dot{1}} \delta_1^\beta k^\nu + \delta_\alpha^2 \delta_{\dot{2}}^\beta k^{*e} + \delta_\alpha^{\dot{2}} \delta_2^\beta k^e\right) \delta_I^1 \delta_1^J \\
&+ \left(\delta_\alpha^1 \delta_{\dot{1}}^\beta k^{*u} + \delta_\alpha^{\dot{1}} \delta_1^\beta k^u + \delta_\alpha^2 \delta_{\dot{2}}^\beta k^{*d} + \delta_\alpha^{\dot{2}} \delta_2^\beta k^d\right) \delta_I^i \delta_j^J \delta_i^j \\
(D_F)_{\alpha I}{}^{\beta' K'} &= \delta_\alpha^{\dot{1}} \delta_{\dot{1}'}^{\beta'} \delta_I^1 \delta_{1'}^{K'} k^{*\nu_R}
\end{aligned}$$

where k^{ν_R} are Yukawa couplings for the right-handed neutrinos. One can also consider the special case of lepton and quark unification by equating

$$k^{\nu} = k^{u}, \qquad k^{e} = k^{d}$$

where we expect some simplifications. From the previous discussion, it will be clear that the Dirac operator D_A including inner fluctuations $U = u\,Ju\,J^{-1}, u \in \mathcal{U}(\mathcal{A})$ would not obey the first order condition.

3 Dirac operator and Inner fluctuations on $\mathbb{H}_R \oplus \mathbb{H}_L \oplus M_4(\mathbb{C})$

When one considers inner fluctuations of the Dirac operator one finds that the gauge transformation takes the form

$$D_A \to U D_A U^{*}, \qquad U = u\,Ju\,J^{-1}, \qquad u \in \mathcal{U}(\mathcal{A})$$

which implies that

$$A \to u\,Au^{*} + u\delta(u^{*})\,.$$

This in turn gives

$$A_{(1)} \to uA_{(1)}u^* + u\left[D, u^*\right] \in \Omega_D^1\left(\mathcal{A}\right)$$
$$A_{(2)} \to Ju\,J^{-1}A_{(2)}Ju^*\,J^{-1} + Ju\,J^{-1}\left[u\left[D, u^*\right], Ju^*J^{-1}\right]$$

where the $A_{(2)}$ in the right hand side is computed using the gauge transformed $A_{(1)}$. Thus $A_{(1)}$ is a one-form and behaves like the usual gauge transformations. On the other hand $A_{(2)}$ transforms non-linearly and includes terms with quadratic dependence on the gauge transformations.

We now proceed to compute the Dirac operator on the product space $M \times F$. The initial operator is given by

$$D = \gamma^\mu D_\mu \otimes 1 + \gamma_5 D_F$$

where $\gamma^\mu D_\mu = \gamma^\mu\left(\partial_\mu + \frac{1}{4}\omega_\mu^{\ \ ab}\gamma_{ab}\right)$ is the Dirac operator on the four dimensional spin manifold. Then the Dirac operator including inner fluctuations is given by

$$D_A = D + A_{(1)} + JA_{(1)}J^{-1} + A_{(2)}$$

$$A_{(1)} = \sum a\,[D, b]$$
$$A_{(2)} = \sum a\left[JA_{(1)}J^{-1}, b\right].$$

The computation is very involved thus for clarity we shall collect all the details in the appendix and only quote the results in what follows. The different components of the operator D_A are then given by

$$\left(D_A\right)_{\dot{a}I}^{\dot{b}J} = \gamma^{\mu}\left(D_{\mu}\delta_{\dot{a}}^{\dot{b}}\delta_I^J - \frac{i}{2}g_R W_{\mu R}^{\alpha}\left(\sigma^{\alpha}\right)_{\dot{a}}^{\dot{b}}\delta_I^J - \delta_{\dot{a}}^{\dot{b}}\left(\frac{i}{2}gV_{\mu}^{m}\left(\lambda^{m}\right)_I^{\ J} + \frac{i}{2}gV_{\mu}\delta_I^J\right)\right)$$
$$\left(D_A\right)_{aI}^{bJ} = \gamma^{\mu}\left(D_{\mu}\delta_{a}^{b}\delta_I^J - \frac{i}{2}g_L W_{\mu L}^{\alpha}\left(\sigma^{\alpha}\right)_{a}^{b}\delta_I^J - \delta_{a}^{b}\left(\frac{i}{2}gV_{\mu}^{m}\left(\lambda^{m}\right)_I^{\ J} + \frac{i}{2}gV_{\mu}\delta_I^J\right)\right)$$

where the fifteen 4×4 matrices $\left(\lambda^{m}\right)_I^{\ J}$ are traceless and generate the group $SU(4)$ and $W_{\mu R}^{\alpha}$, $W_{\mu L}^{\alpha}$, V_{μ}^{m} are the gauge fields of $SU(2)_R$, $SU(2)_L$, and $SU(4)$. The requirement that A is unimodular implies that

$$\mathrm{Tr}\,(A) = 0$$

which gives the condition

$$V_\mu = 0.$$

In addition we have

$$(D_A)^{bJ}_{\dot{a}I} = \gamma_5 \left(\left(k^\nu \phi^b_{\dot{a}} + k^e \widetilde{\phi}^b_{\dot{a}} \right) \Sigma^J_I + \left(k^u \phi^b_{\dot{a}} + k^d \widetilde{\phi}^b_{\dot{a}} \right) \left(\delta^J_I - \Sigma^J_I \right) \right) \equiv \gamma_5 \Sigma^{bJ}_{\dot{a}I} \tag{18}$$

$$(D_A)^{\dot{b}'J'}_{\dot{a}I} = \gamma_5 k^{*\nu_R} \Delta_{\dot{a}J} \Delta_{\dot{b}I} \equiv \gamma_5 H_{\dot{a}I\dot{b}J}$$

where the Higgs field $\phi^b_{\dot{a}}$ is in the $\left(2_R, \overline{2}_L, 1\right)$ of the product gauge group $SU(2)_R \times SU(2)_L \times SU(4)$, and $\Delta_{\dot{a}J}$ is in the $(2_{R,}, 1_L, 4)$ representation while Σ^J_I is in the $(1_R, 1_L, 1 + 15)$ representation. The field $\widetilde{\phi}^b_{\dot{a}}$ is not an independent field and is given by

$$\widetilde{\phi}^b_{\dot{a}} = \tau_2 \overline{\phi}^b_{\dot{a}} \tau_2.$$

Note that the field Σ^J_I decouples (and set to $\delta^1_I \delta^J_1$) in the special case when there is lepton and quark unification of the couplings

$$k^\nu = k^u, \qquad k^e = k^d.$$

In case when the initial Dirac operator satisfies the order one condition, then the $A_{(2)}$

part of the connection becomes a composite Higgs field where the Higgs field $\Sigma_{\dot{a}I}^{bJ}$ is formed out of the products of the fields $\phi_{\dot{a}}^{b}$ and Σ_{I}^{J} while the Higgs field $H_{\dot{a}I\dot{b}J}$ is made from the product of $\Delta_{\dot{a}J}\Delta_{\dot{b}I}$. For generic initial Dirac operators, the field $\left(A_{(2)}\right)_{\dot{a}I}^{bJ}$ becomes independent. The fields $\Sigma_{\dot{a}I}^{bJ}$ and $H_{\dot{a}I\dot{b}J}$ will then not be defined through equation 18 and will be in the $(2_R, 2_L, 1+15)$ and $(3_R, 1_L, 10) + (1_R, 1_L, 6)$ representations of $SU(2)_R \times SU(2)_L \times SU(4)$. In addition, for generic Dirac operator one also generate the fundamental field $(1, 2_L, 4)$. The fact that inner automorphisms form a semigroup implies that the cases where the Higgs fields contained in the connections $A_{(2)}$ are either independent fields or depend quadratically on the fundamental Higgs fields are disconnected. The interesting question that needs to be addressed is whether the structure of the connection is preserved at the quantum level. This investigation must be performed in such a way as to take into account the noncommutative structure of the space. At any rate, we have here a clear advantage over grand unified theories which suffers of having arbitrary and complicated Higgs representations . In the noncommutative geometric setting, this problem is now solved by having minimal representations of the Higgs fields. Remarkably, we note that a very close model to the one deduced here is the one considered by Marshak and Mohapatra where the $U(1)$ of the left-right model is identified with the $B-L$ symmetry. They proposed the same Higgs fields that would result starting with a generic initial Dirac operator not satisfying the first order condition. Although the broken generators of the $SU(4)$ gauge fields can mediate lepto-quark interactions leading to proton decay, it was shown that in all such types of models with partial unification, the proton is stable. In addition this type of model arises in the first phase of breaking of $SO(10)$ to $SU(2)_R \times SU(2)_L \times SU(4)$ and these have been extensively studied. The recent work in considers noncommutative grand unification based on the $k=8$ algebra $M_4(\mathbb{H}) \oplus M_8(\mathbb{C})$ keeping the first order condition.

4 The Spectral Action for the $SU(2)_R \times SU(2)_L \times SU(4)$ model

Having determined the Dirac operator acting on the Hilbert space of spinors in terms of the gauge fields of $SU(2)_R \times SU(2)_L \times SU(4)$ and Higgs fields, some of which are fundamental while others are composite, the next step is to study the dynamics of these fields as governed by the spectral action principle. The geometric invariants of the noncommutative space are encoded in the spectrum of the Dirac operator D_A. The bosonic action is given by

$$\mathrm{Trace}\,\left(f\,\left(D_A/\Lambda\right)\right)$$

where Λ is some cutoff scale and the function f is restricted to be even and positive. Using heat kernel methods the trace can be expressed in terms of Seeley-de Witt coefficients a_n :

$$\mathrm{Trace}\,f\,\left(D_A/\Lambda\right) = \sum_{n=0}^{\infty} F_{4-n}\Lambda^{4-n}a_n$$

where the function F is defined by $F(u) = f\,(v)$ where $u = v^2$, thus $F(D^2) = f\,(D)$. We define

$$f_k = \int_0^{\infty} f\,(v)\,v^{k-1}dv, \qquad k > 0$$

then

$$
\begin{aligned}
F_4 &= \int_0^\infty F(u)udu = 2\int_0^\infty f(v)v^3dv = 2f_4 \\
F_2 &= \int_0^\infty F(u)du = 2\int_0^\infty f(v)vdv = 2f_2 \\
F_0 &= F(0) = f\,(0) = f_0 \\
F_{-2n} &= (-1)^n F^{(n)}(0) = \left[(-1)^n \left(\frac{1}{2v}\frac{d}{dv}\right)^n f\right](0) \qquad n \geq 1.
\end{aligned}
$$

Using the same notation and formulas as in reference [**?**], the first Seeley-de Witt coefficient is

$$a_0 = \frac{1}{16\pi^2}\int d^4x\sqrt{g}\text{Tr}\,(1)$$
$$= \frac{1}{16\pi^2}\,(4)\,(32)\,(3)\int d^4x\sqrt{g}$$
$$= \frac{24}{\pi^2}\int d^4x\sqrt{g}$$

where the numerical factors come, respectively, from the traces on the Clifford algebra, the dimensions of the Hilbert space and number of generations. The second coefficient is

$$a_2 = \frac{1}{16\pi^2}\int d^4x\sqrt{g}\mathrm{Tr}\left(E + \frac{1}{6}R\right)$$

where E is a 384×384 matrix over Hilbert space of three generations of spinors, whose components are derived and listed in the appendix. Taking the various traces we get

$$\begin{aligned} a_2 &= \frac{1}{16\pi^2}\int d^4x\sqrt{g}\left((R(-96+64)-8\left(H_{\dot{a}I\dot{c}K}H^{\dot{c}K\dot{a}I}+2\Sigma_{\dot{a}I}^{cK}\Sigma_{cK}^{\dot{a}I}\right)\right) \\ &= -\frac{2}{\pi^2}\int d^4x\sqrt{g}\left(R+\frac{1}{4}\left(H_{\dot{a}I\dot{c}K}H^{\dot{c}K\dot{a}I}+2\Sigma_{\dot{a}I}^{cK}\Sigma_{cK}^{\dot{a}I}\right)\right). \end{aligned}$$

It should be understood in the above formula and in what follows, that whenever the matrices $k^{\nu}, k^{u}, k^{e}, k^{d}$ and k^{ν_R} appear in an action, one must take the trace over generation space. When the initial Dirac operator without fluctuations is taken to satisfy the order one condition, the fields $H_{\dot{a}I\dot{c}K}$ and $\Sigma_{\dot{a}I}^{cK}$ will become dependent on the fundamental Higgs fields. In this case, the mass terms can be expressed in terms of the fundamental Higgs field to give

$$H_{\dot{a}I\dot{c}K}H^{\dot{c}K\dot{a}I} = \left|k^{\nu_R}\right|^2\left(\Delta_{\dot{a}K}\overline{\Delta}^{\dot{a}K}\right)^2$$

and

$$2\Sigma_{\dot{a}I}^{cK}\Sigma_{cK}^{\dot{a}I} = 2\left(\left(\left(k^{\nu}-k^{u}\right)\phi_{\dot{a}}^{c}+\left(k^{e}-k^{d}\right)\widetilde{\phi}_{\dot{a}}^{c}\right)\Sigma_{I}^{K}+\left(k^{u}\phi_{\dot{a}}^{c}+k^{d}\widetilde{\phi}_{\dot{a}}^{c}\right)\delta_{I}^{K}\right)$$
$$\left(\left(\left(k^{*\nu}-k^{*u}\right)\phi_{c}^{\dot{a}}+\left(k^{*e}-k^{*d}\right)\widetilde{\phi}_{c}^{\dot{a}}\right)\Sigma_{K}^{I}+\left(k^{*u}\phi_{c}^{\dot{a}}+k^{*d}\widetilde{\phi}_{c}^{\dot{a}}\right)\delta_{K}^{I}\right).$$

The next coefficient is

$$a_4=\frac{1}{16\pi^2}\int d^4x\sqrt{g}\mathrm{Tr}\left(\frac{1}{360}\left(5R^2-2R_{\mu\nu}^2+2R_{\mu\nu\rho\sigma}^2\right)1+\frac{1}{2}\left(E^2+\frac{1}{3}RE+\frac{1}{6}\Omega_{\mu\nu}^2\right)\right)$$

where $\Omega_{\mu\nu}$ is the 384×384 curvature matrix of the connection ω_{μ}. Using the expressions for the matrices E and $\Omega_{\mu\nu}$ derived in the appendix, and taking the traces, we get

$$a_4=\frac{1}{2\pi^2}\int d^4x\sqrt{g}\left[-\frac{3}{5}C_{\mu\nu\rho\sigma}^2+\frac{11}{30}R^*R^*+g_L^2\left(W_{\mu\nu L}^{\alpha}\right)^2+g_R^2\left(W_{\mu\nu R}^{\alpha}\right)^2+g^2\left(V_{\mu\nu}^{m}\right)^2\right.$$
$$+\nabla_{\mu}\Sigma_{aI}^{\dot{c}K}\nabla^{\mu}\Sigma_{\dot{c}K}^{aI}+\frac{1}{2}\nabla_{\mu}H_{\dot{a}I\dot{b}J}\nabla^{\mu}H^{\dot{a}I\dot{b}J}+\frac{1}{12}R\left(H_{\dot{a}I\dot{c}K}H^{\dot{c}K\dot{a}I}+2\Sigma_{\dot{a}I}^{cK}\Sigma_{cK}^{\dot{a}I}\right)$$
$$\left.+\frac{1}{2}\left|H_{\dot{a}I\dot{c}K}H^{\dot{c}K\dot{b}J}\right|^2+2H_{\dot{a}I\dot{c}K}\Sigma_{bJ}^{\dot{c}K}H^{\dot{a}I\dot{d}L}\Sigma_{\dot{d}L}^{bJ}+\Sigma_{aI}^{\dot{c}K}\Sigma_{\dot{c}K}^{bJ}\Sigma_{bJ}^{\dot{d}L}\Sigma_{\dot{d}L}^{aI}\right]$$

where $C_{\mu\nu\rho\sigma}$ is the Weyl tensor. Thus the bosonic spectral action to second order is given by

$$S=F_4\Lambda^4a_0+F_2\Lambda^2a_2+F_0a_4+\cdots$$

which finally gives

$$\begin{aligned}
S_{\mathrm{b}}=&\frac{24}{\pi^2}F_4\Lambda^4\int d^4x\sqrt{g}\\
&-\frac{2}{\pi^2}F_2\Lambda^2\int d^4x\sqrt{g}\left(R+\frac{1}{4}\left(H_{\dot{a}I\dot{c}K}H^{\dot{c}K\dot{a}I}+2\Sigma_{\dot{a}I}^{cK}\Sigma_{cK}^{\dot{a}I}\right)\right)\\
&+\frac{1}{2\pi^2}F_0\int d^4x\sqrt{g}\left[\frac{1}{30}\left(-18C_{\mu\nu\rho\sigma}^2+11R^*R^*\right)+g_L^2\left(W_{\mu\nu L}^{\alpha}\right)^2+g_R^2\left(W_{\mu\nu R}^{\alpha}\right)^2+g^2\left(V_{\mu\nu}^{m}\right)^2\right.\\
&+\ \nabla_\mu\Sigma_{aI}^{\dot{c}K}\nabla^\mu\Sigma_{\dot{c}K}^{aI}+\frac{1}{2}\nabla_\mu H_{\dot{a}I\dot{b}J}\nabla^\mu H^{\dot{a}I\dot{b}J}+\frac{1}{12}R\left(H_{\dot{a}I\dot{c}K}H^{\dot{c}K\dot{a}I}+2\Sigma_{\dot{a}I}^{cK}\Sigma_{cK}^{\dot{a}I}\right)\\
&\left.+\frac{1}{2}\left|H_{\dot{a}I\dot{c}K}H^{\dot{c}K\dot{b}J}\right|^2+2H_{\dot{a}I\dot{c}K}\Sigma_{bJ}^{\dot{c}K}H^{\dot{a}I\dot{d}L}\Sigma_{\dot{d}L}^{bJ}+\Sigma_{aI}^{\dot{c}K}\Sigma_{\dot{c}K}^{bJ}\Sigma_{bJ}^{\dot{d}L}\Sigma_{\dot{d}L}^{aI}\right].
\end{aligned}$$

The physical content of this action is a cosmological constant term, the Einstein Hilbert term R, a Weyl tensor square term $C_{\mu\nu\rho\sigma}^2$, kinetic terms for the $SU(2)_R\times SU(2)_L\times$

$SU(4)$ gauge fields, kinetic terms for the composite Higgs fields $H_{\dot{a}I\dot{b}J}$ and $\Sigma_{bJ}^{\dot{c}K}$ as well as mass terms and quartic terms for the Higgs fields. This is a grand unified Pati-Salam type model with a completely fixed Higgs structure which we expect to spontaneously break at very high energies to the $U(1) \times SU(2) \times SU(3)$ symmetry of the SM. We also notice that this action gives the gauge coupling unification

$$g_R = g_L = g.$$

A test of this model is to check whether this relation when run using RG equations would give values consistent with the values of the gauge couplings for electromagnetic, weak and strong interactions at the scale of the Z -boson mass. Having determined the full Dirac operators, including fluctuations, we can write all the fermionic interactions including the ones with the gauge vectors and Higgs scalars. It is given by

$$\int d^4x\sqrt{g}\left\{\psi^*_{\dot{a}I}\gamma^\mu\left(D_\mu\delta^{\dot{b}}_{\dot{a}}\delta^J_I-\frac{i}{2}g_R W^\alpha_{\mu R}\left(\sigma^\alpha\right)^{\dot{b}}_{\dot{a}}\delta^J_I-\delta^{\dot{b}}_{\dot{a}}\left(\frac{i}{2}gV^m_\mu\left(\lambda^m\right)^J_I+\frac{i}{2}gV_\mu\delta^J_I\right)\right)\psi_{\dot{b}J}\right.$$

$$+\psi^*_{aI}\gamma^\mu\left(D_\mu\delta^b_a\delta^J_I-\frac{i}{2}g_L W^\alpha_{\mu L}\left(\sigma^\alpha\right)^b_a\delta^J_I-\delta^b_a\left(\frac{i}{2}gV^m_\mu\left(\lambda^m\right)^J_I+\frac{i}{2}gV_\mu\delta^J_I\right)\right)\psi_{bJ}$$

$$\left.+\psi^*_{\dot{a}I}\gamma_5\Sigma^{bJ}_{\dot{a}I}\psi_{bJ}+\psi^*_{aI}\gamma_5\Sigma^{\dot{b}J}_{aI}\psi_{\dot{b}J}+C\psi_{\dot{a}I}\gamma_5 H^{\dot{a}I\dot{b}J}\psi_{\dot{b}J}+\mathrm{h.c}\right\}$$

Truncation to the Standard Model.

It is easy to see that this model truncates to the Standard Model. The Higgs field $\phi^b_{\dot{a}}$ $=(2_R,2_L,1)$ must be truncated to the Higgs doublet H by writing

$$\phi^b_{\dot{a}}=\delta^{\dot{1}}_{\dot{a}}\epsilon^{bc}H_c.$$

The other Higgs field $\Delta_{\dot{a}I}=(2_R,1,4)$ is truncated to a real singlet scalar field

$$\Delta_{\dot{a}I}=\delta^{\dot{1}}_{\dot{a}}\delta^1_I\sqrt{\sigma}.$$

These then imply the relations

$$
\begin{aligned}
\Sigma_{\dot{a}I}^{bJ} &= \left(\delta_{\dot{a}}^{\dot{1}} k^{\nu} \epsilon^{bc} H_c + \delta_{\dot{a}}^{\dot{2}} \overline{H}^b k^e\right) \delta_I^1 \delta_1^J + \left(\delta_{\dot{a}}^{\dot{1}} k^u \epsilon^{bc} H_c + \delta_{\dot{a}}^{\dot{2}} k^d \overline{H}^b\right) \delta_I^i \delta_j^J \delta_i^j \\
H_{\dot{a}I\dot{b}J} &= \delta_{\dot{a}}^{\dot{1}} \delta_{\dot{b}}^{\dot{1}} k^{\nu_R} \delta_I^1 \delta_1^J \sigma \\
g_R W_{\mu R}^3 &= g_1 B_\mu, \qquad W_{\mu R}^{\pm} = 0 \\
\sqrt{\frac{3}{2}} g V_\mu^{15} &= -g_1 B_\mu \qquad (V_\mu)_1^i = 0
\end{aligned}
$$

where V_μ^{15} is the $SU(4)$ gauge field corresponding to the generator

$$
\lambda^{15} = \frac{1}{\sqrt{6}} \mathrm{diag}\left(3, -1, -1, -1\right)
$$

which could be identified with the $B - L$ generator. In particular the components $(D_A)_{\dot{1}1}^{\dot{1}1}$

and $(D_A)^{\dot{2}1}_{\dot{2}1}$ of the Dirac operator simplify to

$$\begin{aligned}(D_A)^{\dot{1}1}_{\dot{1}1} &= \gamma^\mu \left(D_\mu - \frac{i}{2} g_R W^\alpha_{\mu R} (\sigma^\alpha)^{\dot{1}}_{\dot{1}} - \left(\frac{i}{2} g V^m_\mu (\lambda^m)^1_1 \right) \right) \\ &= \gamma^\mu \left(D_\mu - \frac{i}{2} g_R W^3_{\mu R} - \left(\frac{i}{2} g V^{15}_\mu \sqrt{\frac{3}{2}} \right) \right) \\ &= \gamma^\mu D_\mu\end{aligned}$$

$$\begin{aligned}(D_A)^{\dot{2}1}_{\dot{2}1} &= \gamma^\mu \left(D_\mu - \frac{i}{2} g_R W^\alpha_{\mu R} (\sigma^\alpha)^{\dot{2}}_{\dot{2}} - \left(\frac{i}{2} g V^m_\mu (\lambda^m)^1_1 \right) \right) \\ &= \gamma^\mu \left(D_\mu + \frac{i}{2} g_R W^3_{\mu R} - \left(\frac{i}{2} g V^{15}_\mu \sqrt{\frac{3}{2}} \right) \right) \\ &= \gamma^\mu (D_\mu + i g_1 B_\mu)\end{aligned}$$

which are identified with the Dirac operators acting on the right-handed neutrino and right-handed electron. Similar substitutions give the action of the Dirac operators on the remaining fermions and give the expected results. We now compute the various terms in the spectral action. First for the mass terms we have

$$\begin{aligned}
\frac{1}{4}H_{\dot{a}I\dot{b}J}H^{\dot{b}J\dot{a}I} &= \frac{1}{4}\left(\delta_{\dot{a}}^{1}\delta_{\dot{b}}^{1}k^{\nu_R}\delta_{I}^{1}\delta_{1}^{J}\sigma\right)\left(\delta_{1}^{\dot{a}}\delta_{1}^{\dot{b}}\delta_{1}^{I}\delta_{1}^{J}k^{*\nu_R}\sigma\right) \\
&= \frac{1}{4}\mathrm{tr}\left|k^{\nu_R}\right|^2\sigma^2 = \frac{1}{4}c\sigma^2 \\
\frac{1}{2}\Sigma_{\dot{a}I}^{cK}\Sigma_{cK}^{\dot{a}I} &= \frac{1}{2}\left|\left(\delta_{\dot{a}}^{\dot{1}}k^{\nu}\epsilon^{bc}H_c+\delta_{\dot{a}}^{\dot{2}}\overline{H}^{b}k^{e}\right)\delta_{I}^{1}\delta_{1}^{J}+\left(\delta_{\dot{a}}^{\dot{1}}k^{u}\epsilon^{bc}H_c+\delta_{\dot{a}}^{\dot{2}}k^{d}\overline{H}^{b}\right)\delta_{I}^{i}\delta_{j}^{J}\delta_{i}^{j}\right|^2 \\
&= \frac{1}{2}a\overline{H}H
\end{aligned}$$

where

$$\begin{aligned}
a &= \mathrm{tr}\left(k^{*\nu}k^{\nu}+k^{*e}k^{e}+3\left(k^{*u}k^{u}+k^{*d}k^{d}\right)\right) \\
c &= \mathrm{tr}\left(k^{*\nu_R}k^{\nu_R}\right)
\end{aligned}$$

Next for the a_4 term, starting with the gauge kinetic energies we have

$$g_L^2\left(W_{\mu\nu L}^{\alpha}\right)^2+g_R^2\left(W_{\mu\nu R}^{\alpha}\right)^2+g^2\left(V_{\mu\nu}^{m}\right)^2\rightarrow g_L^2\left(W_{\mu\nu L}^{\alpha}\right)^2+\frac{5}{3}g_1^2B_{\mu\nu}^2+g_3^2\left(V_{\mu\nu}^{m}\right)^2$$

where $m = 1,\cdots,8$ for $V_{\mu\nu}^{m}$ restricted to the $SU(3)$ gauge group. Next for the Higgs kinetic and quartic terms we have

$$
\begin{aligned}
\nabla_\mu \Sigma_{aI}^{\dot{c}K} \nabla^\mu \Sigma_{\dot{c}K}^{aI} &\rightarrow a \nabla_\mu \overline{H} \nabla^\mu H \\
\frac{1}{2} \nabla_\mu H_{\dot{a}I\dot{b}J} \nabla^\mu H^{\dot{a}I\dot{b}J} &\rightarrow \frac{1}{2} c \partial_\mu \sigma \partial^\mu \sigma \\
\frac{1}{12} R \left(H_{\dot{a}I\dot{c}K} H^{\dot{c}K\dot{a}I} + 2 \Sigma_{\dot{a}I}^{cK} \Sigma_{cK}^{\dot{a}I} \right) &\rightarrow \frac{1}{12} R \left(2a \overline{H} H + c \sigma^2 \right) \\
\frac{1}{2} \left| H_{\dot{a}I\dot{c}K} H^{\dot{c}K\dot{b}J} \right|^2 &\rightarrow \frac{1}{2} d \sigma^4 \\
2 H_{\dot{a}I\dot{c}K} \Sigma_{bJ}^{\dot{c}K} H^{\dot{a}I\dot{d}L} \Sigma_{\dot{d}L}^{bJ} &\rightarrow 2e \overline{H} H \sigma^2 \\
\Sigma_{aI}^{\dot{c}K} \Sigma_{\dot{c}K}^{bJ} \Sigma_{bJ}^{\dot{d}L} \Sigma_{\dot{d}L}^{aI} &\rightarrow b \left(\overline{H} H \right) .^2
\end{aligned}
$$

Collecting all terms we end up with the bosonic action for the Standard Model:

$$
\begin{aligned}
S_{\mathrm{b}} = & \frac{24}{\pi^2}F_4\Lambda^4\int d^4x\sqrt{g} \\
& -\frac{2}{\pi^2}F_2\Lambda^2\int d^4x\sqrt{g}\left(R+\frac{1}{2}a\overline{H}H+\frac{1}{4}c\sigma^2\right) \\
& +\frac{1}{2\pi^2}F_0\int d^4x\sqrt{g}\left[\frac{1}{30}\left(-18C^2_{\mu\nu\rho\sigma}+11R^*R^*\right)+\frac{5}{3}g_1^2B^2_{\mu\nu}+g_2^2\left(W^\alpha_{\mu\nu}\right)^2+g_3^2\left(V^m_{\mu\nu}\right)^2\right. \\
& \qquad \left.+\frac{1}{6}aR\overline{H}H+b\left(\overline{H}H\right)^2+a\left|\nabla_\mu H_a\right|^2+2e\overline{H}H\,\sigma^2+\frac{1}{2}d\,\sigma^4+\frac{1}{12}cR\sigma^2+\frac{1}{2}c\left(\partial_\mu\sigma\right)^2\right]
\end{aligned}
$$

where

$$
\begin{aligned}
b &= \mathrm{tr}\left(\left(k^{*\nu}k^{\nu}\right)^2+\left(k^{*e}k^{e}\right)^2+3\left(\left(k^{*u}k^{u}\right)^2+\left(k^{*d}k^{d}\right)^2\right)\right) \\
d &= \mathrm{tr}\left(\left(k^{*\nu_R}k^{\nu_R}\right)^2\right) \\
e &= \mathrm{tr}\left(k^{*\nu}k^{\nu}k^{*\nu_R}k^{\nu_R}\right).
\end{aligned}
$$

This action completely agrees with the results in reference.

5 The potential and symmetry breaking

We now study the resulting potential and try to investigate the possible minima:

$$V = \frac{F_0}{2\pi^2}\left(\frac{1}{2}\left|H_{\dot{a}I\dot{c}K}H^{\dot{c}K\dot{b}J}\right|^2 + 2H_{\dot{a}I\dot{c}K}\Sigma_{bJ}^{\dot{c}K}H^{\dot{a}I\dot{d}L}\Sigma_{\dot{d}L}^{bJ} + \Sigma_{aI}^{\dot{c}K}\Sigma_{\dot{c}K}^{bJ}\Sigma_{bJ}^{\dot{d}L}\Sigma_{\dot{d}L}^{aI}\right)$$
$$-\frac{F_2}{2\pi^2}\left(H_{\dot{a}I\dot{c}K}H^{\dot{c}K\dot{a}I} + 2\Sigma_{\dot{a}I}^{cK}\Sigma_{cK}^{\dot{a}I}\right).$$

However, the Higgs field here are not fundamental and we have to express the potential in terms of the fundamental Higgs fields $\phi_{\dot{a}}^{c}$, $\Delta_{\dot{a}K}$ and Σ_{K}^{I}. Expanding the composite Higgs fields in terms of the fundamental ones, we have for the quartic terms

$$\frac{1}{2}\left|H_{\dot{a}I\dot{c}K}H^{\dot{c}K\dot{b}J}\right|^2 = \frac{1}{2}\left|k^{\nu_R}\right|^4\left(\Delta_{\dot{a}K}\overline{\Delta}^{\dot{a}L}\Delta_{\dot{b}L}\overline{\Delta}^{\dot{b}K}\right)^2$$

$$\Sigma_{aI}^{\dot{c}K}\Sigma_{\dot{c}K}^{bJ}\Sigma_{bJ}^{\dot{d}L}\Sigma_{\dot{d}L}^{aI}=\left(\left(\left(k^{*\nu}-k^{*u}\right)\phi_{a}^{\dot{c}}+\left(k^{*e}-k^{*d}\right)\widetilde{\phi}_{a}^{\dot{c}}\right)\Sigma_{I}^{K}+\left(k^{*u}\phi_{a}^{\dot{c}}+k^{*d}\widetilde{\phi}_{a}^{\dot{c}}\right)\delta_{I}^{K}\right)$$

$$\left(\left(\left(k^{\nu}-k^{u}\right)\phi_{\dot{c}}^{b}+\left(k^{e}-k^{d}\right)\widetilde{\phi}_{\dot{c}}^{b}\right)\Sigma_{K}^{J}+\left(k^{u}\phi_{\dot{c}}^{b}+k^{d}\widetilde{\phi}_{\dot{c}}^{b}\right)\delta_{K}^{J}\right)$$

$$\left(\left(\left(k^{*\nu}-k^{*u}\right)\phi_{b}^{\dot{d}}+\left(k^{*e}-k^{*d}\right)\widetilde{\phi}_{b}^{\dot{d}}\right)\Sigma_{J}^{L}+\left(k^{*u}\phi_{b}^{\dot{d}}+k^{*d}\widetilde{\phi}_{b}^{\dot{d}}\right)\delta_{J}^{L}\right)$$

$$\left(\left(\left(k^{\nu}-k^{u}\right)\phi_{\dot{d}}^{a}+\left(k^{e}-k^{d}\right)\widetilde{\phi}_{\dot{d}}^{a}\right)\Sigma_{L}^{I}+\left(k^{u}\phi_{\dot{d}}^{a}+k^{d}\widetilde{\phi}_{\dot{d}}^{a}\right)\delta_{L}^{I}\right)$$

$$2H_{\dot{a}I\dot{c}K}\Sigma_{bJ}^{\dot{c}K}H^{\dot{a}I\dot{d}L}\Sigma_{\dot{d}L}^{bJ}=2\left|k^{\nu_{R}}\right|^{2}\left(\Delta_{\dot{a}K}\overline{\Delta}^{\dot{a}L}\Delta_{\dot{c}I}\overline{\Delta}^{\dot{d}I}\right)$$

$$\left(\left(\left(k^{*\nu}-k^{*u}\right)\phi_{b}^{\dot{c}}+\left(k^{*e}-k^{*d}\right)\widetilde{\phi}_{b}^{\dot{c}}\right)\Sigma_{J}^{K}+\left(k^{*u}\phi_{b}^{\dot{c}}+k^{*d}\widetilde{\phi}_{b}^{\dot{c}}\right)\delta_{J}^{K}\right)$$

$$\left(\left(\left(k^{\nu}-k^{u}\right)\phi_{\dot{d}}^{b}+\left(k^{e}-k^{d}\right)\widetilde{\phi}_{\dot{d}}^{b}\right)\Sigma_{L}^{J}+\left(k^{u}\phi_{\dot{d}}^{b}+k^{d}\widetilde{\phi}_{\dot{d}}^{b}\right)\delta_{L}^{J}\right).$$

Next we have the mass terms

$$H_{\dot{a}I\dot{c}K}H^{\dot{c}K\dot{a}I}=\left|k^{\nu_{R}}\right|^{2}\left(\Delta_{\dot{a}K}\overline{\Delta}^{\dot{a}K}\right)^{2}$$

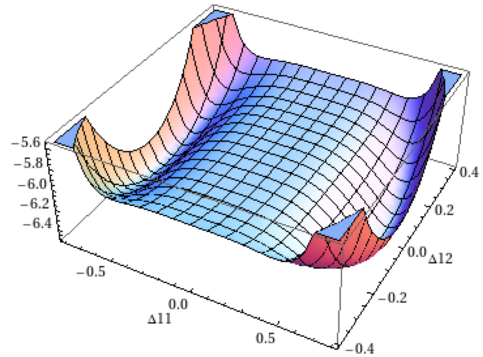
and

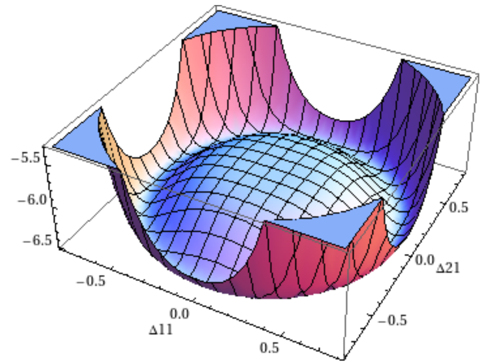
$$2\Sigma_{\dot{a}I}^{cK}\Sigma_{cK}^{\dot{a}I}=2\left(\left(\left(k^{\nu}-k^{u}\right)\phi_{\dot{a}}^{c}+\left(k^{e}-k^{d}\right)\widetilde{\phi}_{\dot{a}}^{c}\right)\Sigma_{I}^{K}+\left(k^{u}\phi_{\dot{a}}^{c}+k^{d}\widetilde{\phi}_{\dot{a}}^{c}\right)\delta_{I}^{K}\right)$$
$$\left(\left(\left(k^{*\nu}-k^{*u}\right)\phi_{c}^{\dot{a}}+\left(k^{*e}-k^{*d}\right)\widetilde{\phi}_{c}^{\dot{a}}\right)\Sigma_{K}^{I}+\left(k^{*u}\phi_{c}^{\dot{a}}+k^{*d}\widetilde{\phi}_{c}^{\dot{a}}\right)\delta_{K}^{I}\right).$$

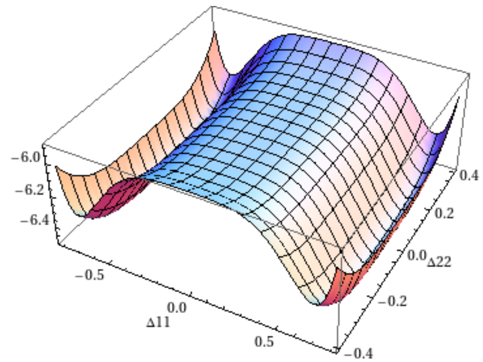
The potential must be analyzed to determine all the possible minima that breaks the symmetry $SU(2)_R\times SU(2)_L\times SU(4)$. In this respect it is useful to determine whether the symmetries of this model break correctly at high energies to the Standard Model. Needless to say that it is difficult to determine all allowed vacua of this potential, especially since there is dependence of order eight on the fields. It is possible, however, to expand this potential around the vacuum that we started with which breaks the gauge symmetry directly from $SU(2)_R\times SU(2)_L\times SU(4)$ to $U(1)_{\mathrm{em}}\times SU(3)_{\mathrm{c}}$. Explicitly, this vacuum is given by

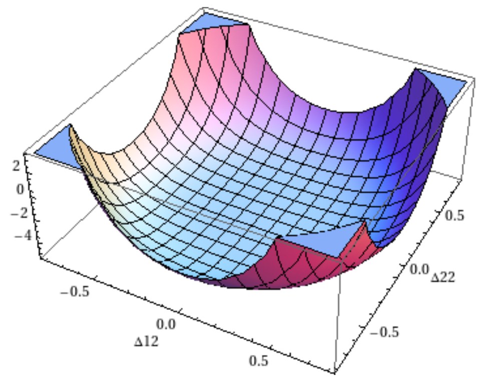
$$\left\langle\phi_{\dot{a}}^{b}\right\rangle=v\delta_{\dot{a}}^{\dot{1}}\delta_{1}^{b}\qquad\left\langle\Sigma_{J}^{I}\right\rangle=u\delta_{1}^{I}\delta_{J}^{1}\qquad\left\langle\Delta_{\dot{a}J}\right\rangle=w\delta_{\dot{a}}^{\dot{1}}\delta_{J}^{1}.$$

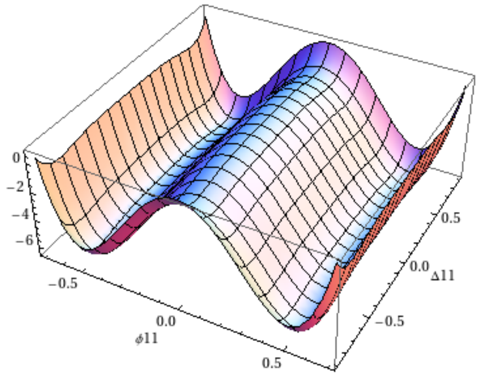
We have included several plots of the scalar potential in the $\Delta_{\dot{a}J}$-directions in Figure. A computation of the Hessian in the Δ-directions shows that the SM-vev is indeed a local minimum.


−5.6
−5.8
−6.0
−6.2
−6.4
−0.5
0.0
Δ11
0.5
0.4
0.2
0.0
Δ12
−0.2
−0.4


−5.5
−6.0
−6.5
−0.5
0.0
Δ11
0.5
0.5
0.0
Δ21
−0.5


−6.0
−6.2
−6.4
−0.5
0.0
Δ11
0.5
0.4
0.2
0.0
Δ22
−0.2
−0.4


2
0
−2
−4
−0.5
0.0
Δ12
0.5
0.5
0.0
Δ22
−0.5

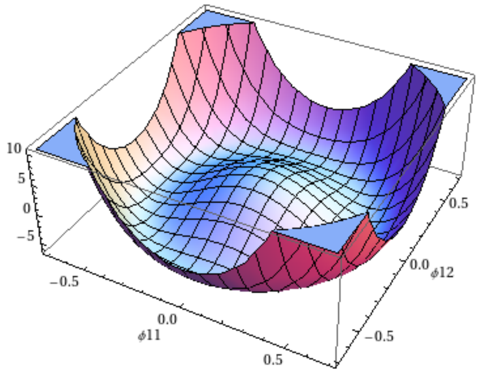


The first order condition now arises as a vacuum solution of the spectral action as follows. We let the Δ-fields take their vev according to the scalar potential, *i.e.* $\Delta_{\dot{a}J} = w\delta_{\dot{a}}^{\dot{1}}\delta_J^1$. Since $\Delta_{\dot{a}J}$ is in the $(2_R, 1_L, 4)$ representation of $SU(2)_R \times SU(2)_L \times SU(4)$, this vacuum solution is only invariant under the subgroup

$$\left\{ \left(\begin{pmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{pmatrix}, u_L, \lambda \oplus \lambda^{-1/3} u \right) : \lambda \in U(1), u_L \in SU(2), u \in SU(3) \right\} \subset SU(2)_R \times SU(2)_L \times SU(4).$$

This is the spontaneous symmetry breaking to $U(1) \times SU(2)_L \times SU(3)_{\mathrm{c}}$, thus selecting the subalgebra (**??**). Note that unimodularity on $\mathcal{U}(\mathcal{A})$ naturally induces unimodularity of the spectral Standard Model, hence it generates the correct hypercharges for the fermions.

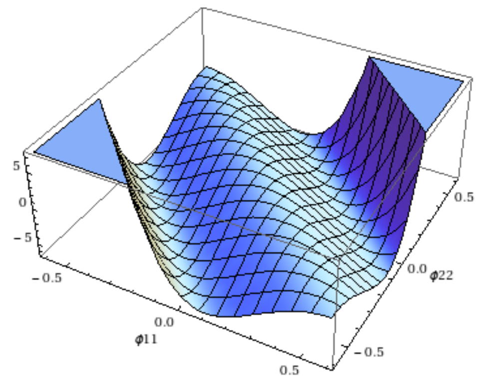
After the Δ and Σ-fields have acquired their vevs, there is a remaining scalar potential for the ϕ-fields, which is depicted in Figure



. As with the Standard Model Higgs sector, the selection of a minimum further breaks the symmetry from $U(1) \times SU(2)_L \times SU(3)_{\mathrm{c}}$ to $U(1)_{\mathrm{em}} \times SU(3)_{\mathrm{c}}$. The plot on the right in Figure
suggests that, instead of the SM-vacuum, the vevs of the ϕ-fields can also be taken of the form

$$\left\langle \phi_{\dot{a}}^{b} \right\rangle = v\delta_{\dot{a}}^{\dot{1}}\delta_{1}^{b} + v'\delta_{\dot{a}}^{\dot{2}}\delta_{2}^{b}.$$

Let us see which of the gauge fields acquire non-zero mass after spontaneous sym-


5
0
−5
−0.5
0.0
ϕ11
0.5
0.5
0.0
ϕ22
−0.5

metry breaking, by expanding around the Standard Model vacuum

$$\phi_{\dot{a}}^{b} = v\delta_{\dot{a}}^{\dot{1}}\delta_{1}^{b} + H_{\dot{a}}^{b}$$
$$\Sigma_{J}^{I} = u\delta_{1}^{I}\delta_{J}^{1} + M_{I}^{J}$$
$$\Delta_{\dot{a}J} = w\delta_{\dot{a}}^{\dot{1}}\delta_{J}^{1} + N_{\dot{a}J}$$

and keep only terms of up to order 4.

6 Conclusions

Relaxing the order one condition which may be required in the process of renormalizing the spectral action leads uniquely to the Pati-Salam model with $SU(2)_R \times SU(2)_L \times SU(4)$ symmetry unifying leptons and quarks with the lepton number as the fourth color. The Higgs fields are fixed and belong to the 16×16 and $16 \times \overline{16}$ products with respect to the Pati-Salam group. Because of the semi-group structure of the inner fluctuations the Higgs fields may all be independent of each other, or the $A_{(2)}$ part of the connection depending on the $A_{(1)}$ parts provided that the initial Dirac operator is taking to satisfy the order one condition with respect to the SM algebra. The model, unlike other unification models does not suffer from prton decay and is not ruled out experimentally. A lot of work remains to be done to investigate this model and study all its possible breakings from the high energy to low energies. Of interest is to determine whether the additional fields present will modify the running of the gauge couplings allowing for the meetings of these couplings at very high energies.