Detecting regularity using cyclic cocycles and singular traces

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joint work with Heiko Gimperlein and Ryszard Nest

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160406 “Gauge Theory and Noncommutative Geometry”, Nijmegen

Regularity in spectral triples

Spectral triples

Recall that a spectral triple is a collection $(\mathcal{A}, \mathcal{H}, D)$ where

1. $A := \overline{\mathcal{A}}^{C^*}$ acts on $\mathcal{H}$;
2. D is a self-adjoint operator on $\mathcal{H}$ with $(i + D)^{-1} \in \mathbb{K}(\mathcal{H})$;
3. $\mathcal{A} \subseteq \mathrm{Lip}_D(A) := \{a \in A : a\mathrm{Dom}(D) \subseteq \mathrm{Dom}(D),\ [D, a] \text{ bounded}\}$.

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Prototypical example

Take a closed Riemannian manifold M, a Clifford bundle $S \to M$ and a Dirac operator D on S. Then $(\mathcal{A}, L^2(M, S), D)$ is a spectral triple for any

$$C^\infty(M) \subseteq \mathcal{A} \subseteq \mathrm{Lip}(M) := \{a \in C(M) : \exists C,\ |a(x) - a(y)| \leq C\mathrm{d}(x, y)\}.$$

Structures on manifolds

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$\mathcal{A} = C^1(M)$

1. If M_1 and M_2 are two C^k-structures on M, $C^k(M_1) \cong C^k(M_2)$ iff $M_1 \cong M_2$ as C^k-manifolds.
2. Any C^1-structure on M gives rise to a unique real analytic structure (Whitney), so if $M_1 \cong M_2$ as C^1-manifolds then $M_1 \cong M_2$ as C^∞-manifolds.
3. A spectral triple $(C^\infty(M), L^2(M, S), D)$ (plus additional data) determines M with its C^∞-structure by Connes' reconstruction theorem.

Hölder continuous functions

Let M be a d-dimensional smooth closed Riemannian manifold and

$$C^\alpha(M) := \{a \in C(M) : \exists C,\ |a(x) - a(y)| \leq C\mathrm{d}(x,y)^\alpha\},\ \alpha \in (0,1).$$

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- If $D \in \Psi^s(M,E)$, $s \in (0,\alpha)$ is elliptic, $(C^\alpha(M), L^2(M,S), D)$ is a spectral triple.
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Motivation

1. A source for non-examples in NCG.
2. Differential topological invariants for low regularity functions, used when solving non-linear PDE arising from field equations (e.g. Skyrme's model).
3. Gromov's question on optimal bounds on the Hölder exponent of isometric embeddings of euclidean balls into a contact manifold with its sub-Riemannian metric.

The structure of $C^\alpha(M)$

An operator algebra is a closed sub-algebra of a C^*-algebra. For instance, $\mathrm{Lip}(M)$ has an operator algebra structure defined from a Dirac operator D acting on some Clifford bundle $S \to M$. This uses the homomorphism

$$\pi_D : \mathrm{Lip}(M) \to L^\infty(M, \mathrm{End}(S \oplus S)), \quad \pi_D(a) := \begin{pmatrix} a & 0 \\ [D, a] & a \end{pmatrix}.$$

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Operator algebra structure on $C^\alpha(M)$

Set $X := M \times M \setminus \Delta_M$. For $\alpha \in (0, 1]$, define $\pi_\alpha : C^\alpha(M) \to C_b(X, M_2(\mathbb{C}))$ by

$$\pi_\alpha(a) := \begin{pmatrix} \pi_L(a) & 0 \\ \delta_\alpha(a) & \pi_R(a) \end{pmatrix} = \begin{pmatrix} a(x) & 0 \\ \frac{a(x)-a(y)}{\mathrm{d}(x,y)^\alpha} & a(y) \end{pmatrix}.$$

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We set $h^\alpha(M) := \overline{C^\infty(M)}^{C^\alpha}$ and note that

$$h^\alpha(X) = \{a \in C^\alpha(M) : \delta_\alpha(a) \in C_0(X)\}, \quad \text{for } \alpha < 1.$$

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Letting Ω denote the Stone-Cech boundary of X, i.e. $C(\Omega) = C_b(X)/C_0(X)$, and $q : C_b(X) \to C(\Omega)$ the quotient, $h^\alpha(X) = \ker q \circ \delta_\alpha$.

The structure of $C^\alpha(M)$ continued

Non-separability

1. The operator algebra $\mathrm{Lip}(M)$ is closed in the strong operator topology in $L^\infty(M, \mathrm{End}(S \oplus S))$.
2. For $\alpha < 1$, $C^\alpha(M) = h^\alpha(M)^{**}$ (Weaver).
3. If $d > 0$, $C^\alpha(M)$ is non-separable.

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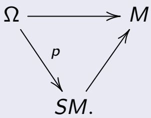
Non-commutative "vector-fields"

There are inclusions $C(M) \subseteq C(SM) \subseteq C(\Omega)$ so, Ω is a "thickening" of SM in the sense that there are mappings



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$$\begin{array}{ccc} \Omega & \longrightarrow & M \\ & {\scriptstyle p}\searrow \quad \nearrow & \\ & SM. & \end{array}$$

If $x \in \Omega$ and $\alpha = 1$, then

$$\delta_1(a)(x) = p(x).a(x), \quad \text{for } a \in C^\infty(M).$$

In particular, for $v \in SM$ the set $p^{-1}(v) \subseteq \Omega$ is that of extensions of "directional derivatives along v" to $C^\alpha(M)$.

Cyclic (co)-homology

We consider a unital Frechet algebra $\mathcal{A}$. Set

$$C_k(\mathcal{A}) := \mathcal{A}^{\hat{\otimes} k+1} \quad \text{and} \quad C_k^\lambda(\mathcal{A}) := \mathcal{A}^{\hat{\otimes} k+1}/(1-\lambda)\mathcal{A}^{\hat{\otimes} k+1},$$

where $\lambda(a_0 \otimes \cdots \otimes a_k) = (-1)^k a_k \otimes a_0 \otimes \cdots \otimes a_{k-1}$. There is a differential

$$b(a_0 \otimes \cdots \otimes a_k) = \sum_{j=0}^{k-1} (-1)^j a_0 \otimes \cdots a_{j-1} \otimes a_j a_{j+1} \otimes \cdots \otimes a_k + (-1)^k a_k a_0 \otimes a_1 \otimes \cdots \otimes a_{k-1}.$$

We set $\quad HC_*(\mathcal{A}) := H_*(C_*^\lambda(\mathcal{A}), b), \quad HH_*(\mathcal{A}) := H_*(C_*(\mathcal{A}), b)$

and $\quad HC^*(\mathcal{A}) := H^*(C^*_\lambda(\mathcal{A}), b^*), \quad HH^*(\mathcal{A}) := H^*(C^*(\mathcal{A}), b^*)$

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The SBI-sequence

The periodicity operator $S : HC_{*+2}(\mathcal{A}) \to HC_*(\mathcal{A})$ fits into a long exact sequence with Hochschild homology:

$$\cdots \xrightarrow{B} HH_{*+2}(\mathcal{A}) \xrightarrow{I} HC_{*+2}(\mathcal{A}) \xrightarrow{S} HC_*(\mathcal{A}) \xrightarrow{B} HH_{*+1}(\mathcal{A}) \xrightarrow{I} HC_{*+1}(\mathcal{A}) \xrightarrow{S} \cdots,$$

where $B : HC_*(\mathcal{A}) \to HH_{*+1}(\mathcal{A})$ denotes the Connes differential. Analogous sequences are exact on the dual side.

Dictionary for smooth manifolds

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The CHKR-isomorphisms

Let M be a closed manifold, $\mathcal{A} = C^\infty(M)$ and let $\Omega_k(M)$ denote the space of k-forms on M:

- $HC_k(\mathcal{A}) \cong \oplus_{j=1}^{\infty} H_{dR}^{k-2j}(M) \oplus \Omega^k(M)/B^k(M)$, where $H_{dR}^*(M)$ is the de Rham cohomology and $B^k(M)$ the space of exact k-forms.
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The index character of a Fredholm module

A Fredholm module $(\mathcal{A}, \mathcal{H}, F)$ satisfying $F^2 = 1$ and $[F, a] \in \mathcal{L}^k(\mathcal{H})$ for $a \in \mathcal{A}$ gives rise to a cyclic k-cocycle:

$$\mathrm{ch}_F^k(a_0, a_1, \ldots, a_k) = c_k \mathrm{tr}(\gamma a_0 [F, a_1] \cdots [F, a_k]).$$

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The Hochschild character of a spectral triple

A spectral triple $(\mathcal{A}, \mathcal{H}, D)$ satisfying $D^{-1} \in \mathcal{L}^{d,\infty}(\mathcal{H})$ and a singular state $\omega \in (\ell^\infty / c_0)^*$ give rise to a Hochschild cocycle:

$$\tau_{D,\omega}(a_0, a_1, \dots, a_d) = c_d \mathrm{tr}_\omega(\gamma a_0 [D, a_1] \cdots [D, a_d] |D|^{-d}).$$

Cyclic theories and Hölder functions

Recall the following facts:

- If $a \in C^\alpha(M)$ and $F \in \Psi^0(M, E)$, $[F, a] \in \mathcal{L}^{d/\alpha,\infty}(L^2(M, E))$. In particular, if $k + 1 > d/\alpha$ and $F^2 = 1$, we obtain a k-cocycle:

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The mapping $C^\infty(M) \to C^\alpha(M)$ in cyclic theories

If $k > d/\alpha$ the mappings induced by the inclusion $C^\infty(M) \to C^\alpha(M)$

$$HC^k(C^\alpha(M)) \to HC^k(C^\infty(M)) \quad \text{and} \quad HC_k(C^\infty(M)) \to HC_k(C^\alpha(M)),$$

are surjective and injective, respectively.

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Open questions

- What happens with surjectivity/injectivity for $k+1 \leq d/\alpha$? E.g. is $\mathrm{ch}_j : K_j(C^\alpha(M)) \to HC_j(C^\alpha(M))$ injective for $j \leq d/\alpha$?
- What aspects of $HC^*(C^\alpha(M))$ are computable?

"Singular" cyclic cocycles

A singular state $\omega \in (\ell^\infty(\mathbb{N})/c_0(\mathbb{N}))^*$ gives rise to a singular trace

$$\mathrm{tr}_\omega : \mathcal{L}^{1,\infty}(\mathcal{H}) \to \mathbb{C}, \quad \mathrm{tr}_\omega(T) := \omega\left(\frac{\sum_{k=1}^{N} \mu_k(T)}{\log(2+N)}\right)_N, \quad \text{for } T \geq 0.$$

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"Singular" Chern characters

Assume that $(\mathcal{A}, \mathcal{H}, F)$ is a (k, ∞)-summable Fredholm module with $F^2 = 1$ and $\omega \in (\ell^\infty(\mathbb{N})/c_0(\mathbb{N}))^*$ is a singular state. We define $c_{k-1,\omega} \in C_\lambda^{k-1}$ and $\xi_{k,\omega} \in C^k$ by

$$c_{k,\omega}(a_0, a_1, \dots, a_{k-1}) := \frac{c_k}{2}\mathrm{tr}_\omega(\gamma F[F, a_0][F, a_1]\cdots[F, a_{k-1}]).$$

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$$\mathrm{tr}_\omega : \mathcal{L}^{1,\infty}(\mathcal{H}) \to \mathbb{C}, \quad \mathrm{tr}_\omega(T) := \omega\left(\frac{\sum_{k=1}^N \mu_k(T)}{\log(2+N)}\right)_N, \quad \text{for } T \geq 0.$$

"Singular" Chern characters

Assume that $(\mathcal{A}, \mathcal{H}, F)$ is a (k,∞)-summable Fredholm module with $F^2 = 1$ and $\omega \in (\ell^\infty(\mathbb{N})/c_0(\mathbb{N}))^*$ is a singular state. We define $c_{k-1,\omega} \in C_\lambda^{k-1}$ and $\xi_{k,\omega} \in C^k$ by

$$c_{k,\omega}(a_0, a_1, \ldots, a_{k-1}) := \frac{c_k}{2}\mathrm{tr}_\omega(\gamma F[F,a_0][F,a_1]\cdots[F,a_{k-1}]).$$

$$\xi_{k,\omega}(a_0, a_1, \ldots, a_k) := \frac{c_k}{2}\mathrm{tr}_\omega(\gamma F a_0[F,a_1]\cdots[F,a_k]).$$

General properties

- Both $c_{k,\omega}$ and $\xi_{k,\omega}$ are closed giving rise to classes $[c_{k,\omega}] \in HC^{k-1}(\mathcal{A})$ and $[\xi_{k,\omega}] \in HH^k(\mathcal{A})$.
- $S[c_{k,\omega}] = 0$ and $B[\xi_{k,\omega}] = [c_{k,\omega}]$.

The situation on manifolds

Henceforth, assume $F \in \Psi^0(M, E)$ satisfies $F^2 = 1$ (e.g. $F = \not{D}|\not{D}|^{-1}$).

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The Fredholm module $(\mathrm{Lip}(M), L^2(M,E), F)$ is (d,∞)-summable. Set $\sigma := \sigma_0(F) \in C^\infty(S^*M, \mathrm{End}(\pi^*E))$. Then

$$c_{k,\omega}(a_0, a_1, \ldots, a_{k-1}) := c_d \int_{S^*M} \mathrm{tr}_E(\gamma\sigma \prod_{j=0}^{k-1} \{\sigma, a_j\}) \mathrm{d}S$$

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Some analytic subtleties

- For $f \in W^{1,d}(M)$, $\|[F,f]\|_{\mathcal{L}^{d,\infty}} \sim \|\nabla f\|_{L^d}$ (Rochberg-Semmes, Connes-Sullivan-Teleman).
- On the other hand, for $p > d$, $\|[F,f]\|_{\mathcal{L}^{p,\infty}} \sim \|f\|_{B^{d/p}_{p,\infty}}$ (Rochberg-Semmes). Note $C^\alpha(M) = B^\alpha_{\infty,\infty}(M) \subseteq B^\alpha_{d/\alpha,\infty}(M)$.

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$C^\infty(M)$ is dense in $W^{1,d}(M)$ but not in $B^\alpha_{d/\alpha,\infty}(M)$!

An example on S^1

To compute Dixmier traces, we need additional mapping properties

Sobolev mapping properties

Let $F \in \Psi^0(M)$. If $\alpha \in (0,1)$, $s \in (-\alpha, 0)$ and $a \in C^\alpha(M)$ then $[F, a]$ extends to a continuous operator

$$[F, a] : W^s(M) = \Delta^{-s/2}L^2(M) \to W^{s+\alpha}(M) = \Delta^{-(s+\alpha)/2}L^2(M).$$

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Computing Dixmier traces

If $F_j \in \Psi^0(M)$ and $a_j \in C^{\alpha_j}(M)$ for $j = 0,1,\ldots,k$ and $\sum_{j=1}^k \alpha_j = d$, then for any singular state $\omega \in (\ell^\infty/c_0)^*$

$$\mathrm{tr}_\omega(F_0a_0[F_1,a_1]\cdots[F_k,a_k]) = \omega\left(\frac{\sum_{k=1}^N \langle F_0a_0[F_1,a_1]\cdots[F_k,a_k]e_k, e_k\rangle_{L^2}}{\log(2+N)}\right),$$

where $(e_k)_{k\in\mathbb{N}}$ is any orthonormal eigenbasis associated with an elliptic operator on M.

An example on S^1, continued

Consider $F \in \Psi^0(S^1)$, where $S^1 \subseteq \mathbb{C}$ defined by

$$Ff(z) := \frac{\mathrm{p.v.}}{\pi i} \int_{S^1} \frac{f(w)}{z-w} \mathrm{d}w.$$

The Szegö projection $P := (F+1)/2$ projects onto the Hardy space $H^2(S^1) \subseteq L^2(S^1)$.

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Ultraviolet divergence of $H^{1/2}$-mapping degree

If $a, b \in C^{1/2}(S^1)$ and $\omega \in (\ell^\infty/c_0)^*$,

$$c_{2,\omega}(a,b) = \omega\left(\frac{\sum_{k=0}^N k(a_k b_{-k} - a_{-k}b_k)}{\log(2+N)}\right),$$

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$$\deg_{H^{1/2}}(a) := \frac{1}{2\pi} \int_{S^1} a^* \mathrm{d}a = \mathrm{tr}((2P-1)[P,a][P,a^*]) = \sum_{k=0}^{\infty} k(|a_k|^2 - |a_{-k}|^2).$$

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In fact, if $x \in K_1(C^{1/2}(S^1))$ has Chern character $\mathrm{ch}_{2k+1}(x) \in HC^{2k+1}(C^{1/2}(S^1))$, then

$$\langle c_{2,\omega}, \mathrm{ch}_1(x) \rangle = \langle c_{2,\omega}, S\mathrm{ch}_3(x) \rangle = \langle Sc_{2,\omega}, \mathrm{ch}_3(x) \rangle = 0.$$

An example on S^1, continued

For $\mu \in \ell^\infty(\mathbb{N})$, we define $w_\mu \in C^{1/2}(S^1)$ by

$$w_\mu(z) := \sum_{n=0}^{\infty} 2^{-n/2}\mu_n(z^{2^n} + z^{-2^n}).$$

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$$\mathrm{tr}_\omega(Pw_\mu(1-P)w_{\mu'}P) = 2^\omega\left(\frac{\sum_{n=0}^{N}\mu_n\mu'_n}{(N+1)\cdot\log(2)}\right), \quad \mu,\mu' \in \ell^\infty(\mathbb{N}).$$

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- For $\mu = \mu' = 1$, $\mathrm{tr}_\omega(Pw_1(1-P)w_1P) = (\log(2))^{-1}$.
- $c_{2,\omega}(Pw_\mu, (1-P)w_{\mu'}) = \mathrm{tr}_\omega(Pw_\mu(1-P)w_{\mu'}P)$.
- The linear span of

$$\{[c_{2,\omega}] : \omega \in (\ell^\infty/c_0)^*\} \subseteq \ker(HC^1(C^{1/2}(S^1)) \to HC^1(C^\infty(S^1)))$$

  is infinite-dimensional and pairs with $HC_1(C^{1/2}(S^1))$ through non-measurable operators.

Thanks

Thanks for your attention!

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Sums of self-adjoint operators: Kasparov products and applications