A tale of 2 Renormalizations: E-G + BPHZ.

Goal: Extend the Connes-Kreimer Hopf algebra to one that is compatible with EG renorm.
Conclusion: There is a context in which the need to verify local counterterms disappears from the Connes-Kreimer P.O.V.

Part 1| A Review of Hopf algebras:

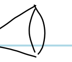
M background manifold, Scalar field theory.

Superficial degree of divergence: $\omega(\Gamma) = \sum_{v \in V(\Gamma)} \left[\left(\frac{d-2}{2}\right) n_v - \delta_v - d \right] + d$

A (bosonic) theory is renormalizable if $\frac{d-2}{2} n_v - \delta_v - d \leq 0 \quad \forall v \in V(\Gamma)$

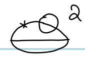
① Connes-Kreimer, φ^4 (renormalizable)

$$\mathcal{H}_{CK} = \mathbb{Q}[\Gamma \mid 1PI, \text{ 2, 4 valent vertices, no self loops, } \underbrace{\omega(\Gamma) \leq 0}_{E(\Gamma) \leq 4}]$$

Draw graphs without external legs. Eg.  , 

product: disjoint union.
graded by first betti # (loop # of graph).
counit ε. $\operatorname{Ker}(\varepsilon) = \mathcal{H}^{i>0}$

$\Delta : \sum_{\gamma} \underbrace{\Gamma/\!/\gamma}_{\text{piecewise contraction to new vertex.}} \otimes \gamma \qquad \gamma = \underbrace{\text{complete subgraph}}_{\text{no self energy}} \text{ on } (V(\gamma))$

Δ  $= \underbrace{1}_{\Gamma/\!/\Gamma : \bullet \to 1} \otimes \Gamma + \Gamma \otimes 1 +$  $^2 \otimes \ominus + \ominus \otimes \ominus^2$

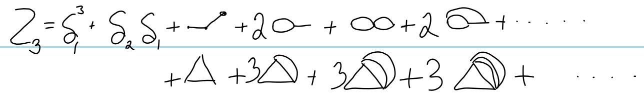
Short coming! What about non sup-div graphs?

② Epstein Glazier Hopf algebra.

$$\mathcal{H}_{EG} = \mathbb{Q}[\delta^n \mid n \in \mathbb{N}] \qquad \begin{array}{l} \delta^1 = 1 \\ \delta^n = \sum \Gamma \text{ scalar, no self loop, } |V(\Gamma)| = n \end{array}$$

$$\Delta \delta^n = \sum \Delta \Gamma = \sum \sum_{\substack{\text{part } P \\ \text{of } V(\Gamma)}} \Gamma/\!/\gamma_1 \ldots \gamma_{|P|} \otimes \prod_{I \in P} \gamma_I \qquad \gamma_I = \overset{\text{Complete}}{\text{Subgraph}} \text{ of } \Gamma \text{ on } I.$$

$$= \sum_{k=1}^{n!} \frac{\delta^k}{k!} \otimes \sum_{j_1 \cdots j_k = n} \frac{\delta^{j_1}}{j_1!} \cdots \frac{\delta^{j_k}}{j_k!} \rightsquigarrow \mathcal{H}_{EG} = \text{Faa di Bruno!}$$

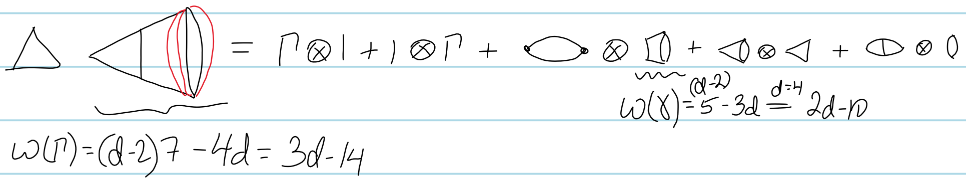
Eg


Note includes ① 1PR graphs
② $\omega(\Gamma) \geq 0$
③ All valencies.

③ Middle ground: Gen of work by Garcia-Bondia, Lazarini ; Pinter

φ theory. (All valency)

$\mathcal{H} = \mathbb{Q}[\Gamma \mid \Gamma \text{ 1PI}]$ continue ignoring ext. legs.



$\mathcal{H}$ bigraded by loop #, vertex #

$\mathcal{H} = \underbrace{\mathcal{H}^{1,0}}_{\mathbb{Q}} \oplus \bigoplus_{V=2, l=1}^{\infty} \mathcal{H}^{V,l}$ connected, graded, $\therefore$ S (antipode) defined

Part 2 | Review of Affine group schemes:

Let $\mathcal{H}$ be a (pro) finite dim, comm Hopf alg.
Then $\mathcal{H}$ defines a fin-dim Lie group: $G_{\mathcal{H}} = \mathrm{Hom}_{alg}(\mathcal{H}, -)$

$G_{\mathcal{H}}$ functor from alg $\to$ groups.
$\mathcal{H} = \mathbb{Q}G$, ring of regular functions on G. Not! $\underbrace{K[G] \text{ group ring}}_{\text{Frobenious alg.}}$
a.k.a. $G = \mathrm{Spec}\, \mathcal{H}$.

properties.
$\varphi \in G_{\mathcal{H}}$, $h \in \mathcal{H}$. $\varphi(hh') = \mu_A(\varphi(h) \otimes \varphi(h'))$

$\varphi * \varphi'(h) = \mu_G(\varphi \otimes \varphi')(h) = \mu_A(\varphi \otimes \varphi' \Delta h)$

$\hookrightarrow$ convoletion

$e_G = \varepsilon$

$\varphi^{-1}(h) = \varphi(S(h))$. $\varphi^{-1} * \varphi(h) = \varphi\, \mu_n(\underbrace{(S \otimes id)\Delta h}_{\varepsilon})$

Part 3 | Renormalization schemes

① BPHZ

Where is Hopf algebra story most obvious?

BPHZ recursion formula:

Bogoliubov preparation map: $\tilde{R}(\Gamma) = U(\Gamma) - \sum_{\gamma \subset \Gamma} U(\Gamma/\gamma)\, C(\gamma)$

$R(\Gamma) = (1-\pi)\tilde{R}(\Gamma)$; $C(\Gamma) = -\pi \tilde{R}(\Gamma)$

π is a projection operator onto singular part of algebra.

Note: $U(\Gamma), R(\Gamma), C(\Gamma)$ live in meromorphic functions over some space, $\Rightarrow$ all good.

Manchon, Eb-F. have shown that A need only be R-B alg of certain type...
stick w/ meromorphic for now.

$$A = \mathrm{Sym}(\mathcal{D}'(M))[z^{-1}][[z]] \supset \underbrace{\mathrm{Hom}_{lin}(S\mathcal{F}_{loc}, S\mathcal{F}_{loc})[z^{-1}][[z]]}_{\mathcal{L}}$$

Consider $G_{CK}(A)$

Thm| Kreimer, E-F, others. (Birkhoff decomp)
A meromorphic $\therefore \forall \varphi \in G_{CK}(A)$, $\underbrace{\varphi = \varphi_+ * \varphi_-^{-1}}_{\text{unique}}$ $\quad \varphi_+ \in G(A_+)$, $\varphi_- \in G(A_-)$

$$A = \underset{\sum_0^{\infty} a_i z^i}{A_+} \oplus \underset{\sum_{-n}^{0} a_j z^j}{A_-}$$

$\varphi_+(\Gamma) = R(\Gamma) \quad \varphi_-(\Gamma) = C(\Gamma) \quad \varphi(\Gamma) = U(\Gamma)$

② What about non-sup. div. graphs?

$\mathcal{H}_{CK} \hookrightarrow \mathcal{H} \quad \therefore \rho: G(A) \twoheadrightarrow G_{CK}(A) \quad \ker(\rho)$ defined by $\varphi'|_{\mathcal{H}_{CK}} = \varphi|_{\mathcal{H}_{CK}}$

$\exists$ subgrp $G_0 \subset G$

s.t. $\varphi_-(\Gamma) = 0$ if $\omega(\Gamma) < 0$, $\Gamma \neq 1$

③ $G_{EG}(\mathcal{L}) = \underbrace{Maps(\mathcal{F}_{loc}, \mathcal{F}_{loc})\Big|_{\substack{Z(0)=0 \\ Z^{(1)}(0)=id}} [\lambda][[\lambda^{-1}]]}$

Because $\mathcal{H}_{EG}$ is FdB.

$\exists\, \rho' : G/(\mathcal{L}) \twoheadrightarrow G_{EG}(\mathcal{L})$ ker: φ agree on sums.

doesn't have 1PR diagrams!

Thm | A.R. can resum such that 1PR contribution to $Z \in G_{EG}$ is 0.

Define 2 more alg. $\mathcal{L}'_{EG} : Lin(\mathcal{F}_{loc}, \mathcal{F}_{loc})\Big|_{\substack{EG \\ \text{conditions}}} [z^{-1}][[z]]$

$\cup$

$\mathcal{L}_{EG} = \mathcal{L}\Big|_{\substack{EG \\ \text{conditions}}}$

<u>Main thm of Renormalization</u> | BDF

Given any S in $G_{EG}(\mathcal{L}'_{EG})$, $\exists$ $\tilde{S} \in G_{EG}(\mathcal{L}'_{EG+})$

$\cup$

$Z \in G_{EG}(\mathcal{L}_{EG-})$

s.t. $S = \underbrace{\tilde{S}}_{\substack{\lim \\ \lambda \to 0 \\ \text{fin.}}} * \underbrace{Z^{-1}}_{\substack{\text{local} \\ \text{counterterm}}}$

$\therefore$ Write $G_0(\mathcal{L}'_{EG}) \longrightarrow$ all elems have local counterterms!