Duality and Modular invariance in Quantum Field Theory and String Theory

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There are five elementary arithmetical operations: addition, subtraction, multiplication, division, and … modular forms.

Modular invariance arise naturally in many physical context

adapted from Terras

1. String Theory
2. Black hole in Quantum gravity
3. Quantum field theory
4. Quantum Chaos
5. Solid state physics …

In this talk we will describe how modular invariance enters in an essential way in quantum field theory and string theory

Modular invariance by relating perturbative and non-perturbative regime of the theory connects the important question and fundamental questions about the consistency of quantum gravity (e.g. high energy behaviour of $\mathcal{N} = 8$ supergravity . . .) to deep and cute properties of automorphic representations.

We will see how string theory identifies some interesting modular and automorphic forms and allows to address difficult mathematical questions in representation theory

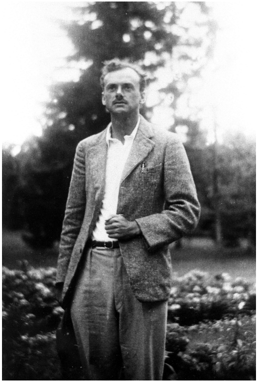
Based on work done with Michael B. Green, Stephen D. Miller, Jorge Russo

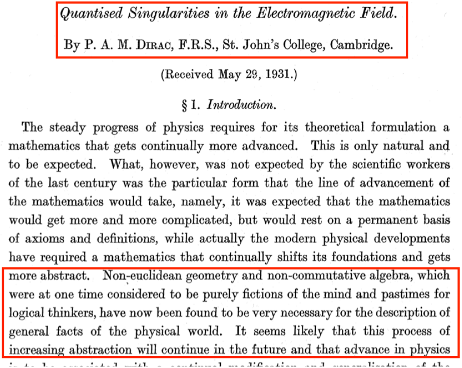


Part I

Dirac Charge quantization

Dirac Charge Quantization





Quantised Singularities in the Electromagnetic Field.

By P. A. M. Dirac, F.R.S., St. John's College, Cambridge.

(Received May 29, 1931.)

§ 1. *Introduction.*

The steady progress of physics requires for its theoretical formulation a mathematics that gets continually more advanced. This is only natural and to be expected. What, however, was not expected by the scientific workers of the last century was the particular form that the line of advancement of the mathematics would take, namely, it was expected that the mathematics would get more and more complicated, but would rest on a permanent basis of axioms and definitions, while actually the modern physical developments have required a mathematics that continually shifts its foundations and gets more abstract. Non-euclidean geometry and non-commutative algebra, which were at one time considered to be purely fictions of the mind and pastimes for logical thinkers, have now been found to be very necessary for the description of general facts of the physical world. It seems likely that this process of increasing abstraction will continue in the future and that advance in physics

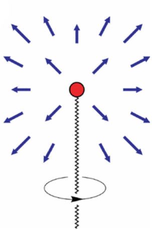
In this paper Dirac provided an elegant argument for the quantization of electric charges.

This idea still has important consequences on our understanding of quantum gravity.

Dirac charge quantization

Schrödinger's equation for the electron wavefunction

$$-\frac{\hbar^2}{2m}\left(\vec{\partial}+ie\vec{A}\right)^2\Psi(t,\vec{x}) = i\hbar\frac{\partial\Psi(t,\vec{x})}{\partial t}$$



- Magnetic monopole of charge g

$$\vec{A}_+ - \vec{A}_- = \vec{\nabla}\left(\frac{g}{2\pi}\phi\right) = \vec{\nabla}\chi$$

- Wavefunction gauge transformation

$$\Psi(t,\vec{x}) \to e^{-ie\chi}\Psi(t,\vec{x}) = e^{-ieg\frac{\phi}{2\pi}}\Psi(t,\vec{x})$$

Single valueness of the wavefunction implies Dirac quantization

$$eg \in 2\pi\mathbb{Z}$$

We have a discrete lattice Γ of electric and magnetic charges

Dirac charge lattice

The electron and the magnetic monopole are dual solution of the four-dimensional Maxwell equations

$$\int_{S^2} F = g \qquad \int_{S^2} \star F = e$$

A dyon is a bound state $d_1 = (e_1, g_1)$ electrically and magnetically charged

Two dyons satisfy the Dirac-Zwanziger-Schwinger quantization

$$e_1 g_2 - e_2 g_1 \in 2\pi\mathbb{Z}$$

This condition is invariant under $SL(2,\mathbb{Z})$ transformation

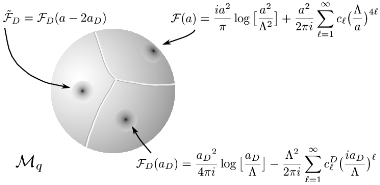
$$\begin{pmatrix} e \\ g \end{pmatrix} \to \gamma \begin{pmatrix} e \\ g \end{pmatrix} \qquad \text{for} \qquad \gamma \in \Gamma = SL(2,\mathbb{Z})$$

What about quantum corrections?

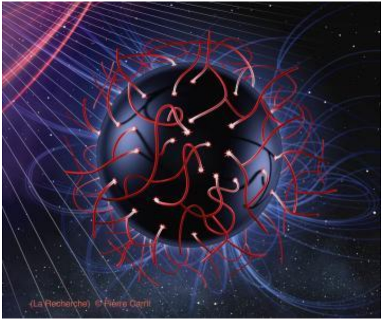
Can this symmetry be a symmetry of a quantum theory; i.e. not being destroyed by quantum corrections?

In the years 1994-1998 it was understood that gauge theories have this remarkable property `[Witten, Vafa, Sen, Seiberg,...]`

- Exact symmetry of $\mathcal{N}=4$ SYM
- Symmetry of Seiberg-Witten $\mathcal{N}=2$ SYM relating different phases of the theory



What about quantum corrections?



Which was soon extended to a fundamental symmetry of String theory

- String theory S-duality and then U-dualities
- Quantum gravity and black hole physics

[Witten, Sen, Schwarz, Green, Hull, Townsend,...]

Modular invariance of Abelian Theory

Witten has clarified the origin of the modular invariance in an Abelian gauge theory

$$\mathcal{L} = -\frac{1}{4e^2}\,(F \wedge \star F) + \frac{\theta}{32\pi^2}\,(F \wedge F)$$

Let's L a $U(1)$-principal bundle with connection A_μ: $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ over a 4 dimensional manifold X

Invariance under $\theta \to \theta + 2\pi$ requires that $\left[\frac{F}{2\pi}\right] = m \in \Lambda$ Lattice

$F \to \star F$ induces the map $m \to *m \in \Lambda$

Modular invariance of Abelian Theory

With $(\cdot, \cdot)$ is the intersection form on $H^2(X)$

$$(m, *m) = \frac{1}{16\pi^2} \int_X F \wedge F; \qquad (m, *m) = \frac{1}{8\pi^2} \int_X F \wedge \star F$$

with $q = \exp(2i\pi\Omega)$ and $\Omega = \frac{\theta}{2\pi} + i\frac{4\pi}{e^2}$

$$\mathcal{Z} = \frac{1}{\mathrm{Vol}(U(1))} \int DAe^{-\int_X \mathcal{L}} = (\Im\mathrm{m}\Omega)^{\frac{b_1-1}{2}} \sum_{m\in\Lambda} q^{\frac{(m,*m)-(m,m)}{4}} \bar{q}^{\frac{(m,*m)-(m,m)}{4}}$$

This is a modular form for $\Gamma = SL(2, \mathbb{Z})$

$$\mathcal{Z}(-\frac{1}{\Omega}) = \Omega^{\frac{\chi+\sigma}{4}} \bar{\Omega}^{\frac{\chi-\sigma}{4}} \mathcal{Z}(\Omega)$$

χ is Euler characteristic and σ signature of X

Black Hole Charge Quantization

Supergravity theories are supersymmetric extensions of the Einstein gravity theory. The massless spectrum of supergravity theories contains the graviton, (many) scalar fields, (many) vector fields, and fermions.

For the case of $D = 10$ $\mathcal{N} = 2b$ supergravity we have

$$\mathcal{L}_{2b} = \frac{1}{2\kappa_{10}^2} |-g|^{\frac{1}{2}} \left(\mathcal{R} - \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2} e^{2\phi} (\partial_\mu C^{(0)})^2 \right)$$

Setting $\Omega = C^{(0)} + ie^{-\phi}$ parametrizes the coset $SL(2,\mathbb{R})/SO(2)$

$$\mathcal{L}_{2b} = \frac{1}{2\kappa_{10}^2} |-g|^{\frac{1}{2}} \left(\mathcal{R} - \frac{1}{2} \frac{\partial_\mu \Omega \partial^\mu \bar{\Omega}}{\Omega_2^2} \right)$$

Again the classical values of Ω parametrize the vacuum of the theory

Black Hole Charge Quantization

D-instantons are finite energy solution [Green, Perry, Gibbons]

$$ds^2 = \left(e^{\Phi_\infty} + \frac{c}{r^8}\right)^{\frac{1}{2}} \left(dr^2 + r^2 d\Omega_9\right)$$

$$e^\Phi = e^{\Phi_\infty} + \frac{c}{r^8}; \quad C^{(0)} = C^{(0)}_\infty + e^{\Phi_\infty} - e^\Phi$$

Charge $Q^{(-1)} = \int_{S^9} e^{2\Phi} \star dC^{(0)} \propto e^{\Phi_\infty} c$

Completely localized object in the $D = 10$ space-time. Their magnetic dual are 7-brane with charge $Q^{(7)} = \oint dC^{(0)}$
Dirac charge quantization condition

$$Q^{(-1)} Q^{(7)} \in 2\pi\mathbb{Z}$$

The classical $SL(2,\mathbb{R})$ symmetry is broken to $\Gamma = SL(2,\mathbb{Z})$ (or a subgroup)

Duality symmetries and the S-matrix

Scattering amplitudes and S-matrix elements depend on the classical background θ and g

$$f(\theta, g) = \sum_{n \geqslant 0} c_n \, g^n + \sum_{n>0} d_n(g) \, e^{-2\pi \frac{n}{g} + 2i\pi n\theta}$$

- Finite or infinite number of perturbative contributions c_n
- Non-perturbative contributions from instantons

Duality symmetries and the S-matrix

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$$f(\theta, g) = \sum_{n\geqslant 0} c_n\, g^n + \sum_{n>0} d_n(g)\, e^{-2\pi\frac{n}{g}+2i\pi n\theta}$$

- The coefficients c_n can be computed (in principle) from the Feynman rule deduced from the Lagrangian of the theory.
- This is an asymptotic expansion with zero radius of convergence $c_n \sim n!$ or $(2n)!$
- The theory is controlled by large fields classical solutions: instantons
- Instanton corrections of energy $2\pi n/g$ and charge $n\theta$ with fluctuations $d_n(g)$
- needed to give a prescription for a complete consistent theory

Duality symmetries and the S-matrix

Scattering amplitudes and S-matrix elements depend on the classical background θ and g

$$f(\Omega) = \sum_{n\geqslant 0} c_n\, \Omega_2^n + \sum_{n\neq 0} d_n(\Omega_2)\, q^n + c.c.$$

Modular invariance allows to complete the perturbative result with the non-perturbative contributions

Allows to reach non-perturbative information very difficult to compute directly

Part II

S-duality in string theory

The type IIB string case

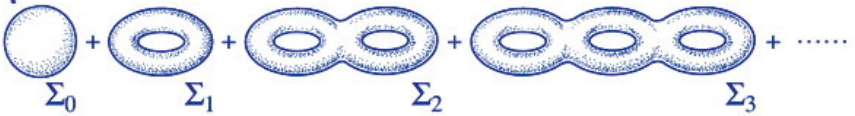
2-dimensional lagrangian

$$S^{(D)}_{\text{eff}}(g_{\mu\nu}, A_\mu, ..)$$

$$= \sum_{\Sigma_\gamma} e^{-\phi\chi(\Sigma_\gamma)} \int_{M(\Sigma_\gamma)} \int d\psi dX..\, e^{\int d^2 z \mathcal{L}_{2d}(\psi, X, .., g_{\mu\nu}, A_\mu, ..)}$$

$$= \int d^D x \sqrt{g} e^{-2\phi} \underbrace{[R + \text{Tr} F_{\mu\nu} F^{\mu\nu} + \dots]}_{\text{general relativity, gauge theory, etc}} + \mathcal{O}({m_{\text{planck}}}^{-1})$$

small string corrections



The string theory induced corrections to the Einstein-Hilbert action in $D = 10$ for the type IIB theory read

$$\delta\mathcal{L}_{IIb} = \mathcal{E}_{(0)}(\Omega)\ell_{10}^{6}\mathcal{R}^4 + \mathcal{E}_{(2)}(\Omega)\ell_{10}^{10}D^4\mathcal{R}^4 + \mathcal{E}_{(3)}(\Omega)\ell_{10}^{12}D^6\mathcal{R}^4 + \cdots$$

The corrections are modular function invariant under the action of $SL(2,\mathbb{Z})$

$$\mathcal{E}_{(k)}(\gamma \cdot \Omega) = \mathcal{E}_{(k)}(\Omega) \qquad \gamma \in SL(2,\mathbb{Z})$$

[Green, Gutperle, Russo, Miller, Vanhove, ...]

The type IIB string case

They satisfy two important constraints

- Their constant term must reproduce string perturbation
- They satisfy second order differential equations

• The boundary data from string perturbation (recall $\Omega_2 = \Im\mathrm{m}\Omega = e^{-\phi_\infty}$ is the string coupling constant)

$$\int_0^1 \mathcal{E}_{(k)}(\Omega)\, d\Omega_1 = \Omega_2^{\frac{k-1}{2}} \left(a_0^k \Omega_2^2 + a_1^k + a_2^k\, \Omega_2^{-2} + \cdots + O(e^{-\Omega_2}) \right)$$

The coefficients a_g arise from the evaluation of the 4-graviton amplitude on a genus g Riemann surface.

The $\exp(-\Omega_2)$ contributions are from the D-instanton described before.

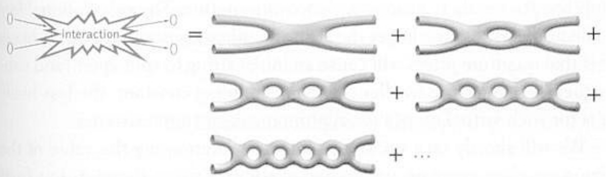
The type IIB string case

Explicit computations gives for the perturbative contributions

[Green, Vanhove, Russo, D'Hoker, Pioline, ...]

$$
\begin{aligned}
\int_0^1 \mathcal{E}_{(0)}(\Omega)\, d\Omega_1 &= \Omega_2^{-\frac{1}{2}} \left(2\zeta(3)\,\Omega_2^2 + 4\zeta(2)\right)\\
\int_0^1 \mathcal{E}_{(2)}(\Omega)\, d\Omega_1 &= \Omega_2^{\frac{1}{2}} \left(\zeta(5)\,\Omega_2^2 + \frac{8\zeta(4)}{3}\,\Omega_2^{-2}\right)\\
\int_0^1 \mathcal{E}_{(3)}(\Omega)\, d\Omega_1 &= \Omega_2 \left(\frac{2}{3}\zeta(3)^2\Omega_2^2 + \frac{4\zeta(2)\zeta(3)}{3} + \frac{4\zeta(4)}{\Omega_2^2} + \frac{4\zeta(6)}{26\Omega_2^4}\right)\\
&+ \quad O(e^{-\Omega_2})
\end{aligned}
$$

Contribution from genus g amplitude $\Omega_2^{2(2-g)}$



The type IIB string case

Supersymmetry of the Lagrangian $\delta\mathcal{L} = 0$

$$\delta = \sum_r \ell_{10}^r \delta^r; \qquad \delta^0 \mathcal{L}^n + \sum_{r_1+r_2=n} \delta^{r_1} \mathcal{L}^{r_2} = \delta^n \mathcal{L}^0$$

implies the differential equations $\Delta = \Omega_2^2(\partial_{\Omega_1}^2 + \partial_{\Omega_2}^2)$

$$\begin{aligned}
\delta^0 \mathcal{L}^6 \simeq 0 \qquad & (\Delta - \frac{3}{4})\, \mathcal{E}_{(0)} &=& \quad 0;\\
\delta^0 \mathcal{L}^{10} \simeq 0 \qquad & (\Delta - \frac{5}{4})\, \mathcal{E}_{(2)} &=& \quad 0;\\
\delta^0 \mathcal{L}^{12} + \delta^{12} \mathcal{L}^{12} \simeq 0 \qquad & (\Delta - 12)\, \mathcal{E}_{(3)} &=& \quad -(\mathcal{E}_{(0)})^2
\end{aligned}$$

[Green, Sethi, Vanhove, Sinha, ...]

The type IIB string case

These equations together with the boundary conditions imply that

$$\begin{aligned}\mathcal{E}_{(0)}(\Omega) &= 2\zeta(3)E_{\frac{3}{2}}(\Omega)\\ \mathcal{E}_{(2)}(\Omega) &= \zeta(5)E_{\frac{5}{2}}(\Omega)\end{aligned}$$

Eisenstein series

$$E_s(\Omega)=\sum_{\gamma\in\Gamma_\infty\backslash SL(2,\mathbb{Z})}(\Im\mathrm{m}(\gamma\cdot\Omega))^s=\sum_{gcd(m,n)=1}\frac{(\Im\mathrm{m}\Omega)^s}{|m\Omega+n|^{2s}}$$

The type IIB string case

$E_{(3)}(\Omega)$ is not an Eisenstein series

$$(\Delta - 12)\,\mathcal{E}_{(3)} = -(2\zeta(3)E_{\frac{3}{2}})^2$$

The solution to the differential equation is given by

$$\mathcal{E}_{(3)}(\Omega) = \frac{2\zeta(3)^2}{3}E_{\frac{3}{2}}(\Omega) + \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \Phi(\gamma\Omega)$$

$$\Phi(x+iy) = 4\zeta(3)\int_{\mathbb{R}} \left(\sum_{n\in\mathbb{Z}} \sigma_{-2}(|n|)\, e^{2i\pi n(x+u)}\right) h\left(\frac{x}{y}\right) du$$

where $h(x)$ is the unique smooth even real function with $h(x) \sim_{x\to\pm\infty} 1/(6|x|^3)$ solving

$$\left(\frac{d}{dx}(1+x^2)\frac{d}{dx} - 12\right) h(x) = -\frac{1}{(1+x^2)^{\frac{3}{2}}}$$

Part III

U-dualities and and UV properties of maximal supergravity

Higher Rank Groups

In lower-dimensions the symmetry group of string theory increases and the theory combines in a higher rank duality group the duality we have discussed on the scalar Ω from the gravity sector and the one from the Abelian theory.

One important example is $\mathcal{N} = 8$ supergravity with 70 scalar parametrizing the coset space $E_{7(7)}(\mathbb{R})/(SU(8)/\mathbb{Z}_2)$ and 56 electromagnetic charges which Dirac charge quantization imply that the only the discrete subgroup $E_{7(7)}(\mathbb{Z}) = E_{7(7)}(\mathbb{R}) \cap Sp(56, \mathbb{Z})$

Dirac quantization of the electric/magnetic charges leads to the *same* lattice Γ as the one constructed by Chevalley method

Duality symmetries

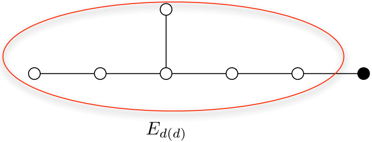
[Hull, Townsend]

D	$E_{11-D(11-D)}(\mathbb{R})$	K_D	$E_{11-D(11-D)}(\mathbb{Z})$
10A	$\mathbb{R}^+$	1	1
10B	$Sl(2,\mathbb{R})$	$SO(2)$	$Sl(2,\mathbb{Z})$
9	$Sl(2,\mathbb{R})\times\mathbb{R}^+$	$SO(2)$	$Sl(2,\mathbb{Z})$
8	$Sl(3,\mathbb{R})\times Sl(2,\mathbb{R})$	$SO(3)\times SO(2)$	$Sl(3,\mathbb{Z})\times Sl(2,\mathbb{Z})$
7	$Sl(5,\mathbb{R})$	$SO(5)$	$Sl(5,\mathbb{Z})$
6	$SO(5,5,\mathbb{R})$	$SO(5)\times SO(5)$	$SO(5,5,\mathbb{Z})$
5	$E_{6(6)}(\mathbb{R})$	$USp(8)$	$E_{6(6)}(\mathbb{Z})$
4	$E_{7(7)}(\mathbb{R})$	$SU(8)/\mathbb{Z}_2$	$E_{7(7)}(\mathbb{Z})$
3	$E_{8(8)}(\mathbb{R})$	$Spin(16)/\mathbb{Z}_2$	$E_{8(8)}(\mathbb{Z})$

- String theory on d-torus $T^d = S^1(R_1)\times\cdots\times S^1(R_d)$
- $E_{11-D(11-D)}$ real split forms, K_D maximal compact subgroup.
- Automorphy $\mathcal{E}_{(k)}(\gamma\cdot\vec{\varphi}) = \mathcal{E}_{(k)}(\vec{\varphi})$ for $\gamma\in\Gamma = E_{d(d)}(\mathbb{Z})$

From geometry to group theory

The duality group in $D+1$ dimensions E_d is embedded the maximal parabolic subgroup $P_{\alpha_{d+1}} = L_{\alpha_{d+1}} U$ where $L_{\alpha_{d+1}} = GL(1) \cdot E_d$ of E_{d+1}



The $GL(1)$ parameter is the radius of circle compactification of the theory from dimensions $D+1$ to D

From geometry to group theory

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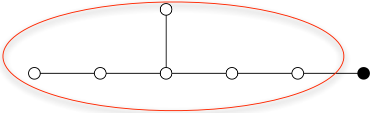
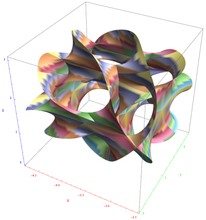
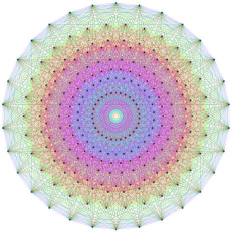


$E_{d(d)}$

In this construction all the information about the extra space time dimensions is contained in the higher rank duality group





Differential equations and critical dimensions

Recursion relation between the various dimensions implies [Green, Russo, Vanhove]

$$\left(\Delta - \frac{3(11-D)(D-8)}{D-2}\right)\mathcal{E}_{(0)} = 6\pi\delta_{D-8,0}$$

$$\left(\Delta - \frac{5(12-D)(D-7)}{D-2}\right)\mathcal{E}_{(2)} = 80\zeta(2)\delta_{D-7,0}$$

$$\left(\Delta - \frac{6(14-D)(D-6)}{D-2}\right)\mathcal{E}^{(D)}_{(3)} = -(\mathcal{E}_{(0)})^2 + \zeta(3)\delta_{D-6,0}$$

Vanishing eigenvalues in the critical dimensions for UV divergence at L-loop

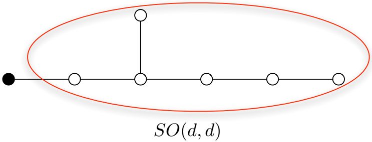
$$D_c = \begin{cases} 8 & \textit{for } L = 0 \\ 4 + 6/L & \textit{for } 2 \leqslant L \leqslant 4 \end{cases}$$

In $D = D_c$ $\mathcal{E}_{(k)} = c_{\mathrm{UV}} \log(g_s) + \cdots$ where c_{UV} UV counter-term

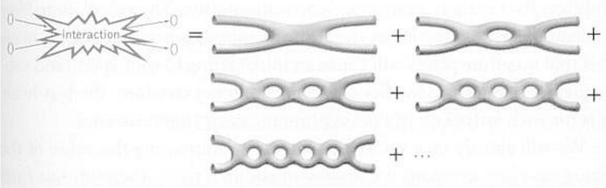
Complete agreement with $\mathcal{N} = 8$ multi-loop computations by [Bern et al.]

Boundary conditions in lower dimensions I

The symmetries of string perturbation (T-duality group $SO(d,d)$) is recovered from the maximal parabolic subgroup $P_{\alpha_1} = L_{\alpha_1} U_{\alpha_1}$ of E_{d+1} where the Levi factor is $GL(1) \cdot SO(d,d)$



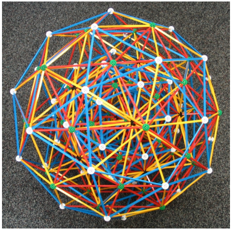
The $GL(1)$ factor is the string coupling constant g_D^2



Boundary conditions in lower dimensions I

In the zero instanton sector (the constant term in a Fourier expansion) we can see the action of supersymmetry in terms of non-renormalisation theorems

$$\begin{aligned}
\mathcal{E}_{(0)}\big|_{\text{pert}} &= g_D^{-2\frac{8-D}{D-2}} \left(\frac{a_{\text{tree}}}{g_D^2} + I_{1-\text{loop}}\right)\\
\mathcal{E}_{(2)}\big|_{\text{pert}} &= g_D^{-4\frac{7-D}{D-2}} \left(\frac{a_{\text{tree}}}{g_D^4} + \frac{1}{g_D^2} I_{1-\text{loop}} + I_{2-\text{loop}}\right)\\
\mathcal{E}_{(3)}\big|_{\text{pert}} &= g_D^{-6\frac{6-D}{D-2}} \left(\frac{a_{\text{tree}}}{g_D^6} + \frac{1}{g_D^4} I_{1-\text{loop}} + \frac{1}{g_D^2} I_{2-\text{loop}} + I_{3-\text{loop}} + O(e^{-\frac{1}{g_D}})\right)
\end{aligned}$$



We could have as many term as elements of the Weyl group $|W(E_7)| = 2903040$; $|W(E_8)| = 696729600$ but the answer picked by string theory is much simpler

String theory picks remarkably simple solutions

- Boundary conditions from string/M-theory allow to determine the *unique* solution [Green, Russo, Miller, Vanhove]

$E_{d+1}(\mathbb{Z})$	$\mathcal{E}_{(0)}$	$\mathcal{E}_{(1)}$
$E_{8(8)}(\mathbb{Z})$	$2\zeta(3)\mathbf{E}^{E_8}_{[1\,0^7];\frac{3}{2}}$	$\zeta(5)\,\mathbf{E}^{E_8}_{[1\,0^7];\frac{5}{2}}$
$E_{7(7)}(\mathbb{Z})$	$2\zeta(3)\mathbf{E}^{E_7}_{[1\,0^6];\frac{3}{2}}$	$\zeta(5)\,\mathbf{E}^{E_7}_{[1\,0^6];\frac{5}{2}}$
$E_{6(6)}(\mathbb{Z})$	$2\zeta(3)\mathbf{E}^{E_6}_{[1\,0^5];\frac{3}{2}}$	$\zeta(5)\,\mathbf{E}^{E_6}_{[1\,0^5];\frac{5}{2}}$
$SO(5,5,\mathbb{Z})$	$2\zeta(3)\mathbf{E}^{SO(5,5)}_{[10000];\frac{3}{2}}$	$\zeta(5)\,\widehat{\mathbf{E}}^{SO(5,5)}_{[10000];\frac{5}{2}} + \frac{8\zeta(6)}{45}\widehat{\mathbf{E}}^{SO(5,5)}_{[00001];3}$
$SL(5,\mathbb{Z})$	$2\zeta(3)\mathbf{E}^{SL(5)}_{[1000];\frac{3}{2}}$	$\zeta(5)\,\widehat{\mathbf{E}}^{SL(5)}_{[1000];\frac{5}{2}} + \frac{6\zeta(5)}{\pi^3}\widehat{\mathbf{E}}^{SL(5)}_{[0010];\frac{5}{2}}$
$SL(3,\mathbb{Z})\times SL(2,\mathbb{Z})$	$2\zeta(3)\widehat{\mathbf{E}}^{SL(3)}_{[10];\frac{3}{2}} + 2\widehat{\mathbf{E}}_1(U)$	$\zeta(5)\,\mathbf{E}^{SL(3)}_{[10];\frac{5}{2}} - 8\zeta(4)^{SL(3)}_{[10];-\frac{1}{2}}\,\mathbf{E}_2(U)$
$SL(2,\mathbb{Z})$	$2\zeta(3)\mathbf{E}_{\frac{3}{2}}(\Omega)\,\nu_1^{-\frac{3}{7}} + 4\zeta(2)\,\nu_1^{\frac{4}{7}}$	$\frac{\zeta(5)\mathbf{E}_{\frac{5}{2}}}{\nu_1^{\frac{5}{7}}} + \frac{4\zeta(2)\zeta(3)}{15}\nu_1^{\frac{9}{7}}\mathbf{E}_{\frac{3}{2}} + \frac{4\zeta(2)\zeta(3}{15\nu_1^{\frac{12}{7}}}$
$SL(2,\mathbb{Z})$	$2\zeta(3)\mathbf{E}_{\frac{3}{2}}(\Omega)$	$\zeta(5)\,\mathbf{E}_{\frac{5}{2}}(\Omega)$

Black hole contributions

- The instantonic black hole contributions are obtained from the Fourier modes

- If $\theta_i \in U \cong \mathbb{R}^m$ is a set of continuous commuting charges of non-perturbative contributions we define *the fourier modes*

$$\mathcal{F}_{(k)}[Q,\varphi] := \int_{[0,1]^m} d\theta\, \mathcal{E}_{(k)}\, e^{2i\pi Q\cdot\theta}$$

- In the decompactification limit $r_d \gg \ell_{D+1}$ the Fourier modes take the form

$$\mathcal{F}_{(k)}[Q,\varphi] \sim \sum_{Q\in\mathbb{Z}^n} d_Q f_Q(\varphi)\, e^{-2\pi r_d\, m_Q}$$

Black hole contributions

$$\mathcal{F}_{(k)}[Q,\varphi] \sim \sum_{Q\in\mathbb{Z}^n} d_Q f_Q(\varphi)\, e^{-2\pi r_d\, m_Q}$$

- The Fourier transform induces a condition on the discrete charges Q in the charge lattice $E_{d+1}(\mathbb{Z}) \cap U_{\alpha_{d+1}}$ which lies in *discrete orbits*
- m_Q is the mass of the BPS particle state in dimension $D+1$ that lead to an instanton once wrapped on the (euclidean) circle of radius r_d
- d_Q is a number theoretic function counting the instanton configurations
- $f_Q(\varphi)$, due to the quantum fluctuations, is a function of the moduli invariant under the symmetry group E_d in dimension $D+1$

This solves a very difficult math problem

Comments on the impact of Dynkin's work on current research in representation theory

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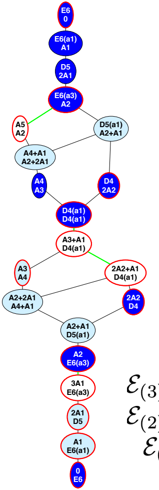
A central problem in the representations of reductive Lie groups is constructing unitary representations attached to the nilpotent coadjoint orbits.

In a related direction, Arthur's conjectures (still unproved) relate homomorphisms of $SL(2)$ to residues of Eisenstein series. Colette Moeglin has done great work in the direction of proving that the residues predicted by Arthur (with Dynkin's tables) actually exist. The residues give rise to interesting unitary automorphic representations that are difficult to construct in any other way. Her first paper on this subject is "Orbites unipotentes



$$\mathcal{E}_{(3)}\, D^6\mathcal{R}^4$$
$$\mathcal{E}_{(2)}\, D^4\mathcal{R}^4$$
$$\mathcal{E}_{(0)}\, \mathcal{R}^4$$
$$1\,\mathcal{R}$$

Conclusion

Modular and automorphic symmetry of string theory is fundamental for getting a consistent theory of quantum gravity.

Any truncation of the theory keeping only a subset of non-perturbative effects breaks the symmetry and cannot be a consistent theory at high energy

The consistency is different from the notion of UV finiteness : A theory can be finite in perturbation but in need of non perturbative contribution for being defined for all ranges of coupling constant and energy