Feynman integrals and the functions associated to them

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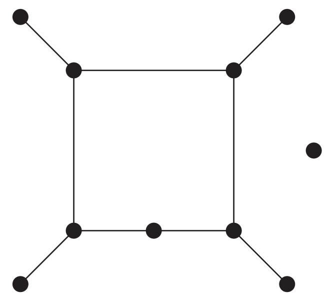
I: **Feynman graph polynomials**

II: **Multiple polylogarithms**

III: **Elliptic curves**

Definition of the Feynman graph polynomials through spanning trees and spanning forests

Graphs



A graph consists of edges and vertices.

The valency of a vertex is the number of edges attached to it.

- Vertices of valency 0 are usually not considered.

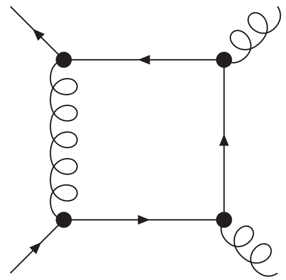
- Vertices of valency 1: The edge attached to an vertex of valency 1 is called an external edge.
  All other edges are called internal edges.
  In physics one usually does not draw a vertex of valency 1.

- Vertices of valency 2 are called mass insertions and usually not considered.

- Therefore in physics it is usually implied that a vertex has valency ≥ 3.

Feynman graphs

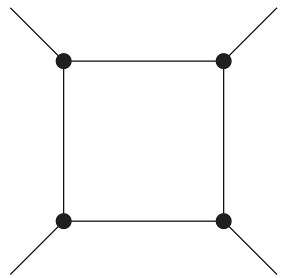
An edge in a Feynman graph represents a propagating particle.
The edge is drawn in a way as to represent the type of the particle.



To each edge we associate a vector, the momentum of the particle.

At each vertex we have momentum conservation:
Sum of all incoming momenta = sum of all outgoing momenta.

Neglecting all this extra information, we speak about the underlying topology.



The loop number

Consider a graph G with n edges and r vertices. Assume that the graph has k connected components.

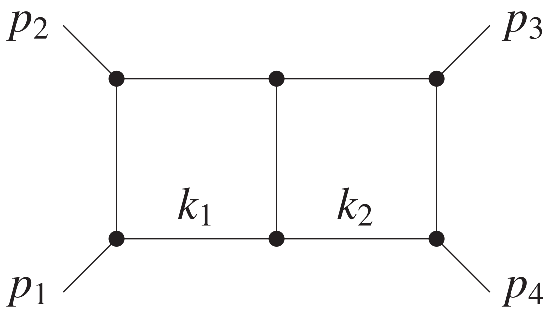
The loop number l is defined by

$$l = n - r + k.$$

This number is also called the first Betti number of the graph or the cyclomatic number.

In a Feynman graph: If we fix all momenta on external lines and impose momentum conservation at each vertex, then the loop number is equal to the number of independent momenta vectors not constrained by momentum conservation.

Example: A two-loop graph



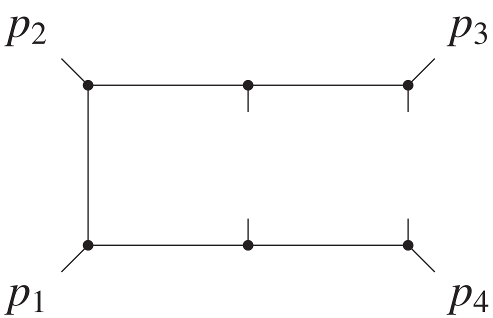
Trees, spanning trees and spanning forests

A connected graph of loop number 0 is called a tree.

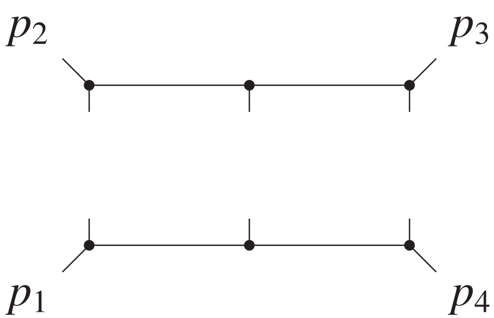
A graph of loop number 0 is called a forest.
If the forest has k connected components, it is called a k-forest.

Given an arbitrary connected graph G, a spanning tree of G is a subgraph, which contains all the vertices of G and which is a tree.

Given an arbitrary connected graph G, a spanning k-forest of G is a subgraph, which contains all the vertices of G and which is a k-forest.

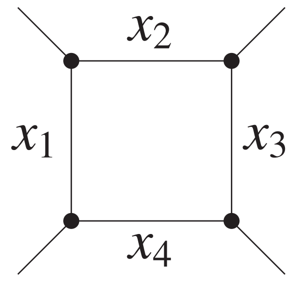


A spanning tree



A spanning 2-forest

Feynman parameters



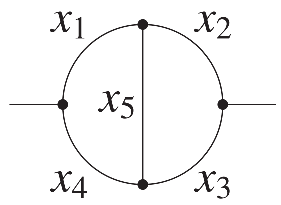
Step 1 for the construction of a graph polynomial:

To each internal edge j we associate a real (or complex) variable x_j.

The variables x_j are called Feynman parameters.

The first Symanzik polynomial

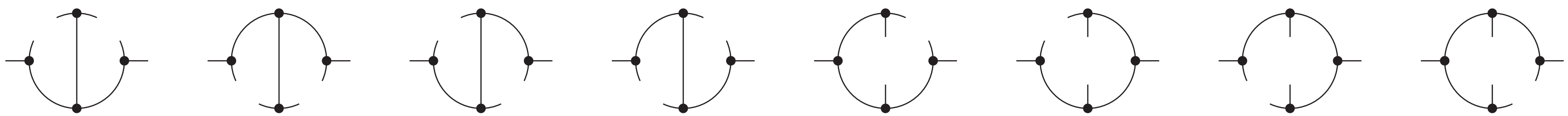
Let G be a connected graph and $\mathcal{T}_1$ the set of its spanning trees.



The first Symanzik polynomial is defined by

$$\mathcal{U} = \sum_{T \in \mathcal{T}_1} \prod_{e_j \notin T} x_j,$$

Example:



$$x_1 x_2 + x_3 x_4 + x_1 x_3 + x_2 x_4 + x_2 x_5 + x_1 x_5 + x_4 x_5 + x_3 x_5$$

The Kirchhoff polynomial

In mathematics, the Kirchhoff polynomial of a graph is better known. It is defined by

$$\mathcal{K} = \sum_{T \in \mathcal{T}_1} \prod_{e_j \in T} x_j$$

Compare this definition to the definition of the first Symanzik polynomial:

$$\mathcal{U} = \sum_{T \in \mathcal{T}_1} \prod_{e_j \notin T} x_j$$

Relation between the Kirchhoff polynomial $\mathcal{K}$ and the first Symanzik polynomial $\mathcal{U}$:

$$\mathcal{U}(x_1, ..., x_n) = x_1 ... x_n \mathcal{K}\left(\frac{1}{x_1}, ..., \frac{1}{x_n}\right)$$

The second Symanzik polynomial

Let G be a connected graph and $\mathcal{T}_2$ the set of its spanning 2-forests.
An element of $\mathcal{T}_2$ is denoted as (T_1, T_2).

Let further denote P_{T_i} the set of external momenta of G attached to T_i.

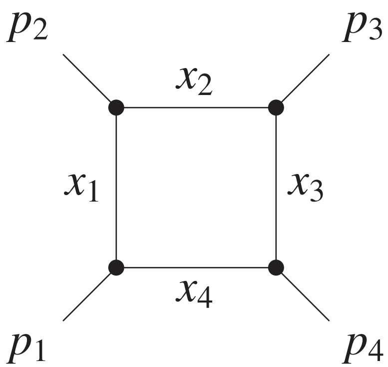
The second Symanzik polynomial is defined for massless particles by

$$\mathcal{F} = \sum_{(T_1,T_2)\in\mathcal{T}_2} \left(\prod_{e_i \notin (T_1,T_2)} x_i \right) \left(\sum_{p_j \in P_{T_1}} \sum_{p_k \in P_{T_2}} \frac{p_j \cdot p_k}{\mu^2} \right)$$

$p_j \cdot p_k$ is the Minkowski scalar product of two momenta vectors.

μ is an arbitrary scale introduced to make the expression dimensionless.

Example

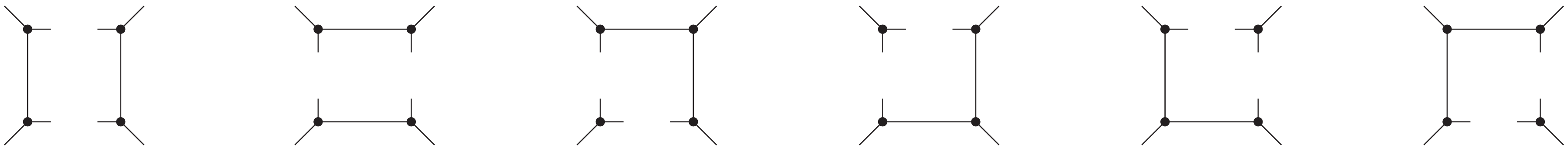


$$p_1 + p_2 + p_3 + p_4 = 0$$

$$s = (p_1 + p_2)^2$$

$$t = (p_2 + p_3)^2$$

For $\mathcal{F}$ we obtain:



$$x_2 x_4 \frac{(-s)}{\mu^2} + x_1 x_3 \frac{(-t)}{\mu^2} + x_1 x_4 \frac{(-p_1^2)}{\mu^2} + x_1 x_2 \frac{(-p_2^2)}{\mu^2} + x_2 x_3 \frac{(-p_3^2)}{\mu^2} + x_3 x_4 \frac{(-p_4^2)}{\mu^2}$$

Basic properites of the Symanzik polynomials

- They are homogeneous in the Feynman parameters, $\mathcal{U}$ is of degree l, $\mathcal{F}$ is of degree $l+1$.
- $\mathcal{U}$ is linear in each Feynman parameter. If all internal masses are zero, then also $\mathcal{F}$ is linear in each Feynman parameter.
- In expanded form each monomial of $\mathcal{U}$ has coefficient $+1$.

Definition of the Feynman graph polynomials as it is done in the text books of physics

Feynman rules

Each part in a Feynman graph corresponds to a mathematical expression.
In the simplest version:

- Edge:
$$\frac{i}{q^2 - m^2}$$
- Vertex:
$$1$$
- External line:
$$1$$
- For each internal momentum not constrained by momentum conservation
$$\int \frac{d^D k}{(2\pi)^D}$$

Feynman integrals

A Feynman graph with m external lines, n internal lines and l loops corresponds (up to prefactors) in D space-time dimensions to the Feynman integral

$$I_G = \left(\mu^2\right)^{n-lD/2} \int \prod_{r=1}^{l} \frac{d^D k_r}{i\pi^{\frac{D}{2}}} \prod_{j=1}^{n} \frac{1}{\left(-q_j^2+m_j^2\right)}$$

The momenta flowing through the internal lines can be expressed through the independent loop momenta k_1, ..., k_l and the external momenta p_1, ..., p_m as

$$q_i = \sum_{j=1}^{l} \lambda_{ij} k_j + \sum_{j=1}^{m} \sigma_{ij} p_j, \qquad \lambda_{ij}, \sigma_{ij} \in \{-1, 0, 1\}.$$

Feynman parametrisation

The Feynman trick:

$$\prod_{j=1}^{n} \frac{1}{P_j} = \Gamma(n) \int_{x_j \geq 0} d^n x \, \delta(1 - \sum_{j=1}^{n} x_j) \frac{1}{\left(\sum_{j=1}^{n} x_j P_j \right)^n}$$

We use this formula with $P_j = -q_j^2 + m_j^2$.
We can write

$$\sum_{j=1}^{n} x_j (-q_j^2 + m_j^2) = -\sum_{r=1}^{l} \sum_{s=1}^{l} k_r M_{rs} k_s + \sum_{r=1}^{l} 2 k_r \cdot Q_r + J,$$

where M is a $l \times l$ matrix with scalar entries and Q is a l-vector with momenta vectors as entries.

Feynman integrals

After Feynman parametrisation the integrals over the loop momenta k_1, ..., k_l can be done:

$$I_G = \Gamma(n - lD/2) \int\limits_{x_j \geq 0} d^n x \, \delta(1 - \sum_{i=1}^{n} x_i) \frac{\mathcal{U}^{n-(l+1)D/2}}{\mathcal{F}^{n-lD/2}},$$

with

$$\mathcal{U} = \det(M), \qquad \mathcal{F} = \det(M) \left(J + Q M^{-1} Q \right) / \mu^2.$$

This provides a second definition of the Feynman graph polynomials $\mathcal{U}$ and $\mathcal{F}$.

Remarks

$$I_G = \Gamma(n - lD/2) \int\limits_{x_j \geq 0} d^n x \, \delta(1 - \sum_{i=1}^{n} x_i) \frac{\mathcal{U}^{n-(l+1)D/2}}{\mathcal{F}^{n-lD/2}},$$

- The integral over the Feynman parameters is a $(n-1)$-dimensional integral, where n is the number of internal edges of the graph.
- The dimension D of space-time enters only in the exponent of the integrand.
- Singularities may arise if the zero sets of $\mathcal{U}$ and $\mathcal{F}$ intersect the region of integration.
- The exponent acts as a regularisation.

Schwinger parametrisation

Feynman parametrisation and Schwinger parametrisation are equivalent:
For $\mathrm{Im}(P) < 0$ one has

$$\frac{1}{P} = i\int_0^\infty d\alpha\, e^{-i\alpha P}$$

and therefore

$$\prod_{j=1}^{n}\frac{1}{P_j} = i^n \int_{\alpha_j\geq 0} d^n\alpha\, e^{-i\sum\limits_{j=1}^{n}\alpha_j P_j}$$

But we can write this with $x_j = \alpha_j/\lambda$ as

$$\prod_{j=1}^{n}\frac{1}{P_j} = i^n \int_{\alpha_j\geq 0} d^n\alpha \int_0^\infty d\lambda\, \delta(\lambda - \sum_{j=1}^{n}\alpha_j)\, e^{-i\sum\limits_{j=1}^{n}\alpha_j P_j} = i^n \int_{x_j\geq 0} d^n x\, \delta(1-\sum_{j=1}^{n}x_j)\int_0^\infty d\lambda\, \lambda^{n-1}\, e^{-i\lambda\sum\limits_{j=1}^{n}x_j P_j}$$

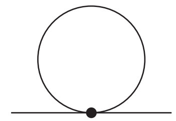
$$= \Gamma(n)\int_{x_j\geq 0} d^n x\, \delta(1-\sum_{j=1}^{n}x_j)\frac{1}{\left(\sum\limits_{j=1}^{n}x_j P_j\right)^n}$$

Definition of the Feynman graph polynomials through the Laplacian of the graph

The matrix tree theorem

For a graph G with n edges and r vertices define the Laplacian L as a $r \times r$-matrix with

$$L_{ij} = \begin{cases} \sum x_k & \text{if } i = j \text{ and edge } e_k \text{ is attached to } v_i \text{ and is not a self-loop,} \\ -\sum x_k & \text{if } i \neq j \text{ and edge } e_k \text{ connects } v_i \text{ and } v_j. \end{cases}$$



self-loop

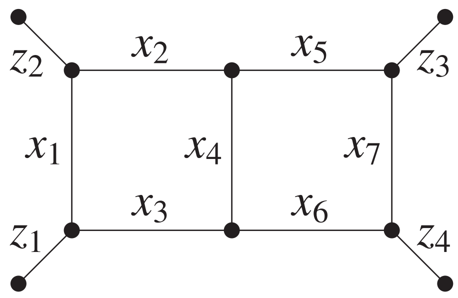
Denote by $L[i]$ the $(r-1) \times (r-1)$-matrix obtained from L by deleting the i-th row and column.

Matrix-tree theorem:

$$\mathcal{K} = \det L[i]$$

The all-minor matrix tree theorem

Consider a graph with n internal edges, r internal vertices $(v_1, ..., v_r)$ and m external legs.



Attach m additional vertices $(v_{r+1}, ..., v_{r+m})$ to the end of the external legs.
Associate parameters z_1, ..., z_m with the external edges.

Consider the polynomial

$$\mathcal{W}(x_1, ..., x_n, z_1, ..., z_m) \quad = \quad \det L[r+1, ..., r+m]$$

Expand $\mathcal{W}$ in polynomials homogeneous in the variables z_j:

$$\mathcal{W} \quad = \quad \mathcal{W}^{(0)} + \mathcal{W}^{(1)} + \mathcal{W}^{(2)} + ... + \mathcal{W}^{(m)}$$
$$\mathcal{W}^{(k)} \quad = \quad \sum_{1 \le j_1 < ... < j_k \le m} \mathcal{W}^{(k)}_{(j_1, ..., j_k)}(x_1, ..., x_n)\, z_{j_1} ... z_{j_k}$$

The all-minor matrix tree theorem

We then have

$$\begin{aligned} \mathcal{W}^{(0)} &= 0, \\ \mathcal{U} &= x_1 \dots x_n \, \mathcal{W}^{(1)}_{(j)} \left(\frac{1}{x_1}, \dots, \frac{1}{x_n} \right) \qquad \text{for any } j. \end{aligned}$$

For massless particles we also have

$$\mathcal{F} = x_1 \dots x_n \sum_{(j,k)} \left(\frac{p_j \cdot p_k}{\mu^2} \right) \cdot \mathcal{W}^{(2)}_{(j,k)} \left(\frac{1}{x_1}, \dots, \frac{1}{x_n} \right)$$

This provides a third definition of the Feynman graph polynomials $\mathcal{U}$ and $\mathcal{F}$.

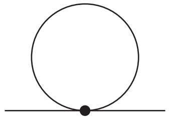
This formulation is particularly well suited for computer algebra.

Feynman graph polynomials, part IV

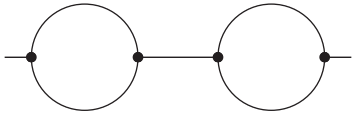
Definition of the Feynman graph polynomials through deletion and contraction properties

Terminology

Regular edge: Neither a self-loop nor a bridge

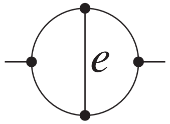


self-loop

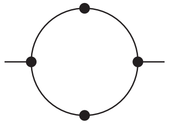


bridge

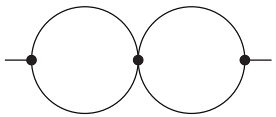
G/e	Graph obtained from G by contracting the regular edge e,
$G-e$	Graph obtained from G by deleting the regular edge e.



G



$G-e$



G/e

Deletion and contraction properties

Recursive definition of the Feynman graph polynomials for massless particles:
For any regular edge e_k we have

$$\begin{aligned} \mathcal{U}(G) &= \mathcal{U}(G/e_k) + x_k \mathcal{U}(G - e_k), \\ \mathcal{F}(G) &= \mathcal{F}(G/e_k) + x_k \mathcal{F}(G - e_k). \end{aligned}$$

Recursion terminates when all edges are either bridges or self-loops.
For a terminal form we have

$$\mathcal{U} = x_r \dots x_n, \qquad \mathcal{F} = x_r \dots x_n \sum_{j=1}^{r-1} x_j \left(\frac{-q_j^2}{\mu^2} \right),$$

where we labelled the edges, which are bridges from 1 to $(r-1)$, and the ones which are self-loops from r to n.
q_j is the momentum flowing throught the bridge j.

Dodgson's identity

Let A be a $n \times n$ matrix.

$A[i]$	$(n-1) \times (n-1)$ matrix obtained from A by deleting the i-th row and column
$A[i;j]$	$(n-1) \times (n-1)$ matrix obtained from A by deleting the i-th row and the j-th column
$A[i,j]$	$(n-2) \times (n-2)$ matrix obtained from A by deleting the rows and columns i and j

Dodgson's identity reads:

$$\det(A)\det(A[i,j]) \quad = \quad \det(A[i])\det(A[j]) - \det(A[i;j])\det(A[j;i])$$

Factorisation theorems

Let e_a and e_b be two regular edges, which share a common vertex.

From Dodgson's identity one obtains the following factorisation theorems (for massless particles):

$$\mathcal{U}(G/e_a - e_b)\,\mathcal{U}(G/e_b - e_a) - \mathcal{U}(G - e_a - e_b)\,\mathcal{U}(G/e_a/e_b) = \left(\frac{\Delta_1}{x_a x_b}\right)^2,$$

$$\begin{aligned} &\mathcal{U}(G/e_a - e_b)\,\mathcal{F}(G/e_b - e_a) - \mathcal{U}(G - e_a - e_b)\,\mathcal{F}(G/e_a/e_b) \\ &+\mathcal{F}(G/e_a - e_b)\,\mathcal{U}(G/e_b - e_a) - \mathcal{F}(G - e_a - e_b)\,\mathcal{U}(G/e_a/e_b) = 2\left(\frac{\Delta_1}{x_a x_b}\right)\left(\frac{\Delta_2}{x_a x_b}\right). \end{aligned}$$

If for all external momenta one has $(p_{i_1} \cdot p_{i_2}) \cdot (p_{i_3} \cdot p_{i_4}) = (p_{i_1} \cdot p_{i_3}) \cdot (p_{i_2} \cdot p_{i_4})$, then

$$\mathcal{F}(G/e_a - e_b)\,\mathcal{F}(G/e_b - e_a) - \mathcal{F}(G - e_a - e_b)\,\mathcal{F}(G/e_a/e_b) = \left(\frac{\Delta_2}{x_a x_b}\right)^2.$$

Remarks

- Factorisation theorems can be used for the computation of Feynman integrals.
- Δ_1 and Δ_2 can be expressed as sums over 2-forests and 3-forests, respectively.
- Generalisation to matroid theory.

Summary on Feynman graph polynomials

- Feynman graph polynomials define the integrand of a Feynman integral.
  The zero sets of the polynomials are related to the divergences of the integral.

- Feynman graph polynomials are related to spanning trees and spanning forests of the corresponding graph.

- Feynman graph polynomials are related to the Laplacian of the graph.

- Feynman graph polynomials have a recursive definition based on deletion and contraction.
  Factorisation theorems can be exploited in computational algorithms.

Part II

Feynman integrals and multiple polylogarithms

Feynman integrals

$$I_G = \Gamma(n - lD/2) \int\limits_{x_j \geq 0} d^n x \, \delta(1 - \sum_{i=1}^{n} x_i) \frac{\mathcal{U}^{n-(l+1)D/2}}{\mathcal{F}^{n-lD/2}},$$

$\mathcal{U}$ is a homogeneous polynomial in the Feynman parameters of degree l, positive definite inside the integration region and positive semi-definite on the boundary.

$\mathcal{F}$ is a homogeneous polynomial in the Feynman parameters of degree $l+1$ and depends in addition on the masses m_i^2 and the momenta $(p_{i_1} + ... + p_{i_r})^2$.
In the euclidean region it is also positive definite inside the integration region and positive semi-definite on the boundary.

Laurent expansion in $\varepsilon = (4-D)/2$:

$$I_G = \sum_{j=-2l}^{\infty} c_j \varepsilon^j.$$

One-loop amplitudes

All one-loop amplitudes can be expressed as a sum of algebraic functions of the spinor products and masses times two transcendental functions, whose arguments are again algebraic functions of the spinor products and the masses.

The two transcendental functions are the logarithm and the dilogarithm:

$$\begin{aligned} \text{Li}_1(x) &= -\ln(1-x) = \sum_{n=1}^{\infty} \frac{x^n}{n} \\ \text{Li}_2(x) &= \sum_{n=1}^{\infty} \frac{x^n}{n^2} \end{aligned}$$

Generalisations of the logarithm

Beyond one-loop, at least the following generalisations occur:

Polylogarithms:

$$\text{Li}_m(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^m}$$

Multiple polylogarithms (Goncharov 1998):

$$\text{Li}_{m_1,m_2,...,m_k}(x_1,x_2,...,x_k) = \sum_{n_1>n_2>...>n_k>0}^{\infty} \frac{x_1^{n_1}}{n_1^{m_1}} \cdot \frac{x_2^{n_2}}{n_2^{m_2}} \cdot \ldots \cdot \frac{x_k^{n_k}}{n_k^{m_k}}$$

This is a nested sum:

$$\ldots \sum_{n_j=1}^{n_{j-1}-1} \frac{x_j^{n_j}}{n_j^{m_j}} \sum_{n_{j+1}=1}^{n_j-1} \ldots$$

Multiple ζ-values

The values of the multiple polylogarithms at $x_1 = \dots x_k = 1$ are called multiple ζ-values:

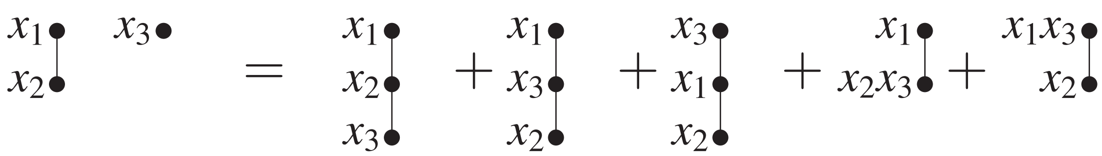
$$\begin{aligned}\zeta_{m_1,\dots,m_k} &= \mathrm{Li}_{m_1,m_2,\dots,m_k}(1,1,\dots,1)\\ &= \sum_{n_1>n_2>\dots>n_k>0}^{\infty} \frac{1}{n_1^{m_1}} \cdot \frac{1}{n_2^{m_2}} \cdot \dots \cdot \frac{1}{n_k^{m_k}}\end{aligned}$$

Multiplication

Multiple polylogarithms obey an algebra:

$$\begin{aligned}
&\text{Li}_{m_1,m_2}(x_1,x_2)\cdot\text{Li}_{m_3}(x_3) = \\
&\quad = \quad \text{Li}_{m_1,m_2,m_3}(x_1,x_2,x_3)+\text{Li}_{m_1,m_3,m_2}(x_1,x_3,x_2)+\text{Li}_{m_3,m_1,m_2}(x_3,x_1,x_2) \\
&\qquad\quad +\text{Li}_{m_1,m_2+m_3}(x_1,x_2x_3)+\text{Li}_{m_1+m_3,m_2}(x_1x_3,x_2)
\end{aligned}$$

Pictorial representation:



The multiplication law corresponds to a quasi-shuffle algebra (Hoffman '99), also called stuffle algebra (Broadhurst), mixed shuffle algebra (Guo) or mould symmetrel (Ecalle).

Hopf algebras

The multiple polylogarithms form actually a Hopf algebra.

- An algebra has a multiplication $\cdot$ and a unit e.

- A coalgebra has a comultiplication Δ and a counit $\bar{e}$.

$$\Delta\left(\begin{array}{l}\bullet x_1\\ \bullet x_2\\ \bullet x_3\end{array}\right) = 1 \otimes \begin{array}{l}\bullet x_1\\ \bullet x_2\\ \bullet x_3\end{array} + \bullet x_3 \otimes \begin{array}{l}\bullet x_1\\ \bullet x_2\end{array} + \begin{array}{l}\bullet x_2\\ \bullet x_3\end{array} \otimes \bullet x_1 + \begin{array}{l}\bullet x_1\\ \bullet x_2\\ \bullet x_3\end{array} \otimes 1.$$

- A Hopf algebra is an algebra and a coalgebra at the same time, such that the two structures are compatible with each other.

  In addition, there is an antipode S.

Iterated integrals

Define the functions G by

$$G(z_1,...,z_k;y) = \int_0^y \frac{dt_1}{t_1 - z_1} \int_0^{t_1} \frac{dt_2}{t_2 - z_2} ... \int_0^{t_{k-1}} \frac{dt_k}{t_k - z_k}.$$

Scaling relation:

$$G(z_1,...,z_k;y) = G(xz_1,...,xz_k;xy)$$

Short hand notation:

$$G_{m_1,...,m_k}(z_1,...,z_k;y) = G(\underbrace{0,...,0}_{m_1-1},z_1,...,z_{k-1},\underbrace{0...,0}_{m_k-1},z_k;y)$$

Conversion to multiple polylogarithms:

$$\text{Li}_{m_1,...,m_k}(x_1,...,x_k) = (-1)^k G_{m_1,...,m_k}\left(\frac{1}{x_1},\frac{1}{x_1 x_2},...,\frac{1}{x_1 ... x_k};1\right).$$

Shuffle algebra

The functions $G(z_1,...,z_k;y)$ fulfill a shuffle algebra.

Example:

$$G(z_1,z_2;y)G(z_3;y) = G(z_1,z_2,z_3;y)+G(z_1,z_3,z_2;y)+G(z_3,z_1,z_2;y)$$

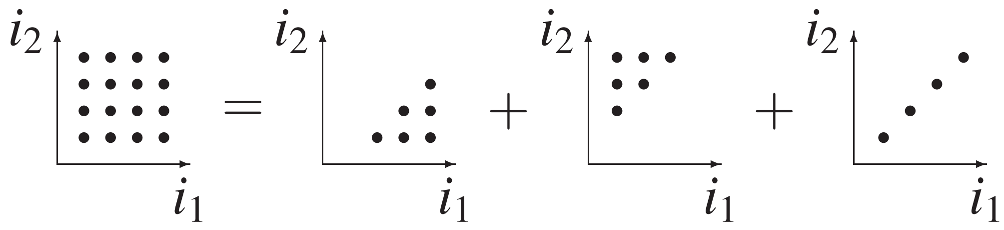
This algebra is different from the quasi-shuffle algebra already encountered and provides the second Hopf algebra for multiple polylogarithms.

A shuffle algebra is also called a mould symmetral (Ecalle).

Shuffle algebra versus quasi-shuffle algebra

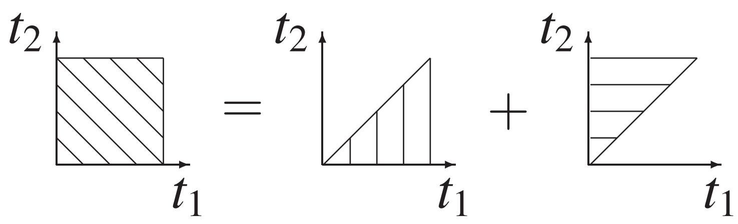
Quasi-shuffle algebra from the sum representation:

$$\mathrm{Li}_{m_1}(x_1)\mathrm{Li}_{m_2}(x_2) \quad = \quad \mathrm{Li}_{m_1,m_2}(x_1,x_2)+\mathrm{Li}_{m_2,m_1}(x_2,x_1)+\mathrm{Li}_{m_1+m_2}(x_1x_2).$$



Shuffle algebra from the integral representation:

$$G(z_1;y)G(z_2;y) \quad = \quad G(z_1,z_2;y)+G(z_2,z_1;y)$$



Mellin-Barnes

Mellin-Barnes transformation:

$$(A_1+A_2+\ldots+A_n)^{-c} = \frac{1}{\Gamma(c)} \frac{1}{(2\pi i)^{n-1}} \int\limits_{-i\infty}^{i\infty} d\sigma_1 \ldots \int\limits_{-i\infty}^{i\infty} d\sigma_{n-1}$$

$$\times \Gamma(-\sigma_1) \ldots \Gamma(-\sigma_{n-1}) \Gamma(\sigma_1+\ldots+\sigma_{n-1}+c) \; A_1^{\sigma_1} \ldots A_{n-1}^{\sigma_{n-1}} A_n^{-\sigma_1-\ldots-\sigma_{n-1}-c}$$

The contour is such that the poles of $\Gamma(-\sigma)$ are to the right and the poles of $\Gamma(\sigma+c)$ are to the left.

Converts a sum into products and is therefore the “inverse” of Feynman parametrization.

Smirnov; Tausk; Davydychev; Bierenbaum, S.W.; Czakon; Anastasiou, Daleo; Gluza, Kajda, Riemann;

Higher transcendental functions

More generally, we get the following types of infinite sums:

- Type A:
$$\sum_{i=0}^{\infty} \frac{\Gamma(i+a_1)...\Gamma(i+a_k)}{\Gamma(i+a'_1)...\Gamma(i+a'_k)} x^i$$

  Example: Hypergeometric functions ${}_{J+1}F_J$ (up to prefactors).

- Type B:
$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{\Gamma(i+a_1)...\Gamma(i+a_k)}{\Gamma(i+a'_1)...\Gamma(i+a'_k)} \frac{\Gamma(j+b_1)...\Gamma(j+b_l)}{\Gamma(j+b'_1)...\Gamma(j+b'_l)} \frac{\Gamma(i+j+c_1)...\Gamma(i+j+c_m)}{\Gamma(i+j+c'_1)...\Gamma(i+j+c'_m)} x^i y^j$$

  Example: First Appell function F_1.

- Type C:
$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \binom{i+j}{j} \frac{\Gamma(i+a_1)...\Gamma(i+a_k)}{\Gamma(i+a'_1)...\Gamma(i+a'_k)} \frac{\Gamma(i+j+c_1)...\Gamma(i+j+c_m)}{\Gamma(i+j+c'_1)...\Gamma(i+j+c'_m)} x^i y^j$$

  Example: Kampé de Fériet function S_1.

- Type D:
$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \binom{i+j}{j} \frac{\Gamma(i+a_1)...\Gamma(i+a_k)}{\Gamma(i+a'_1)...\Gamma(i+a'_k)} \frac{\Gamma(j+b_1)...\Gamma(j+b_l)}{\Gamma(j+b'_1)...\Gamma(j+b'_l)} \frac{\Gamma(i+j+c_1)...\Gamma(i+j+c_m)}{\Gamma(i+j+c'_1)...\Gamma(i+j+c'_m)} x^i y^j$$

  Example: Second Appell function F_2.

All a, b, c's are of the form "integer $+$ const $\cdot\, \varepsilon$".

Introducing nested sums

- Definition of Z-sums:

$$Z(n;m_1,...,m_k;x_1,...,x_k) = \sum_{n \geq i_1 > i_2 > ... > i_k > 0} \frac{x_1^{i_1}}{i_1^{m_1}} \frac{x_2^{i_2}}{i_2^{m_2}} \cdots \frac{x_k^{i_k}}{i_k^{m_k}}.$$

- Multiple polylogarithms ($n = \infty$) are a special subset

$$\text{Li}_{m_1,...,m_k}(x_1,...,x_k) = \sum_{i_1 > i_2 > ... > i_k > 0}^{\infty} \frac{x_1^{i_1}}{i_1^{m_1}} \frac{x_2^{i_2}}{i_2^{m_2}} \cdots \frac{x_k^{i_k}}{i_k^{m_k}}$$

- Euler-Zagier sums ($x_1 = ... = x_k = 1$) are a special subset

$$Z_{m_1,...,m_k}(n) = \sum_{i_1 > i_2 > ... > i_k > 0}^{n} \frac{1}{i_1^{m_1}} \frac{1}{i_2^{m_2}} \cdots \frac{1}{i_k^{m_k}}$$

- Multiple ζ-values ($n = \infty, x_1 = ... = x_k = 1$) are a special subset

$$\zeta_{m_1,...,m_k} = \sum_{i_1 > i_2 > ... > i_k > 0}^{\infty} \frac{1}{i_1^{m_1}} \frac{1}{i_2^{m_2}} \cdots \frac{1}{i_k^{m_k}}$$

Expansion of Gamma functions

Euler-Zagier sums (or harmonic sums) occur in the expansion for Γ functions: For positive integers n we have

$$\Gamma(n+\varepsilon) = \Gamma(1+\varepsilon)\Gamma(n) \cdot \left(1+\varepsilon Z_1(n-1)+\varepsilon^2 Z_{11}(n-1)+\varepsilon^3 Z_{111}(n-1)+...+\varepsilon^{n-1} Z_{11...1}(n-1)\right).$$

Z-sums interpolate between Goncharov's multiple polylogarithms and Euler-Zagier sums.

Algorithms

Multiplication:

$$Z(n;m_1,\ldots;x_1,\ldots)\cdot Z(n;m'_1,\ldots;x'_1,\ldots)$$

Convolution: Sums involving i and $n-i$

$$\sum_{i=1}^{n-1}\frac{x_1^i}{i^{m_1}}Z(i-1;m_2\ldots;x_2,\ldots)\frac{{x'_1}^{n-i}}{(n-i)^{m'_1}}Z(n-i-1;m'_2,\ldots;x'_2,\ldots)$$

Conjugations:

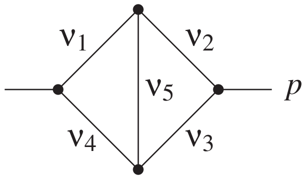
$$-\sum_{i=1}^{n}\binom{n}{i}(-1)^i\frac{x_0^i}{i^{m_0}}Z(i;m_1,\ldots,m_k;x_1,\ldots,x_k)$$

Conjugation and convolution: Sums involving binomials and $n-i$

$$-\sum_{i=1}^{n-1}\binom{n}{i}(-1)^i\frac{x_1^i}{i^{m_1}}Z(i;m_2\ldots;x_2,\ldots)\frac{{x'_1}^{n-i}}{(n-i)^{m'_1}}Z(n-i;m'_2,\ldots;x'_2,\ldots)$$

(Moch, Uwer, S.W., '01)

The two-loop two-point function



$$(1-2\varepsilon)\hat{I}^{(2,5)}(2-\varepsilon,1+\varepsilon,1+\varepsilon,1+\varepsilon,1+\varepsilon,1+\varepsilon) =$$
$$6\zeta_3 + 9\zeta_4\varepsilon + 372\zeta_5\varepsilon^2 + \left(915\zeta_6 - 864\zeta_3{}^2\right)\varepsilon^3$$
$$+(18450\zeta_7 - 2592\zeta_4\zeta_3)\varepsilon^4 + (50259\zeta_8 - 76680\zeta_5\zeta_3 - 2592\zeta_{6,2})\varepsilon^5$$
$$+\left(905368\zeta_9 - 200340\zeta_6\zeta_3 - 130572\zeta_5\zeta_4 + 66384\zeta_3{}^3\right)\varepsilon^6$$
$$+O(\varepsilon^7).$$

Theorem: Multiple zeta values are sufficient for the Laurent expansion of the two-loop integral $\hat{I}^{(2,5)}(m-\varepsilon,\nu_1,\nu_2,\nu_3,\nu_4,\nu_5)$, if all powers of the propagators are of the form $\nu_j = n_j + a_j\varepsilon$, where the n_j are positive integers and the a_j are non-negative real numbers.

I. Bierenbaum, S.W., (2003)

Part III

Feynman integrals and elliptic curves

Periodic functions

Let us consider a non-constant meromorphic function f of a complex variable z.

A period ω of the function f is a constant such that for all z:

$$f(z+\omega) = f(z)$$

The set of all periods of f forms a lattice, which is either

- trivial (i.e. the lattice consists of $\omega = 0$ only),
- a simple lattice, $\Lambda = \{n\omega \mid n \in \mathbb{Z}\}$,
- a double lattice, $\Lambda = \{n_1\omega_1 + n_2\omega_2 \mid n_1, n_2 \in \mathbb{Z}\}$.

Examples of periodic functions

- Singly periodic function: Exponential function

$$\exp(z)\,.$$

$\exp(z)$ is periodic with peridod $\omega = 2\pi i$.

- Doubly periodic function: Weierstrass's $\wp$-function

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \in \Lambda \setminus \{0\}} \left(\frac{1}{(z+\omega)^2} - \frac{1}{\omega^2} \right), \qquad \Lambda = \{ n_1\omega_1 + n_2\omega_2 | n_1, n_2 \in \mathbb{Z} \}\,,$$

$$\mathrm{Im}(\omega_2/\omega_1) \neq 0.$$

$\wp(z)$ is periodic with periods ω_1 and ω_2.

Inverse functions

The corresponding inverse functions are in general multivalued functions.

- For the exponential function $x = \exp(z)$ the inverse function is the logarithm

$$z = \ln(x).$$

- For Weierstrass's elliptic function $x = \wp(z)$ the inverse function is an elliptic integral

$$z = \int\limits_x^\infty \frac{dt}{\sqrt{4t^3 - g_2 t - g_3}}, \qquad g_2 = 60 \sum_{\omega \in \Lambda \backslash \{0\}} \frac{1}{\omega^4}, \quad g_3 = 140 \sum_{\omega \in \Lambda \backslash \{0\}} \frac{1}{\omega^6}.$$

Periods as integrals over algebraic functions

In both examples the periods can be expressed as integrals involving only algebraic functions.

- Period of the exponential function:

$$2\pi i \quad = \quad 2i \int_{-1}^{1} \frac{dt}{\sqrt{1-t^2}}.$$

- Periods of Weierstrass's $\wp$-function: Assume that g_2 and g_3 are two given algebraic numbers. Then

$$\omega_1 = 2\int_{t_1}^{t_2} \frac{dt}{\sqrt{4t^3 - g_2 t - g_3}}, \qquad \omega_2 = 2\int_{t_3}^{t_2} \frac{dt}{\sqrt{4t^3 - g_2 t - g_3}},$$

where t_1, t_2 and t_3 are the roots of the cubic equation $4t^3 - g_2 t - g_3 = 0$.

Numerical periods

Kontsevich and Zagier suggested the following generalisation:

A numerical period is a complex number whose real and imaginary parts are values of absolutely convergent integrals of rational functions with rational coefficients, over domains in $\mathbb{R}^n$ given by polynomial inequalities with rational coefficients.

Remarks:

- One can replace “rational” with “algebraic”.
- The set of all periods is countable.
- Example: $\ln 2$ is a numerical period.

$$\ln 2 \quad = \quad \int_1^2 \frac{dt}{t}.$$

Feynman integrals and periods

Laurent expansion in $\varepsilon = (4-D)/2$:

$$I_G = \sum_{j=-2l}^{\infty} c_j \varepsilon^j.$$

Question: What can be said about the coefficients c_j ?

Theorem: For rational input data in the euclidean region the coefficients c_j of the Laurent expansion are numerical periods.

(Bogner, S.W., '07)

Next question: Which periods ?

Differential equations for Feynman integrals

If it is not feasible to compute the integral directly:

Pick one variable t from the set s_{jk} and m_i^2.

1. Find a differential equation for the Feynman integral.

$$\sum_{j=0}^{r} p_j(t) \frac{d^j}{dt^j} I_G(t) = \sum_i q_i(t) I_{G_i}(t)$$

   Inhomogeneous term on the rhs consists of simpler integrals I_{G_i}.

   $p_j(t)$, $q_i(t)$ polynomials in t.

2. Solve the differential equation.

Kotikov; Remiddi, Gehrmann; Laporta; Argeri, Mastrolia, S. Müller-Stach, S.W., R. Zayadeh; Henn; ...

Differential equations: The case of multiple polylogarithms

Suppose the differential operator factorises into linear factors:

$$\sum_{j=0}^{r} p_j(t) \frac{d^j}{dt^j} = \left(a_r(t) \frac{d}{dt} + b_r(t) \right) \dots \left(a_2(t) \frac{d}{dt} + b_2(t) \right) \left(a_1(t) \frac{d}{dt} + b_1(t) \right)$$

Iterated first-order differential equation.

Denote homogeneous solution of the j-th factor by

$$\psi_j(t) = \exp\left(- \int\limits_0^t ds \frac{b_j(s)}{a_j(s)} \right).$$

Full solution given by iterated integrals

$$I_G(t) = C_1 \psi_1(t) + C_2 \psi_1(t) \int\limits_0^t dt_1 \frac{\psi_2(t_1)}{a_1(t_1)\psi_1(t_1)} + C_3 \psi_1(t) \int\limits_0^t dt_1 \frac{\psi_2(t_1)}{a_1(t_1)\psi_1(t_1)} \int\limits_0^{t_1} dt_2 \frac{\psi_3(t_2)}{a_2(t_2)\psi_2(t_2)} + \dots$$

Multiple polylogarithms are of this form.

Differential equations: Beyond linear factors

Suppose the differential operator

$$\sum_{j=0}^{r} p_j(t) \frac{d^j}{dt^j}$$

does not factor into linear factors.

The next more complicate case:
The differential operator contains one irreducible second-order differential operator

$$a_j(t) \frac{d^2}{dt^2} + b_j(t) \frac{d}{dt} + c_j(t)$$

An example from mathematics: Elliptic integral

The differential operator of the second-order differential equation

$$\left[t\left(1-t^2\right)\frac{d^2}{dt^2}+\left(1-3t^2\right)\frac{d}{dt}-t\right]f(t) \quad = \quad 0$$

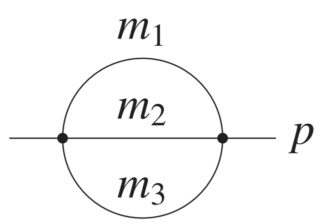
is irreducible.

The solutions of the differential equation are $K(t)$ and $K(\sqrt{1-t^2})$, where $K(t)$ is the complete elliptic integral of the first kind:

$$K(t) \quad = \quad \int_0^1 \frac{dx}{\sqrt{(1-x^2)\,(1-t^2x^2)}}.$$

An example from physics: The two-loop sunrise integral

$$S\left(p^2, m_1^2, m_2^2, m_3^2\right) \quad =$$



- Two-loop contribution to the self-energy of massive particles.
- Sub-topology for more complicated diagrams.

The two-loop sunrise integral: Prior art

Integration-by-parts identities allow to derive a coupled system of 4 first-order differential equations for S and S_1, S_2, S_3, where

$$S_i = \frac{\partial}{\partial m_i^2} S$$

(Caffo, Czyz, Laporta, Remiddi, 1998).

This system reduces to a single second-order differential equation in the case of equal masses $m_1 = m_2 = m_3$

(Broadhurst, Fleischer, Tarasov, 1993).

Dimensional recurrence relations relate integrals in $D = 4$ dimensions and $D = 2$ dimensions

(Tarasov, 1996, Baikov, 1997, Lee, 2010).

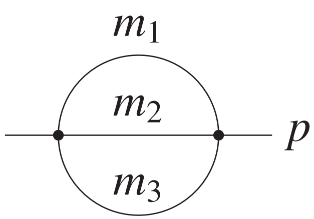
Analytic result in the equal mass case known up to quadrature, result involves elliptic integrals

(Laporta, Remiddi, 2004).

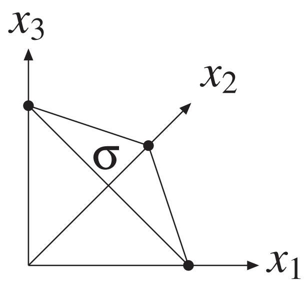
The two-loop sunrise integral in two dimensions

The two-loop sunrise integral with non-zero masses in two-dimensions ($t = p^2$):

$$S(t) =$$



$$= \int\limits_{\sigma} \frac{\omega}{\mathcal{F}},$$



$$\omega = x_1 dx_2 \wedge dx_3 + x_2 dx_3 \wedge dx_1 + x_3 dx_1 \wedge dx_2,$$

$$\mathcal{F} = -x_1 x_2 x_3 t + \left(x_1 m_1^2 + x_2 m_2^2 + x_3 m_3^2 \right) \left(x_1 x_2 + x_2 x_3 + x_3 x_1 \right)$$

Algebraic geometry studies the zero sets of polynomials.

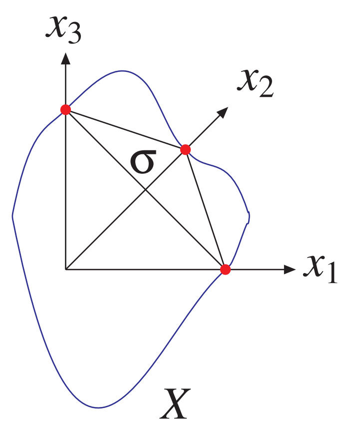
In this case look at the set $\mathcal{F} = 0$.

The two-loop sunrise integral

From the point of view of algebraic geometry there are two objects of interest:

- the domain of integration σ,
- the zero set X of $\mathcal{F} = 0$.

X and σ intersect at three points:



The elliptic curve

Algebraic variety X defined by the polynomial in the denominator:

$$-x_1x_2x_3t + \left(x_1m_1^2 + x_2m_2^2 + x_3m_3^2\right)\left(x_1x_2 + x_2x_3 + x_3x_1\right) = 0.$$

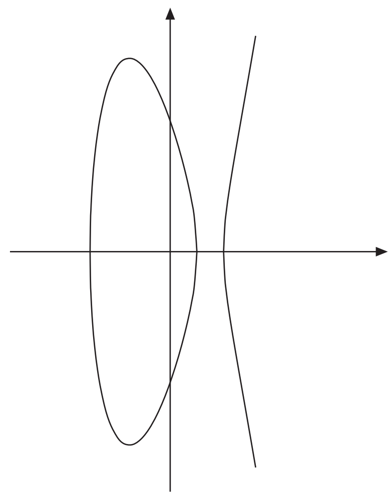
This defines (together with a choice of a rational point as origin) an elliptic curve.

Change of coordinates $\rightarrow$ Weierstrass normal form

$$y^2z - 4x^3 + g_2(t)xz^2 + g_3(t)z^3 = 0.$$

In the chart $z = 1$ this reduces to

$$y^2 - 4x^3 + g_2(t)x + g_3(t) = 0.$$



$y^2 = 4x^3 - 28x + 24$

The curve varies with t.

Abstract periods

Input:

- X a smooth algebraic variety of dimension n defined over $\mathbb{Q}$,
- $D \subset X$ a divisor with normal crossings (i.e. a subvariety of dimension $n-1$, which looks locally like a union of coordinate hyperplanes),
- ω an algebraic differential form on X of degree n,
- σ a singular n-chain on the complex manifold $X(\mathbb{C})$ with boundary on the divisor $D(\mathbb{C})$.

To each quadruple (X, D, ω, σ) associate the period

$$P(X, D, \omega, \sigma) \quad = \quad \int_{\sigma} \omega.$$

The motive

P: Blow-up of $\mathbb{P}^2$ in the three points, where X intersects σ.

Y: Strict transform of the zero set X of $\mathcal{F} = 0$.

B: Total transform of $\{x_1 x_2 x_3 = 0\}$.

Mixed Hodge structure:

$$H^2(P \backslash Y, B \backslash B \cap Y)$$

(S. Bloch, H. Esnault, D. Kreimer, 2006)

We need to analyse $H^2(P \backslash Y, B \backslash B \cap Y)$.
We can show that essential information is given by $H^1(X)$.

(S. Müller-Stach, S.W., R. Zayadeh, 2011)

The second-order differential equation

In the Weierstrass normal form $H^1(X)$ is generated by

$$\eta = \frac{dx}{y} \quad \text{and} \quad \dot{\eta} = \frac{d}{dt}\eta.$$

$\ddot{\eta} = \frac{d^2}{dt^2}\eta$ must be a linear combination of η and $\dot{\eta}$:

$$p_0(t)\ddot{\eta} + p_1(t)\dot{\eta} + p_2(t)\eta \quad = \quad 0.$$

Differential equation:

$$\left[p_0(t)\frac{d^2}{dt^2} + p_1(t)\frac{d}{dt} + p_2(t)\right] S(t) \quad = \quad p_3(t)$$

p_0, p_1, p_2 and p_3 are polynomials in t.

Periods of an elliptic curve

In the Weierstrass normal form, factorise the cubic polynomial in x:

$$y^2 = 4(x-e_1)(x-e_2)(x-e_3).$$

Holomorphic one-form is $\frac{dx}{y}$, associated periods are

$$\psi_1(t) = 2\int\limits_{e_2}^{e_3} \frac{dx}{y}, \qquad \psi_2(t) = 2\int\limits_{e_1}^{e_3} \frac{dx}{y}.$$

These periods are the solutions of the homogeneous differential equation.

L. Adams, Ch. Bogner, S.W., '13

The full result

- Once the homogeneous solutions are known, variation of the constants yields the full result up to quadrature:
  - Equal mass case: Laporta, Remiddi, ’04
  - Unequal mass case: L. Adams, Ch. Bogner, S.W., ’13

- The full result can be expressed in terms of elliptic dilogarithms:
  - Equal mass case: Bloch, Vanhove, ’13
  - Unequal mass case: L. Adams, Ch. Bogner, S.W., ’14

The elliptic dilogarithm

Recall the definition of the classical polylogarithms:

$$\mathrm{Li}_n(x) = \sum_{j=1}^{\infty} \frac{x^j}{j^n}.$$

Generalisation, the two sums are coupled through the variable q:

$$\mathrm{ELi}_{n;m}(x;y;q) = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \frac{x^j}{j^n}\frac{y^k}{k^m} q^{jk}.$$

Elliptic dilogarithm:

$$\mathrm{E}_{2;0}(x;y;q) = \frac{1}{i}\left[\frac{1}{2}\mathrm{Li}_2(x) - \frac{1}{2}\mathrm{Li}_2\left(x^{-1}\right) + \mathrm{ELi}_{2;0}(x;y;q) - \mathrm{ELi}_{2;0}\left(x^{-1};y^{-1};q\right)\right].$$

(Slightly) different definitions of elliptic polylogarithms can be found in the literature

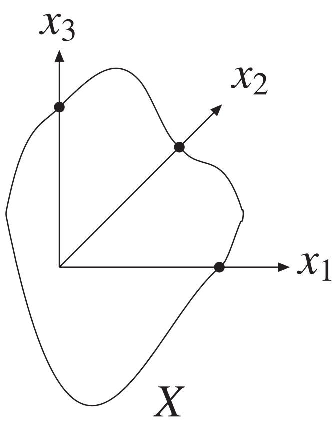
Beilinson ’94, Levin ’97, Brown, Levin ’11, Wildeshaus ’97.

Elliptic curves again

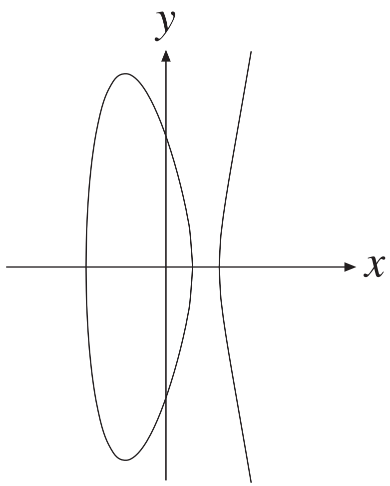
The nome q is given by

$$q = e^{i\pi\tau} \quad \text{with} \quad \tau = \frac{\psi_2}{\psi_1} = i\frac{K(k')}{K(k)}.$$

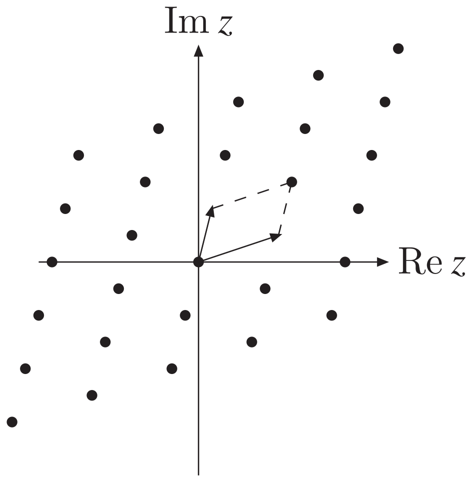
Elliptic curve represented by



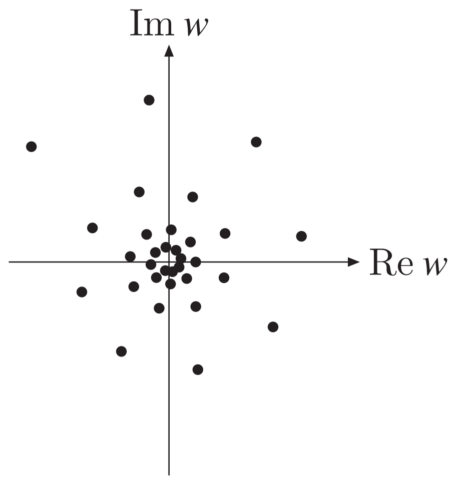
Algebraic variety
$\mathcal{F} = 0$



Weierstrass normal form
$y^2 = 4x^3 - g_2 x - g_3$



Torus
$\mathbb{C}/\Lambda$



Jacobi uniformization
$\mathbb{C}^*/q^{2\mathbb{Z}}$

The arguments of the elliptic dilogarithms

Elliptic curve: Cubic curve together with a choice of a rational point as the origin O.

Distinguished points are the points on the intersection of the cubic curve $\mathcal{F} = 0$ with the domain of integration σ:

$$P_1 = [1:0:0], \quad P_2 = [0:1:0], \quad P_3 = [0:0:1].$$

Choose one of these three points as origin and look at the image of the two other points in the Jacobi uniformization $\mathbb{C}^*/q^{2\mathbb{Z}}$ of the elliptic curve. Repeat for the two other choices of the origin. This defines

$$w_1, w_2, w_3, w_1^{-1}, w_2^{-1}, w_3^{-1}.$$

In other words: $w_1, w_2, w_3, w_1^{-1}, w_2^{-1}, w_3^{-1}$ are the images of P_1, P_2, P_3 under

$$E_i \longrightarrow \mathrm{WNF} \longrightarrow \mathbb{C}/\Lambda \longrightarrow \mathbb{C}^*/q^{2\mathbb{Z}}.$$

The full result in terms of elliptic dilogarithms

The result for the two-loop sunrise integral in two space-time dimensions with arbitrary masses:

$$S = \underbrace{\frac{4}{\left[\left(t-\mu_1^2\right)\left(t-\mu_2^2\right)\left(t-\mu_3^2\right)\left(t-\mu_4^2\right)\right]^{\frac{1}{4}}}}_{\text{algebraic prefactor}} \underbrace{\frac{K(k)}{\pi}}_{\text{elliptic integral}} \underbrace{\sum_{j=1}^{3} \mathrm{E}_{2;0}\left(w_j;-1;-q\right)}_{\text{elliptic dilogarithms}}$$

t	momentum squared
μ_1, μ_2, μ_3	pseudo-thresholds
μ_4	threshold
$K(k)$	complete elliptic integrals of the first kind
k, q	modulus and nome
w_1, w_2, w_3	points in the Jacobi uniformization

Summary

- Feynman integrals and multiple polylogarithms:

  Algebraic structure based on

  – nested sums,
  – iterated integrals

- Feynman integrals beyond multiple polylogarithms:

  Elliptic case:

  – Algebraic prefactors as before.
  – Elliptic integrals generalise the period π.
  – Elliptic (multiple) polylogarithms generalise the (multiple) polylogarithms.
  – Arguments of the elliptic polylogarithms are points in the Jacobi uniformization of the elliptic curve.