The Lorentzian Noncommutative Standard Model

Nadir Bizi

Joint work with Christian Brouder and Fabien Besnard

Introduction

☐ Main issues of noncommutative standard model:

- Still a classical theory
- Still a Riemannian theory

☐ Lorentzian signature ⇒ indefinite inner product spaces - **Krein spaces** - have to be used (See Strohmaier, Paschke, Sitarz, Rennie, Van den Dungen, etc.))

☐ What should the axioms be now ?

Krein Spaces

☐ "Commutative" Dirac operator:

$$\not{D} = -i\gamma^\mu \nabla^S_\mu$$

☐ Self-adjointness of $\not{D}$ ⇒ sesquilinear form on spinors on spinors such the γ^μ are self-adjoint

☐ Riemannian signature ⇔ definite positive inner product on spinors (Hilbert space)

☐ But for non-Riemannian signature ⇒ **indefinite** inner product on spinors! Hence the need for Krein space

$$(\gamma^i)^2 = -1 \Rightarrow (\gamma^i)^\dagger \gamma^i = -1 \Rightarrow \text{Impossible!}$$

☐ Krein space $\mathcal{K}$:

- (complex) vector space $\mathcal{K}$
- *indefinite* non-degenerate sesquilinear form $(\cdot, \cdot)$
- corresponding quadratic form: q

☐ Example: complex Minkowski space $\mathbb{C}^{1,3}$.

$$(u, v) = -\overline{u^0}v^0 + \overline{u^1}v^1 + \overline{u^2}v^2 + \overline{u^3}v^3$$

☐ Krein space (continued):

- $\mathcal{K} = \mathcal{K}_+ \oplus \mathcal{K}_-$
- q = positive definite on $\mathcal{K}_+$
- q = negative definite on $\mathcal{K}_-$
- $(\mathcal{K}_+, \mathcal{K}_-) = 0$
- a proper topology on $\mathcal{K}_\pm$ (that we do not use)

☐ Definite positive inner product from decomposition:

$$\langle \phi, \psi \rangle = (\phi_+, \psi_+) - (\phi_-, \psi_-)$$

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$$\left.\begin{array}{l} \mathcal{K}_+ = \{(0, u^1, u^2, u^3)\} \\ \mathcal{K}_- = \{(u^0, 0, 0, 0)\} \end{array}\right\} \Rightarrow \langle u, v \rangle = \overline{u^0} v^0 + \overline{u^1} v^1 + \overline{u^2} v^2 + \overline{u^3} v^3$$

☐ Fundamental symmetry $\mathcal{J}$:

- $\mathcal{J} = P_+ - P_-$ = "grading" for the decomposition
- satisfies: $\mathcal{J}^2 = 1$ and Krein-self-adjoint
- used to parametrize decompositions

☐ Definite positive inner product:

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$$\left.\begin{array}{l} \mathcal{K}_+ = \{(0, u^1, u^2, u^3)\} \\ \mathcal{K}_- = \{(u^0, 0, 0, 0)\} \end{array}\right\} \Rightarrow \mathcal{J} = \begin{pmatrix} -1 & 0 \\ 0 & I_3 \end{pmatrix}$$

☐ $T^\times$ = adjoint of the operator T:

$$(\phi, T^\times \psi) = (T\phi, \psi)$$

☐ $T^\dagger = \mathcal{J} T^\times \mathcal{J}$ = $\langle \cdot, \cdot \rangle_\mathcal{J}$-adjoint of T

$$\langle \phi, T^\dagger \psi \rangle_\mathcal{J} = \langle T\phi, \psi \rangle_\mathcal{J}$$

(depends on $\mathcal{J}$)

☐ Krein-unitaries: $U^\times U = UU^\times = 1$ = "symmetries":

$$(U\phi, U\psi) = (\phi, \psi)$$

- Two decompositions: $\mathcal{K} = \mathcal{K}_+ \oplus \mathcal{K}_-$ and $\mathcal{K} = \mathcal{K}'_+ \oplus \mathcal{K}'_-$ = isomorphic!

- Isomorphism = preserves the inner product?

- **Theorem:** For finite-dimensional $\mathcal{K}$, the set of all fundamental symmetries is:

$$\mathcal{F} = \{U^\times \mathcal{J} U | U^\times U = UU^\times = 1\},$$

for *any* fundamental symmetry $\mathcal{J}$

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☐ Krein-unitaries map fundamental symmetries to fundamental symmetries, and Hilbert space structures to Hilbert space structures $\Rightarrow$ No preferred Hilbert space structure

☐ Polar decomposition of Krein-unitaries: U = Krein-unitary $\Rightarrow$ unique polar decomposition: $U = F|U|$

- F = unitary operator *and* Krein-unitary
- $|U| = (U^{\dagger\mathcal{J}}U)^{1/2}$ = positive self-adjoint *and* Krein-unitary operator
- with respect to some definite positive inner product $\langle\cdot,\cdot\rangle_{\mathcal{J}}$
- Now: $\mathcal{F} = \{|U|^{\times}\mathcal{J}|U|\}$

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☐ Remains true for Lorentz group on *spinors*!

☐ A relevant example: $\mathrm{Cl}(1,3)$ = real algebra generated by:

$$(\gamma^0)^2 = -1$$
$$(\gamma^i)^2 = +1 \text{ for } i = 1,2,3$$
$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 0 \text{ for } \mu \neq \nu$$

☐ Unique inner product $(\cdot,\cdot)$ on spinors[1] such that: $(\gamma^\mu)^\times = \gamma^\mu$

☐ Chirality operator: $\gamma^5 = \pm i\gamma^0\gamma^1\gamma^2\gamma^3 \;\Rightarrow\; (\gamma^5)^\times = -\gamma^5$

☐ Charge conjugation operator: $\mathcal{C}\gamma^\mu = -\gamma^\mu\mathcal{C} \;\Rightarrow\; \mathcal{C}^\times\mathcal{C} = 1$

☐ Later: $\gamma^5 = \gamma_M$ and $\mathcal{C} = J_M$

[1] P. L. Robinson (1988), *Spinors and canonical hermitian forms*, Glasgow Mathematical Journal, **30**, pp 263-270.

☐ Allowed fundamental symmetry:

$$\mathcal{J} = \pm\gamma^5\gamma^0$$

$$(\mathcal{J} = i\gamma^0 \Leftrightarrow (\gamma^\mu)^\times = -\gamma^\mu)$$

☐ With this choice:

$$(\gamma^0)^\dagger = -\gamma^0$$
$$(\gamma^i)^\dagger = +\gamma^i$$

Compatible representations: Dirac, Weyl, Majorana.

☐ Weyl representation:

$$\gamma^0 = -i\begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}, \quad \gamma^k = -i\begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix}$$

☐ Positive inner product:

$$\langle \phi, \psi \rangle_{\mathcal{J}} = \phi^\dagger \psi = \text{ canonical scalar product}$$

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☐ Charge conjugation:

$$\mathcal{C} = \gamma^5 \gamma^2 \times \text{complex conjugation}$$

$$\left.\begin{aligned} \mathcal{C}^\dagger \mathcal{C} &= 1 \\ \mathcal{C}^\times \mathcal{C} &= 1 \\ \mathcal{C}^2 &= -1 \end{aligned}\right\}$$

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$$\left.\begin{aligned} \mathcal{C}^\dagger \mathcal{C} &= 1 \\ \mathcal{C}^\times \mathcal{C} &= 1 \\ \mathcal{C}^2 &= -1 \end{aligned}\right\} \Rightarrow \text{Does not depend on the representation!!}$$

- More generally for $\mathrm{Cl}(t, s)$:

- Allowed fundamental symmetry:

$$\mathcal{J} = \begin{cases} \pm i^{(s-1)/2}\gamma^{t+1}...\gamma^{d} & \text{for odd } t \\ \pm i^{t/2}\gamma^{1}...\gamma^{t} & \text{for even } t \end{cases}$$

- $(\gamma^\mu)^\dagger = \pm(\gamma^\mu)$ depending on $(\gamma^\mu)^2 = \pm 1$

- Chirality: $\gamma^\times = (-1)^t \gamma$

- Charge conjugation: $\mathcal{C}\gamma^\mu = -\gamma^\mu \mathcal{C}$

$$\mathcal{C}^\times \mathcal{C} = \begin{cases} (-1)^{(s+1)/2} & \text{for odd } t \\ (-1)^{t/2} & \text{for even } t \end{cases}$$

Indefinite Spectral Triples

☐ Usual (real even) spectral triple:

- algebra A
- Hilbert space $\mathcal{H}$
- involutive representation of A on $\mathcal{H}$
- self-adjoint Dirac operator D s.t. $D^\dagger = D$
- real structure J s.t. $J^\dagger J = 1, JD = DJ$ and $J^2 = \epsilon$
- chirality γ s.t. $\gamma^\dagger = \gamma, \gamma^2 = 1$ and $J\gamma = \epsilon'' \gamma J$

KO	0	2	4	6
ϵ	1	-1	-1	1
ϵ''	1	-1	1	-1

☐ **Indefinite (real even) spectral triple:** (see Strohmaier, etc.+real structure)

- algebra A
- *Krein* space $\mathcal{K}$
- involutive representation of A on $\mathcal{K}$: $\pi(a^*) = \pi(a)^\times$
- self-adjoint Dirac operator D s.t. $D^\times = D$
- real structure J s.t. $JD = DJ$ and $J^2 = \epsilon$
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☐ In analogy with Clifford algebras: $\gamma^\times = \pm\gamma, J^\times J = \pm 1$

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☐ Total spectral triple $\Rightarrow$ all signs can be determined!

☐ Fermionic action:

$$S = (\psi, D\psi).$$

With ψ = *anti-commuting variable*

☐ Barrett's conditions[2]:

$$J\psi = \eta\psi \Rightarrow \text{phase absorption} \Rightarrow J\psi = \psi,$$
$$\gamma\psi = \pm\psi \Rightarrow \text{convention} \Rightarrow \gamma\psi = \psi$$

☐ Consequences:

- KO dimension necessarily 0

$$J^2\psi = \psi \Rightarrow J^2 = 1$$
$$\gamma J\psi = J\gamma\psi \Rightarrow \gamma J = J\gamma$$

[2] J. Barrett, *Lorentzian version of the noncommutative geometry of the standard model of particle physics* , J. Math. Phys. 48, 012303 (2007)

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- action nontrivial implies:

$$\gamma^\times = -\gamma$$
$$J^\times J = -1$$

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☐ Assume $J^{\times} J = \kappa$:

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- $S \neq 0$ implies $\kappa = -1$
- Same computation for γ but $\gamma D = -D\gamma$

☐ Convenient choices for the fundamental symmetry:

$$J^\dagger J = 1 \Rightarrow J\mathcal{J} = -\mathcal{J}J$$

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$$\pi(a^*) = \pi(a)^\dagger \Rightarrow [\mathcal{J}, A] = 0$$

☐ We recover the usual axioms for $\mathcal{J}$ found in the literature!

Tensor products of spectral triples

☐ Graded tensor product = built in analogy to Clifford algebras

☐ Grading:

- $\mathcal{K} = \mathcal{K}^0 \oplus \mathcal{K}^1$, with grading given by γ
- Algebra A = even
- Dirac operator D = odd
- Grading of J: $(-1)^{|J|} = \epsilon''$
- Chirality = trivially even
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☐ Initial data: two indefinite spectral triples $(A_1, \mathcal{K}_1, D_1, \gamma_1, J_1)$ and $(A_2, \mathcal{K}_2, D_2, \gamma_2, J_2)$

☐ Tensor product $(A, \mathcal{K}, D, \gamma, J) = ?$

- ☐ Algebra: $A = A_1 \hat{\otimes} A_2$
- ☐ Krein space: $\mathcal{K} = \mathcal{K}_1 \hat{\otimes} \mathcal{K}_2$, with grading:

$$\mathcal{K}^0 \cong \mathcal{K}_1^0 \otimes \mathcal{K}_2^0 \oplus \mathcal{K}_1^1 \otimes \mathcal{K}_2^1$$
$$\mathcal{K}^1 \cong \mathcal{K}_1^0 \otimes \mathcal{K}_2^1 \oplus \mathcal{K}_1^1 \otimes \mathcal{K}_2^0$$

- ☐ Corresponding grading operator: $\gamma = \gamma_1 \hat{\otimes} \gamma_2$
- ☐ Dirac operator: $D = D_1 \hat{\otimes} 1 + 1 \hat{\otimes} D_2$
- ☐ Real structure: $J = J_1 \gamma_1^{|J_2|} \hat{\otimes} J_2 \gamma_2^{|J_1|}$
- ☐ Result =KO dimension is additive !!

- Algebra: $A = A_1 \hat{\otimes} A_2$

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- Inner product on $\mathcal{K}$?

☐ Inner product on $\mathcal{K} \Rightarrow$ such that D is self-adjoint !!

☐ Solution:

$$(\phi_1 \hat{\otimes} \phi_2, \psi_1 \hat{\otimes} \psi_2) = \sigma(\phi_1, \psi_1)_1 (\phi_2, \psi_2)_2$$

with:

$$\sigma = \begin{cases} 1 & \text{if } \gamma_1^\times = \gamma_1 \\ (-1)^{|\psi_2|} & \text{if } \gamma_1^\times = -\gamma_1 \text{ and } \gamma_2^\times = \gamma_2 \\ i(-1)^{|\psi_2|} & \text{if } \gamma_1^\times = -\gamma_1 \text{ and } \gamma_2^\times = -\gamma_2 \end{cases}$$

☐ Added bonus!

$$(T_1 \hat{\otimes} T_2)^\times = (-1)^{|T_1||T_2|} T_1^\times \hat{\otimes} T_2^\times$$

☐ Fundamental symmetry:

$$\mathcal{J} = i^{|\mathcal{J}_1||\mathcal{J}_2|} \gamma_1^{|\mathcal{J}_2|} \mathcal{J}_1 \hat{\otimes} \gamma_2^{|\mathcal{J}_1|} \mathcal{J}_2$$

☐ Non-graded representation of graded products (easier to handle):

$$(T_1 \hat{\otimes} T_2)(\psi_1 \hat{\otimes} \psi_2) = (-1)^{|T_2||\psi_1|} T_1\psi_1 \hat{\otimes} T_2\psi_2$$
$$(S_1 \hat{\otimes} S_2)(T_1 \hat{\otimes} T_2) = (-1)^{|S_2||T_1|} S_1 T_1 \hat{\otimes} S_2 T_2$$
$$\Downarrow$$
$$T_1 \hat{\otimes} T_2 = T_1 \gamma_1^{|T_2|} \otimes T_2$$

☐ For triples:

- $\mathcal{K} = \mathcal{K}_1 \otimes \mathcal{K}_2$
- $A = A_1 \otimes A_2$
- $D = D_1 \otimes 1 + \gamma_1 \otimes D_2$
- $J = J_1 \otimes \gamma_2^{|J_1|} J_2$ (as expected[3])
- $\gamma = \gamma_1 \otimes \gamma_2$
- Inner product ?

[3] F. J. Vanhecke, *On the Product of Real Spectral Triples*, Lett. Math. Phys. **50**, 157-162 (1999)

☐ Inner product:

$$(\phi_1 \hat{\otimes} \phi_2, \psi_1 \hat{\otimes} \psi_2) = (\phi_1, \psi_1)_1 \underbrace{(\phi_2, \beta\psi_2)_2}_{(\phi_2, \psi_2)_{2,\beta}}$$

with:

$$\beta = \begin{cases} 1 & \text{if } \gamma_1^\times = \gamma_1 \\ \gamma_2 & \text{if } \gamma_1^\times = -\gamma_1 \text{ and } \gamma_2^\times = \gamma_2 \\ i\gamma_2 & \text{if } \gamma_1^\times = -\gamma_1 \text{ and } \gamma_2^\times = -\gamma_2 \end{cases}$$

We have $\beta^\times = \beta$.

☐ Fundamental symmetry:

$$\mathcal{J} = (-i)^{|\mathcal{J}_1||\mathcal{J}_2|} \mathcal{J}_1 \hat{\otimes} \gamma_2^{|\mathcal{J}_1|} \mathcal{J}_2$$

A Lorentzian spectral triple: the standard model

☐ Spectral triple as the product of two triples:

- manifold part: commutative spectral triple $\Rightarrow$ KO-dimension $= s - t$
- gauge part: Finite spectral triple $\Rightarrow$ KO-dimension $= 6$

☐ Tensor product of spectral triples: KO-dimension is additive

$$s - t + 6 \equiv 0[8]$$
$$s + t = 4$$

☐ Solution: $(t, s) = (1, 3) \Rightarrow$ Signature $(- + + +)$.

☐ **Commutative spectral triple:** Manifold of signature $(-+++)$

- algebra: $\mathcal{C}^\infty(M)$
- Krein space: $\mathcal{K}_M =$ completion of $\Gamma(S)$
- indefinite inner product $(\cdot,\cdot)_M \Rightarrow$ yet to be determined...
- Dirac operator: $\not{D} = -i\gamma^\mu \nabla^S_\mu$
- chirality: $\gamma_M = \gamma^5$ (locally), obeys: $\gamma_M^\times = -\gamma_M$
- real structure: $J_M = \mathcal{C}$ (locally)

☐ **Finite spectral triple:** almost as usual

- algebra: $A_F = \mathbb{C} \oplus \mathbb{H} \oplus \mathrm{M}_3(\mathbb{C})$
- Krein space: $\mathcal{K}_F \cong \mathbb{C}^{96} (\cong \mathcal{H}_F)$
- Particle content:

$$\mathcal{K}_F = \mathcal{K}_R \oplus \mathcal{K}_L \oplus \mathcal{K}_{\overline{R}} \oplus \mathcal{K}_{\overline{L}}$$
$$\mathcal{K}_R = \mathrm{Span}(\nu_R, e_R, u_R^r, u_R^g, u_R^b, d_R^r, d_R^g, d_R^b) \otimes \mathbb{C}^3$$

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$$(u, v)_F = u^\dagger H v$$

$H =$ self-adjoint $\Rightarrow$ yet to be determined...

- Dirac operator: $D_F \Rightarrow$ yet to be determined...
- chirality: γ_F
- real structure: J_F

☐ **Total spectral triple:**

- algebra: $A = \mathcal{C}^\infty(M) \otimes A_F$
- Krein space: $\mathcal{K} = \mathcal{K}_M \otimes \mathcal{K}_F$
- indefinite inner product
  $(\phi_M \otimes \phi_F, \psi_M \otimes \psi_F) = (\phi_M, \psi_M)_M (\phi_F, \psi_F)_{F\beta}$
- Dirac operator: $D = \not{D} \otimes 1 + \gamma_M \otimes D_F$
- chirality: $\gamma = \gamma_M \otimes \gamma_F$
- real structure: $J = J_M \otimes \gamma_F J_F$

- Hypothesis: $H = 1$ (Riemannian finite part)

- Then $\gamma_F^\times = \gamma_F \Rightarrow \beta = \gamma_F$

- Kinetic terms in fermionic action:

$$S_{\mathrm{Kin}} = (\psi, (\not{D} \otimes 1)\psi) \supset (\psi_L, \not{D}\psi_L) - (\psi_R, \not{D}\psi_R)$$

Absurd!

- Simplest working choice:

$$H = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & -1 & \\ & & & 1 \end{pmatrix} \begin{matrix} R \\ L \\ \overline{R} \\ \overline{L} \end{matrix}$$

☐ With this choice: $\gamma_F^\times = \gamma_F \Rightarrow \beta = \gamma_F$

☐ Works as if the product on the finite part were[4] :

$$(u, v)_{F\beta} = u^\dagger \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} v$$

☐ We have $J^\times J = -1$ and $\gamma^\times = -\gamma \Rightarrow$ nontrivial spectral triple!!!

[4] Koen Van den Dungen, *Krein spectral triples and the fermionic action*, Math. Phys. Anal. Geom. 19 (2016)

☐ Form of the finite Dirac operator:

$$\left.\begin{array}{c} J_F D_F = D_F J_F \\ \gamma_F D_F = -D_F \gamma_F \\ D_F^\times = D_F \end{array}\right\} \Rightarrow D_F = \begin{pmatrix} 0 & -iX^\dagger & \overline{Y} & 0 \\ -iX & 0 & 0 & \overline{Z} \\ Y & 0 & 0 & iX^T \\ 0 & Z & i\overline{X} & 0 \end{pmatrix}$$

With Y, Z symmetric

☐ Order 0,1 & 2 conditions[5] to restrict D_F:

$$[a^o, b] = 0$$
$$[a^o, [D, b]] = 0$$
$$\{[D, a]^o, [D, b]\} \subset J^2$$

$(A^o = JA^\times J^{-1})$

☐ Allowed solution:

$$X = \begin{pmatrix} Y_\nu & 0 & 0 & 0 \\ 0 & Y_e & 0 & 0 \\ 0 & 0 & Y_u \otimes I_3 & 0 \\ 0 & 0 & 0 & Y_d \otimes I_3 \end{pmatrix}$$
$$Y = \begin{pmatrix} a & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & (0) & \\ 0 & & & \end{pmatrix}, Z = 0. \tag{1}$$

[5] Latham Boyle, Shane Farnsworth, *Non-Commutative Geometry, Non-Associative Geometry and the Standard Model of Particle Physics*, New J. Phys. 16, 123027 (2014); CB, NB, FB, *The Standard Model as an extension of the noncommutative algebra of forms*, arXiv:1504.03890 (2015)

☐ Fluctuated Dirac operator:

$$D = \not{D} \otimes 1 + \gamma^\mu \otimes B_\mu + \gamma_M \otimes \Phi,$$

☐ B_μ = usual gauge fields

☐ Higgs part:

$$\Phi = \begin{pmatrix} 0 & -iY(H)^\dagger & \overline{Y} & 0 \\ -iY(H) & 0 & 0 & 0 \\ Y & 0 & 0 & iY(H)^T \\ 0 & 0 & \overline{iY(H)} & 0 \end{pmatrix}$$

With $H = (\alpha, \beta)^T$ = Higgs doublet, and:

$$Y(H) = \begin{pmatrix} Y_\nu \alpha & -Y_e \overline{\beta} & 0 & 0 \\ Y_\nu \beta & Y_e \overline{\alpha} & 0 & 0 \\ 0 & 0 & Y_u \alpha \otimes I_3 & -Y_d \overline{\beta} \otimes I_3 \\ 0 & 0 & Y_u \beta \otimes I_3 & Y_d \overline{\alpha} \otimes I_3 \end{pmatrix}$$

☐ Generic vector in $\mathcal{K}$:

$$\psi = \begin{pmatrix} \psi_R \\ i\psi_L \\ \psi_{\overline{R}} \\ i\psi_{\overline{L}} \end{pmatrix}$$

☐ Barrett's conditions give us:

$$\psi_{\overline{R}} = J_M \psi_R$$
$$\psi_{\overline{L}} = J_M \psi_L$$

☐ Particle and anti-particle "spinors":

$$\psi_P = \psi_R + \psi_L$$
$$\psi_{\overline{P}} = \psi_{\overline{R}} + \psi_{\overline{L}}$$

Related by $\psi_{\overline{P}} = J_M \psi_P$

- Fermionic action: $S = S_{\text{Kin}} + S_B + S_\Phi$

- Kinetic term:

$$\begin{aligned} S_{\text{Kin}} &= (\psi, (\not{D} \otimes 1)\psi) \\ &= (\psi_P, \not{D}\psi_P) - (\psi_{\overline{P}}, \not{D}\psi_{\overline{P}}) \\ &= (\psi_P, \not{D}\psi_P) - (J_M\psi_P, J_M\not{D}\psi_P) \\ &= (\psi_P, \not{D}\psi_P) + (\psi_P, \not{D}\psi_P) \\ &= 2(\psi_P, \not{D}\psi_P) = \text{ expected result} \end{aligned}$$

We used $J_M^\times J_M = 1$.

☐ Gauge term:

$$S_B = (\psi, (\gamma^\mu \otimes B_\mu)\psi) = 2(\psi_P, (\gamma^\mu \otimes B_\mu|_P)\psi_P) = \text{ expected result}$$

where $B_\mu|_P$ = restriction to particle space

☐ Mass term:

$$\begin{aligned} S_\Phi = & a(\psi_{\nu_R}, J_M \psi_{\nu_R}) + a(J_M \psi_{\nu_R}, \psi_{\nu_R}) \\ & + 2(\psi_L, Y(H)\psi_R) + 2(\psi_R, Y(H)^\dagger \psi_L), \end{aligned}$$

(a can be chosen real).

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(a can be chosen real).

☐ The fermionic action matches the standard model !!

A perspective on the Bosonic action

- Spectral action:

$$S = \mathrm{Tr} f(\frac{D^2}{\Lambda^2})$$

- Invariant by unitaries of $\mathcal{H}$: $D \mapsto UDU^\dagger$
- Invariance as a *basis* for defining the action?

A perspective on the Bosonic action

☐ Spectral action:

$$S = \mathrm{Tr} f\left(\frac{D^2}{\Lambda^2}\right)$$

☐ Invariant by unitaries of $\mathcal{H}$: $D \mapsto UDU^\dagger$

☐ Invariance as a *basis* for defining the action?

☐ Bosonic action = most general functional $S[D]$ invariant under $D \mapsto UDU^\times$

☐ With U = *Krein*-unitary

☐ Work in progress

Conclusion

- ☐ Axioms for indefinite spectral triples
- ☐ Krein spaces are necessary
- ☐ The standard model can be recovered! Almost...
- ☐ Bosonic action $\Rightarrow$ Through symmetries ?