Inner perturbations in noncommutative geometry

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Overview

- Gauge theory from spectral triples
- Gauge group, semi-group of inner perturbations
- Examples: Yang–Mills, almost-commutative manifolds, SM

Spectral triples

$$(\mathcal{A}, \mathcal{H}, D)$$

- Extended to real, even spectral triple:
  - $J : \mathcal{H} \to \mathcal{H}$ real structure (anti-unitary)
  - $\gamma : \mathcal{H} \to \mathcal{H}$ grading $\gamma^2 = 1$ (self-adjoint)

  such that

$$J^2 = \pm 1; \qquad JD = \pm DJ, \qquad J\gamma = \pm \gamma J$$

- Action of $\mathcal{A}^{\text{op}}$ on $\mathcal{H}$: $a^{\text{op}} = Ja^* J^{-1}$ and

$$[a^{\text{op}}, b] = 0; \qquad a, b \in \mathcal{A}$$

- D is said to satisfy first-order condition if

$$[[D, a], b^{\text{op}}] = 0$$

Spectral invariants

$$\mathrm{Tr}\ f(D/\Lambda) + \frac{1}{2}\langle J\widetilde{\psi}, D\widetilde{\psi}\rangle$$

- Invariant under unitaries $u \in \mathcal{U}(\mathcal{A})$ acting as

$$D \mapsto UDU^*; \qquad U = uJuJ^{-1}$$

- Gauge group: $\mathcal{G}(\mathcal{A}) := \{uJuJ^{-1} : u \in \mathcal{U}(\mathcal{A})\}$.
- Compute *rhs*:

$$D \mapsto D + u[D, u^*] + \hat{u}[D, \hat{u}^*] + \hat{u}[u[D, u^*], \hat{u}^*]$$

with $\hat{u} = JuJ^{-1}$ and blue term vanishes if D satisfies first-order condition

Semi-group of inner perturbations

$$\mathrm{Pert}(\mathcal{A}) := \left\{ \sum_j a_j \otimes b_j^{\mathrm{op}} : \sum_j a_j b_j = 1, \quad \sum_j a_j \otimes b_j^{\mathrm{op}} = \sum_j b_j^* \otimes a_j^{*\mathrm{op}} \right\}$$

with semi-group law inherited from product in $\mathcal{A} \otimes \mathcal{A}^{\mathrm{op}}$.

- $\mathcal{U}(\mathcal{A})$ maps to $\mathrm{Pert}(\mathcal{A})$ by sending $u \mapsto u \otimes u^{*\mathrm{op}}$.
- $\mathrm{Pert}(\mathcal{A})$ acts on D:

$$D \mapsto \sum_j a_j D b_j$$

- For real spectral triples we use the map $\mathrm{Pert}(\mathcal{A}) \to \mathrm{Pert}(\mathcal{A} \otimes \hat{\mathcal{A}})$ sending $A \mapsto A \otimes \hat{A}$ so that

$$D \mapsto \sum_{i,j} a_i \hat{a}_j D b_i \hat{b}_j$$

Proposition (Chamseddine–Connes–vS, 2013)

If $\sum_j a_j \otimes b_j^{op} \in \mathrm{Pert}(\mathcal{A})$ *then the perturbed operator*

$$D' := \sum_{i,j} a_i \hat{a}_j D b_i \hat{b}_j =: D + A_{(1)} + \widetilde{A}_{(1)} + A_{(2)}$$

where

$$A_{(1)} := \sum_j a_j [D, b_j], \qquad \widetilde{A}_{(1)} := \sum_j \hat{a}_j [D, \hat{b}_j] = \pm J A_{(1)} J^{-1};$$
$$A_{(2)} := \sum_j \hat{a}_j [A_{(1)}, \hat{b}_j] = \sum_{j,k} \hat{a}_j a_k [[D, b_k], \hat{b}_j].$$

Gauge transformations $D' \mapsto U D' U^*$ implemented by

$$A_{(1)} \mapsto u A_{(1)} u^* + u[D, u^*]$$
$$A_{(2)} \mapsto J u J^{-1} A_{(2)} J u^* J^{-1} + J u J^{-1} [u[D, u^*], J u^* J^{-1}]$$

Example: Yang–Mills theory

On a 4-dimensional background:

- $\mathcal{A} = C^\infty(M) \otimes M_n(\mathbb{C})$
- $\mathcal{H} = L^2(S) \otimes M_n(\mathbb{C})$
- $D = \not{\partial} \otimes 1$
- $\gamma = \gamma_5 \otimes 1, \qquad J = C \otimes (.)^*.$

Proposition (Chamseddine-Connes, 1996)

- $\mathrm{Tr}\ f(D)$: *pure gravity*
- *The self-adjoint operator* $A_{(1)} + \widetilde{A}_{(1)}$ *with* $A_{(1)} = \gamma^\mu A_\mu$ *describes an* $\mathfrak{su}(n)$*-gauge field on* M.
- *Gauge group* $\mathcal{G}(\mathcal{A}) \simeq C^\infty(M, SU(n))$
- *The spectral action of perturbed Dirac operator is given by*

$$\mathrm{Tr}\ f(D + A_{(1)} + \widetilde{A}_{(1)}) \sim (\cdots) + \frac{f(0)}{24\pi^2} \int_M \mathrm{Tr}\ F_{\mu\nu} F^{\mu\nu} + \langle \psi, (\not{\partial} + \mathrm{ad} A)\psi \rangle$$

Almost-commutative geometries

A class of examples

$$(C^\infty(M) \otimes A_F, \quad L^2(S) \otimes \mathcal{H}_F, \quad \not\partial \otimes 1 + \gamma_5 \otimes D_F)$$

with grading $\gamma = \gamma_5 \otimes \gamma_F$ and real structure $J = J_M \otimes J_F$.

- Gauge group $\mathcal{G}(C^\infty(M) \otimes A_F) = C^\infty(M, \mathcal{G}(A_F))$
- Inner perturbations:

$$D \mapsto D' = \not\partial \otimes 1 + \gamma^\mu \otimes \mathrm{ad} A_\mu + \gamma_5 \otimes \Phi$$

  with $\mathrm{ad} A_\mu$ a $\mathfrak{g}(A_F)$-gauge potential and $\Phi = D_F + \phi + J_F \phi J_F^{-1}$ a map $\mathcal{H}_F \to \mathcal{H}_F$
- Explicitly,

$$A_\mu = -i \sum_j a_j \partial_\mu(b_j); \qquad \phi = \sum_j a_j [D_F, b_j]$$

- As $\mathcal{G}(A_F)$-representations:

$$A_\mu \mapsto u A_\mu u^* - i u \partial_\mu u^*, \qquad \Phi \mapsto U \Phi U^*$$

Proposition (Van den Dungen–vS, 2012)

In the above setting,

$$\mathrm{Tr}\ \left(f\left(\frac{D'}{\Lambda}\right)\right) \sim (\cdots) + \frac{f(0)}{24\pi^2}\mathrm{Tr}\ (F_{\mu\nu}F^{\mu\nu}) - \frac{2f_2\Lambda^2}{4\pi^2}\mathrm{Tr}\ (\Phi^2) + \frac{f(0)}{8\pi^2}\mathrm{Tr}\ (\Phi^4)) \\ + \frac{f(0)}{48\pi^2}s\mathrm{Tr}\ (\Phi^2) + \frac{f(0)}{8\pi^2}\mathrm{Tr}\ \big((D_\mu\Phi)(D^\mu\Phi)\big).$$

The noncommutative Standard Model

$$(C^\infty(M) \otimes (\mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})), L^2(S) \otimes \mathcal{H}_F, \partial\!\!\!/ \otimes 1 + \gamma_5 \otimes D_F)$$

- Fermions are given by:

$$\mathcal{H}_F := \left(\mathcal{H}_l \oplus \mathcal{H}_{\bar{l}} \oplus \mathcal{H}_q \oplus \mathcal{H}_{\bar{q}}\right)^{\oplus 3}.$$

- Algebra acts as:

$$(\lambda, q, m) \xrightarrow{\mathcal{H}_l} \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \overline{\lambda} & 0 & 0 \\ 0 & 0 & \alpha & \beta \\ 0 & 0 & -\overline{\beta} & \overline{\alpha} \end{pmatrix}, \quad (\lambda, q, m) \xrightarrow{\mathcal{H}_q} \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \overline{\lambda} & 0 & 0 \\ 0 & 0 & \alpha & \beta \\ 0 & 0 & -\overline{\beta} & \overline{\alpha} \end{pmatrix} \otimes 1_3.$$

- Real structure J_F interchanges fermions and anti-fermions.
- Dirac operator is

$$D_F := \begin{pmatrix} S & T^* \\ T & \overline{S} \end{pmatrix}.$$

The noncommutative Standard Model

The finite Dirac operator

$$D_F := \begin{pmatrix} S & T^* \\ T & \overline{S} \end{pmatrix}$$

- The operator S is given by

$$S_l := S|_{\mathcal{H}_l} = \begin{pmatrix} 0 & 0 & Y_\nu & 0 \\ 0 & 0 & 0 & Y_e \\ Y_\nu^* & 0 & 0 & 0 \\ 0 & Y_e^* & 0 & 0 \end{pmatrix}, \quad S_q \otimes 1_3 := S|_{\mathcal{H}_q} = \begin{pmatrix} 0 & 0 & Y_u & 0 \\ 0 & 0 & 0 & Y_d \\ Y_u^* & 0 & 0 & 0 \\ 0 & Y_d^* & 0 & 0 \end{pmatrix}$$

  where Y_ν, Y_e, Y_u and Y_d are 3×3 mass matrices acting on the three generations.
- The symmetric operator T only acts on the right-handed (anti)neutrinos, $T\nu_R = Y_R\overline{\nu_R}$ for a 3×3 symmetric Majorana mass matrix Y_R, and $Tf = 0$ for all other fermions $f \neq \nu_R$.

Proposition (Chamseddine–Connes–Marcolli, 2007)

In the above setting,

- *The unimodular gauge group* $S\mathcal{G}(\mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})) = U(1) \times SU(2) \times SU(3)$
- *The inner perturbations of* $\partial\!\!\!/ \otimes 1 + \gamma_5 \otimes D_F$ *are parametrized by* $U(1), SU(2)$ *and* $SU(3)$ *gauge fields* $\Lambda_\mu, Q_\mu, V_\mu$ *and a Higgs doublet* H
- *The spectral action is given by*

$$\mathrm{Tr}\ f\left(\frac{D'}{\Lambda}\right) \sim (\cdots) + \frac{f(0)}{\pi^2}\left(\frac{10}{3}\Lambda_{\mu\nu}\Lambda^{\mu\nu} + \mathrm{Tr}\ (Q_{\mu\nu}Q^{\mu\nu}) + \mathrm{Tr}\ (V_{\mu\nu}V^{\mu\nu})\right)$$
$$+ \frac{bf(0)}{2\pi^2}|H|^4 + \frac{-2af_2\Lambda^2 + ef(0)}{\pi^2}|H|^2$$
$$- \frac{cf_2\Lambda^2}{\pi^2} + \frac{df(0)}{4\pi^2} + \frac{af(0)}{12\pi^2}s|H|^2 + \frac{cf(0)}{24\pi^2}s + \frac{af(0)}{2\pi^2}|D_\mu H|^2.$$

Example beyond first-order

[Chamseddine–Connes–vS, 2013]

$$A'_F = \mathbb{C}_R \oplus \mathbb{C}_L \oplus M_2(\mathbb{C}),$$

$$H_F = (\mathbb{C}_R \oplus \mathbb{C}_L) \otimes (\mathbb{C}^2)^\circ \oplus \mathbb{C}^2 \otimes (\mathbb{C}_R^\circ \oplus \mathbb{C}_L^\circ),$$

$$J_F = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \circ C \qquad (C : \text{complex conjugation}),$$

$$D_F = \begin{pmatrix} 0 & k_x \otimes 1_2 & \begin{smallmatrix} \overline{k}_y & 0 \\ 0 & 0 \end{smallmatrix} & 0 \\ \overline{k}_x \otimes 1_2 & 0 & 0 & \\ \begin{smallmatrix} k_y & 0 \\ 0 & 0 \end{smallmatrix} & 0 & 0 & 1_2 \otimes \overline{k}_x \\ 0 & 0 & 1_2 \otimes k_x & 0 \end{pmatrix}$$

The algebra action of $(\lambda_R, \lambda_L, m) \in \mathcal{A}$ on $\mathcal{H}$ is given explicitly by

$$\pi(\lambda_R, \lambda_L, m) = \begin{pmatrix} \lambda_R 1_2 & & & \\ & \lambda_L 1_2 & & \\ & & m & \\ & & & m \end{pmatrix}, \pi^\circ(\lambda_R, \lambda_L, m) = \begin{pmatrix} m^t & & & \\ & m^t & & \\ & & \lambda_R 1_2 & \\ & & & \lambda_L 1_2 \end{pmatrix}.$$

Proposition

The largest (even) subalgebra $\mathcal{A}_F \subset \mathcal{A}'_F \equiv \mathbb{C}_R \oplus \mathbb{C}_L \oplus M_2(\mathbb{C})$ *for which the first-order condition holds (for the above* $\mathcal{H}_F, D_F$ *and* J_F*) is given by*

$$\mathcal{A}_F = \left\{ \left(\lambda_R, \lambda_L, \begin{pmatrix} \lambda_R & 0 \\ 0 & \mu \end{pmatrix} \right) : (\lambda_R, \lambda_L, \mu) \in \mathbb{C}_R \oplus \mathbb{C}_L \oplus \mathbb{C} \right\}$$

Proposition

The inner perturbed Dirac operator D' *is parametrized by three complex scalar fields* ϕ, σ_1, σ_2 *entering in* $A_{(1)}$ *and* $A_{(2)}$*:*

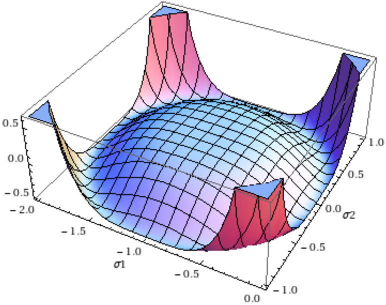
$$D_F + A_{(1)} + \hat{A}_{(1)} + A_{(2)} = \begin{pmatrix} 0 & k_x(1+\phi)\otimes 1_2 & \overline{k}_y \overline{v}\overline{v}^t & 0 \\ \overline{k}_x(1+\overline{\phi})\otimes 1_2 & 0 & 0 & \\ k_y v \ \ v^t & 0 & 0 & 1_2 \otimes \overline{k}_x(1+\overline{\phi}) \\ 0 & 0 & 1_2 \otimes k_x(1+\phi) & 0 \end{pmatrix}$$

with $v = \begin{pmatrix} 1+\sigma_1 \\ \sigma_2 \end{pmatrix}$.

Spectral action

Spectral action gives rise to a scalar potential

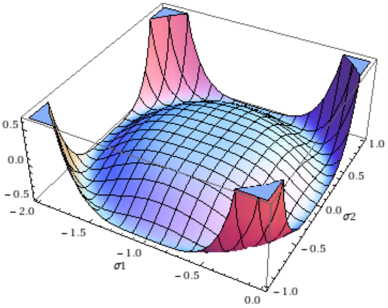
$$V(\phi,\sigma_1,\sigma_2) = -\frac{f_2}{\pi^2}\Lambda^2\left(4|k_x|^2|\phi|^2 + |k_y|^2(|1+\sigma_1|^2+|\sigma_2|^2)^2\right)$$
$$+\frac{f_0}{4\pi^2}\Big(4|k_x|^4|\phi|^4 + 4|k_x|^2|k_y|^2|\phi|^2(|1+\sigma_1|^2+|\sigma_2|^2)^2$$
$$+|k_y|^4(|1+\sigma_1|^2+|\sigma_2|^2)^4\Big)$$



Spontaneous symmetry breaking to first-order

Proposition

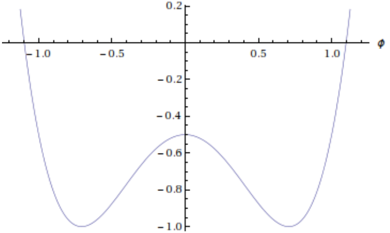
The potential $V(\phi = 0, \sigma_1, \sigma_2)$ has a local minimum at $(\sigma_1, \sigma_2) = (-1 + \sqrt{w}, 0)$ with $w = \sqrt{2 f_2 \Lambda^2 / (f_0 |k_y|^2)}$ and this point spontaneously breaks the symmetry group $\mathcal{U}(A'_F)$ to $\mathcal{U}(A_F)$.



“Usual” SSB

After the fields (σ_1, σ_2) have reached their vevs $(-1+\sqrt{w}, 0)$, there is a remaining potential for the ϕ-field:

$$V(\phi) = -\frac{2f_2}{\pi^2}\Lambda^2|k_x|^2|\phi|^2 + \frac{f_0}{\pi^2}|k_x|^4|\phi|^4.$$



Selecting one of the minima of $V(\phi)$ spontaneously breaks the symmetry further from $\mathcal{U}(A_F) = U(1)_R \times U(1)_L \times U(1)$ to $U(1)_L \times U(1)$, and generates mass terms for the $L - R$ abelian gauge field.

Spectral action: pure gravity

Proposition

For the canonical triple $(C^\infty(M), L^2(M,S), \partial\!\!\!/)$*, the spectral action is*

$$\mathrm{Tr}\,\left(f\left(\frac{\partial\!\!\!/}{\Lambda}\right)\right) \sim \frac{f_4\Lambda^4}{2\pi^2} - \frac{f_2\Lambda^2}{24\pi^2}s + \frac{f(0)}{16\pi^2}\left(\frac{1}{30}\Delta s - \frac{1}{20}C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} + \frac{11}{360}R^*R^*\right).$$

Coefficients NCSM

$$
\begin{aligned}
a &= \mathrm{Tr}\,\left(Y_\nu^* Y_\nu + Y_e^* Y_e + 3Y_u^* Y_u + 3Y_d^* Y_d\right),\\
b &= \mathrm{Tr}\,\left((Y_\nu^* Y_\nu)^2 + (Y_e^* Y_e)^2 + 3(Y_u^* Y_u)^2 + 3(Y_d^* Y_d)^2\right),\\
c &= \mathrm{Tr}\,\left(Y_R^* Y_R\right),\\
d &= \mathrm{Tr}\,\left((Y_R^* Y_R)^2\right),\\
e &= \mathrm{Tr}\,\left(Y_R^* Y_R Y_\nu^* Y_\nu\right).
\end{aligned}
\tag{1}
$$