Matrix geometries

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? $\int e^{iS}$ "dD"

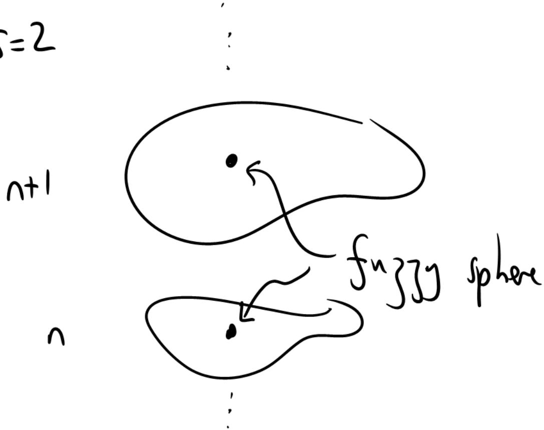
spectral triples

Finite diml examples $\rightsquigarrow$ $\int dD$ makes sense

$A = M_n(\mathbb{C})$. Fix n. KO-dim s

Construct $\mathcal{H}, \gamma, J$ space of $D \subset \mathrm{End}(\mathcal{H})$

Example $s=2$



Gamma matrices $\quad \mathrm{Cliff}(p,q)$ act on V $(\,,\,)$ hermitian

$$\tfrac{1}{2}(\gamma^a\gamma^b + \gamma^b\gamma^a) = g^{ab} = \mathrm{diag}(q^-, p^+)$$

$$(\gamma^a)^2 = -1\,, \quad (\gamma^a)^* = -\gamma^a \quad \text{a.h.}$$

or $\quad +1 \quad\quad +\gamma^a$ herm

Chirality γ $\quad$ Real C $\quad$ $C^2 = \epsilon$ $\quad$ $C\gamma^a = \epsilon' \gamma^a C$

C antilinear $\quad$ $C\gamma = \epsilon'' \gamma C$

s	0	1	2	3	4	5	6	7
ϵ	1	1	−1	−1	−1	−1	1	1
ϵ'	1	−1	1	1	1	−1	1	1
ϵ''	1	1	−1	1	1	1	−1	1

$$(C\psi, C\psi') = \overline{(\psi, \psi')}$$

Spectral triple $\mathcal{H} = V \otimes M_n(\mathbb{C})$ $\quad A = M_n(\mathbb{C})$

$$\langle v \otimes m, v' \otimes m' \rangle = (v, v') \operatorname{tr} m^* m'$$

$$\Gamma = \gamma \otimes 1 \qquad J = C \otimes *$$

$\mathrm{Cliff}(p,q)$ $\quad s = q - p \mod 8$

$\omega^i \in \mathrm{Cliff}(p,q)$ $\quad \gamma \omega^i = -\omega^i \gamma$ (s even) (odd)

Define $\Theta = \sum_{i=1}^{\tilde{n}} \omega^i \otimes X^i$, $\quad X^i \in A$, $\quad \Theta = \Theta^*$

Thm $D = \Theta + \epsilon' J \Theta J^{-1}$ is a real s.triple

including 1st order condition w/o P.D.

Action of Θ: $\Theta\psi = \sum (w^i \otimes x^i)(v \otimes m)$

$= \sum w^i v \otimes x^i m$

$J\Theta J^{-1}$: $J\Theta J^{-1}\psi = \sum \underbrace{c w^i c^{-1}}_{\epsilon' w^i} v \otimes m x^{i*}$

Assume w^i odd:

$D(v \otimes m) = \sum w^i v \otimes (x^i m + m x^{i*})$

Write $\Theta = \sum_j \sigma^j \otimes L^j + \sum_k \tau^k \otimes H^k = \Theta^*$

a/h $\qquad\qquad$ h

$$\mathcal{D} = \sum_j \sigma^j \otimes [L^j, \cdot] + \sum_k \tau^k \{H^k, \cdot\}$$

Examples $\quad p = q = 0 \quad s = 0 \qquad \mathcal{D} = 0$

$q = 1, \; p = 0 . \; s = 1 \quad V = \mathbb{C} \quad \mathcal{H} = M_n(\mathbb{C}) \quad \gamma^1 = i$

$$\mathcal{D} = [\theta, \cdot] \qquad \theta^* = \theta .$$

$p=1,\ q=0 \quad s=7 \qquad \gamma'=1$

$$\mathcal{D} = \{\theta, \cdot\} \qquad \theta^* = \theta$$

$p=1,\ q=1 \quad s=0 \qquad V = \mathbb{C}^2, \quad \mathcal{H} = \mathbb{C}^2 \otimes M_n(\mathbb{C})$

$$J(v \otimes m) = \bar{v} \otimes m^*$$

$$\mathcal{D} = \begin{pmatrix} 0 & d \\ d^* & 0 \end{pmatrix} \qquad d = [L, \cdot] + \{H, \cdot\}$$

$$\Gamma = \begin{pmatrix} \mathbb{1} & \cdot \\ \cdot & -\mathbb{1} \end{pmatrix} \qquad \mathcal{D} = \begin{pmatrix} \cdot & d_7 + i d_1 \\ d_7 - i d_1 & \cdot \end{pmatrix}$$

$q=2$, $p=1$ $\quad s=1 \quad V=\mathbb{C}^2$

$$\mathcal{D} = \gamma^1 \otimes [L^1, \cdot] + \gamma^2 \otimes [L^2, \cdot] + \gamma^3 \otimes \{H^3, \cdot\}$$
$$+ \mathbb{1} \otimes [H, \cdot]$$

$$(\gamma^1)^2 = (\gamma^2)^2 = -1 \quad , \quad (\gamma^3)^2 = 1$$

Fuzzy circle ??

$q=3,\ p=0 \quad s=3 \quad V=\mathbb{C}^2, \quad J\begin{pmatrix}v_0\\ v_1\end{pmatrix}\otimes m = \begin{pmatrix}v_1^*\\ -v_0^*\end{pmatrix}\otimes m^*$

$$D = \sum_{1}^{3} \gamma^i \otimes [L^i, \cdot\,] + \gamma^1\gamma^2\gamma^3 \otimes \{H, \cdot\,\}$$

Example: L^i $su(2)$ generators in $M_n(\mathbb{C})$

$$H = -\frac{1}{2}\mathbb{1}_n$$

$\rightsquigarrow$ Fuzzy S^2 Grosse-Presnajder (almost)

$$D^2 = -\sum_i [L^i,[L^i,\cdot\,]] + \frac{1}{4}$$

Madore

Fuzzy sphere KO-dim $2 = s$

$q=3,\ p=1 \qquad (\gamma^i)^2 = -1 \quad i=1,2,3,\ (\gamma^4)^2 = 1$

$$\gamma^i = \begin{pmatrix} \cdot & \sigma^i \\ \sigma^i & \cdot \end{pmatrix} \qquad \gamma^4 = \begin{pmatrix} \cdot & i\mathbb{1} \\ -i\mathbb{1} & \cdot \end{pmatrix}$$

$$\gamma = \begin{pmatrix} -\mathbb{1} & \cdot \\ \cdot & \mathbb{1} \end{pmatrix} \qquad D = \begin{pmatrix} \cdot & d \\ d^* & \cdot \end{pmatrix}$$

Examples in which $d = d^*$: $\quad D = \begin{pmatrix} \cdot & d_{(0,3)} \\ d_{(0,3)} & \cdot \end{pmatrix}$

$$d = \sigma^i \otimes [L^i, \cdot] + \{H, \cdot\}$$

Doubled fuzzy sphere as KO dim $= 2$ geometry.