# The Standard Model in Noncommutative Geometry: particles as Dirac spinors?

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# Summary

Talk based on:

FD & L. Dabrowski,
*The Standard Model in Noncommutative Geometry and Morita equivalence*,
preprint arXiv:1501.00156 [math-ph]; to appear in J. Noncommut. Geom.

Keywords $\longrightarrow$ Standard Model, Morita equivalence, finite-dimensional spectral triples.

Summary of the talk:

1. Spectral triples (again).
2. An algebraic characterization of Dirac spinors.
3. The finite nc space of the $\nu$SM.

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# Introduction

## Definition

A unital spectral triple $(A, H, D)$ is the datum of:

(i) a (real or complex) unital $C^*$-algebra $A$ of bounded operators on a (separable) complex Hilbert space $H$,

(ii) a selfadjoint operator $D$ on $H$ with compact resolvent,

such that

(iii) the unital $*$-subalgebra

$$\mathrm{Lip}_D(A) = \{\, a \in A \ :\ a \cdot \mathrm{Dom}(D) \subset \mathrm{Dom} D \ \text{ and } \ [D, a] \in \mathcal{B}(H) \,\}$$

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## Example 0 (finite nc spaces)

Take any finite-dimensional $H$, any $A \subset \mathcal{B}(H)$ and $D \in \mathcal{B}(H)$. In this case $\mathrm{Lip}_D(A) \equiv A$ .

# Examples of spectral triples

Let: $(M, g)$ = compact oriented Riemannian manifold without boundary, $E \to M$ herm. vector bundle equipped with a unitary Clifford action $c : C^\infty(M, T^*_{\mathbb{C}}M \otimes E) \to C^\infty(M, E)$ and a connection $\nabla^E$ compatible with $g$. Then:

$$A = C(M) \qquad H = L^2(M, E) \qquad D = c \circ \nabla^E$$

is a spectral triple.

**1. Hodge operator**

$E = \bigwedge^{\text{even}} T^*_{\mathbb{C}}M \oplus \bigwedge^{\text{odd}} T^*_{\mathbb{C}}M$, $D = d + d^*$

$\Rightarrow$ $\mathrm{Index}(D^+)$ = Euler char. of $M$

**2. Signature operator ($\dim M$ even)**

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**3. Dolbeault operator ($M$ complex m.)**

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**4. Dirac operator ($M$ spin)**

$E$ = spinor bundle, $D = \not{D}$ Dirac operator

In all these examples, $H$ carries commuting representations of $A = C(M)$ and $B = \mathcal{C}\ell(M, g)$.

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## 5. The Standard Model spectral triple

The underlying geometry is

$$M \times F$$

(spin manifold) (finite nc space)

with finite-dim. spectral triple $(A_F, H_F, D_F, \gamma_F, J_F)$ given by:

- $H_F \simeq \mathbb{C}^{32n}$ $\rightsquigarrow$ internal degrees of freedom of the elementary fermions. Total nr:

| 2 | × | 4 | × | 2 | × | 2 | × | $n$ | = | $32n$ |
|---|---|---|---|---|---|---|---|---|---|---|
| (weak isospin) | | (lepton + quark in 3 colors) | | (L,R chirality) | | (particle or antiparticle) | | (generations) | | |

- $\gamma_F$ = chirality operator
- $A_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$
  $J_F$ = charge conjugation
  $\rightsquigarrow$ gauge group $\approx U(1) \times SU(2) \times SU(3)$
- $D_F$ encodes the free parameters of the theory.

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# Algebraic characterization of Dirac spinors

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A unital spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is called:

- even if $\exists\, \gamma = \gamma^*$ on $\mathcal{H}$ s.t. $\gamma^2 = 1$, $\gamma D = -D\gamma$ and $[\gamma, a] = 0 \;\forall\, a \in \mathcal{A}$;
- real if $\exists$ an antilinear isometry $J$ on $\mathcal{H}$ s.t. $J^2 = \pm 1$, $JD = \pm DJ$, $J\gamma = \pm\gamma J$ and $\forall\, a, b \in \mathcal{A}$:

$$[a, JbJ^{-1}] = 0 \qquad\qquad [[D, a], JbJ^{-1}] = 0$$

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Once we have $S$, we can canonically introduce the Dirac operator $D$ of the spin$^c$ structure:

3. $M$ is a spin manifold iff $\exists$ a real structure $J$ on $L^2(M, S)$.

# What is a noncommutative spin manifold?

For simplicity, let us focus on finite-dimensional spectral triples.
( $\Rightarrow\ A \equiv \mathrm{Lip}_D(A)$ and we can use the ring-theoretic Morita equivalence.)

**Definition (1-forms)**

If $(A, H, D)$ is a spectral triple, we define $\Omega^1_D \subseteq \mathcal{B}(H)$ as:

$$\Omega^1_D := \mathrm{Span}\{a[D, b] \ : \ a, b \in A\}$$

**Definition (Clifford algebra)** [≈ Lord, Rennie & Várilly, J.Geom.Phys. 2012]

We call $\mathcal{C}\ell_D(A) \subseteq \mathcal{B}(H)$ the algebra generated by $A$, $\Omega^1_D$ and possibly $\gamma$ (in the even case).

Let $A^\circ := \{Ja^*J^{-1} : a \in A\}$. The reality and 1st order cond. are equivalent to the statement

$$A^\circ \subseteq \mathcal{C}\ell_D(A)' := \{b \in \mathcal{B}(H) : [b, \xi] = 0\ \forall\ \xi \in \mathcal{C}\ell_D(A)\}\ . \qquad (\star)$$

**Definition (Dirac condition)**

Elements of $H$ are “Dirac spinors” if $(\star)$ is an equality: $A^\circ = \mathcal{C}\ell_D(A)'$.

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# On a property of “Hodge spinors”

In the geometric examples (slide 3), $[D, f] = c(df)$. In the Hodge example:

$$H = \overline{\Omega^\bullet(M)}^{L^2} \simeq \overline{\mathcal{C}\ell(M, g)}^{L^2} \qquad B := \mathcal{C}\ell_D(A) = \mathcal{C}\ell(M, g)$$

Representation of $B$: by Clifford multiplication on $\Omega^\bullet(M)$, or by left multiplication on itself.

Real structure: $J(\omega) = \omega^*$. The algebra $B^\circ = JBJ^{-1}$ acts by right multiplication on $H$, that up to completion is a self-Morita equivalence $B$-bimodule.

**Definition (2nd order condition)**

$(A, H, D, J)$ satisfies the 2nd order condition if

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Remark: this is the old “order-two” condition by Boyle and Farnsworth (cf. also Besnard, Bizi, Brouder).

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# Spin + 2nd order

**Observation 1.**

Dirac condition + 2nd order condition ⇒ Hodge condition.

In fact:

$$\mathcal{C}\ell_D(A)^\circ \subseteq \mathcal{C}\ell_D(A)' = A^\circ \qquad \Longrightarrow \qquad \boxed{\mathcal{C}\ell_D(A) = A}$$

Therefore:

**Observation 2.**

Dirac condition + 2nd order ⇒ $H$ is a self-Morita equivalence $A$-bimodule (a “line bundle”).

An example of spectral triple satisfying both conditions (Einstein-Yang Mills):

$$A = M_N(\mathbb{C}) \qquad H = A \qquad J(a) = a^* \qquad D = 0$$

# Back to the Standard Model...

Recall that in the ncg approach to the Standard Model,one has:

$$\underset{\text{(spin manifold)}}{M} \times \underset{\text{(finite nc space)}}{F}$$

For the continuous part, elements of $H_M$ are Dirac spinors. What about the finite part?

We have the following dictionary:

| **Geometry** $\longleftrightarrow$ | **Algebra** |
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| Spin$^c$ | $A$-$\mathcal{C}\ell_D(A)$ Morita equivalence |
| Spin | $A$-$\mathcal{C}\ell_D(A)$ Morita equivalence with $J$ |
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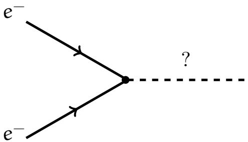
# Postdictions on $\mathrm{D_F}$

Not every $\mathrm{D_F}$ is allowed! $\Rightarrow$ Restrictions on the free parameters/on the interactions.

Constraints of the 1st kind:

1. The parity ($\gamma_F \mathrm{D_F} = -\mathrm{D_F}\gamma_F$) and 1st (or 2nd) order condition put constraints on $\mathrm{D_F}$: some matrix entries must be zero.

   For example, the 1st order cond. does not allow a vertex

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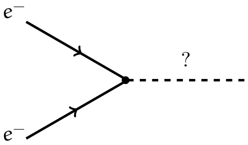


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# The 1st order condition

Let $(\mathcal{A}, \mathrm{H}, \mathrm{J})$ be finite dim. One can completely characterize D's of 1st order:

**Theorem ( ≈ Krajewski)**

- $\mathrm{D} \in \mathrm{End}_{\mathbb{C}}(\mathrm{H})$ satisfies the 1st order condition iff it is of the form

$$\mathrm{D} = \mathrm{D}_0 + \mathrm{D}_1 \qquad (\dagger)$$

  with $\mathrm{D}_0 \in (\mathcal{A}^\circ)'$ and $\mathrm{D}_1 \in \mathcal{A}'$.
- D selfadjoint resp. odd $\Rightarrow$ one can always choose $\mathrm{D}_0$ and $\mathrm{D}_1$ selfadjoint resp. odd.
- $\mathrm{JD} = \mathrm{DJ} \Rightarrow$ one can choose $\mathrm{D}_1 = \mathrm{J}\mathrm{D}_0\mathrm{J}^{-1}$.

*Proof.* Lemma: Let H be finite-dimensional and $V \subset \mathrm{End}(\mathrm{H})$ a $*$-subalgebra. Then, there exists a direct complement $W$ of $V$ in $\mathrm{End}(\mathrm{H})$ such that $[V, W] \subset W$. ♣

For $V = \mathcal{A}'$ let $W$ be the complement above. Write $\mathrm{D} = \mathrm{D}_0 + \mathrm{D}_1$ with $\mathrm{D}_0 \in V$ and $\mathrm{D}_1 \in W$. From the 1st order condition we deduce that in fact $\mathrm{D}_1 \in (\mathcal{A}^\circ)'$. □

Remark: In [Krajewski, J.Geom.Phys. 1998] uniqueness of the decomposition (†) follows from the orientability condition. In the νSM orientability is not satisfied, and the decomposition is not unique.

# The 1st order condition

Let $(A, H, J)$ be finite dim. One can completely characterize $D$'s of 1st order:

**Theorem ( $\approx$ Krajewski)**

- $D \in \text{End}_{\mathbb{C}}(H)$ satisfies the 1st order condition iff it is of the form

$$D = D_0 + D_1 \qquad (\dagger)$$

  with $D_0 \in (A^\circ)'$ and $D_1 \in A'$.
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# The Dirac condition

**Theorem**

If $\gamma_F = \chi$ is the chirality operator, there is no compatible $D_F$ satisfying the Dirac condition.

On the other hand, consider the following grading, given on particles by

$$\gamma_F := (B - L)\chi$$

with $B, L =$ barion/lepton nr. Then it is possible to find $D_F$ satisfying the Dirac condition (we have theorems both with necessary conditions and sufficient conditions).

Remarks:

- 16 free parameters or 25 with the non-standard $\gamma_F$ (for a toy model with 1 generation).
- In the Standard Model: 19 parameters, whose numerical values are established by experiments. One of these is the Higgs mass: $m_H \approx 126$ GeV.
- In Chamseddine-Connes' original spectral triple, $m_H$ is not a free parameter. It was predicted $m_H \approx 170$ GeV, a value ruled out by Tevatron in 2008.

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# On the Higgs mass

Several modifications of the original model have been proposed. One can:

1. enlarge the Hilbert space thus introducing new fermions [Stephan, 2009];
2. turn one element of $D_F$ into a field by hand, rather than getting it as a fluctuation of the metric [Chamseddine & Connes, 2012];
3. break (relax) the 1st order condition, thus allowing more terms in the Dirac operator (or in the algebra) [Chamseddine, Connes & van Suijlekom, 2013];
4. Grand Symmetry + twisted spectral triples [Devastato, Lizzi & Martinetti, 2014].

In 2,3,4: the Majorana mass term of the neutrino is replaced by a new scalar field $\Phi$.

**Theorem**

In order to satisfy the Dirac condition, we must add two terms to Chamseddine-Connes $D_F$. We get:

- $\longrightarrow$ a new scalar field close to the $\Phi$ above (but doesn't break the 1st order condition);
- $\longrightarrow$ a field coupling leptons with quarks.

Physical implications are under investigation (see the talk at this conference by F. Lizzi).

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Questions?