(Not only) Line bundles over noncommutative spaces

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Work done over few years with

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Abstract:

• Pimsner algebras of ‘tautological’ line bundles: Total spaces of principal bundles out of a Fock-space construction

• Gysin-like sequences in KK-theory

• Quantum lens spaces as direct sums of line bundles over weighted quantum projective spaces

• Self-dual connections:
on line bundles: monopole connections
on higher rank bundles: instanton connections

• some hint to T-dual noncommutative bundles

‘grand motivations’ :

Gauge fields on noncommutative spaces

T-duality for noncommutative spaces

Chern-Simons theory

A Gysin sequence for U(1)-bundles

relates H-flux (three-forms on the total space E) to line bundles (two-forms on the base space M) also giving an isomorphism between Dixmier-Douady classes on E and line bundles on M

The classical Gysin sequence

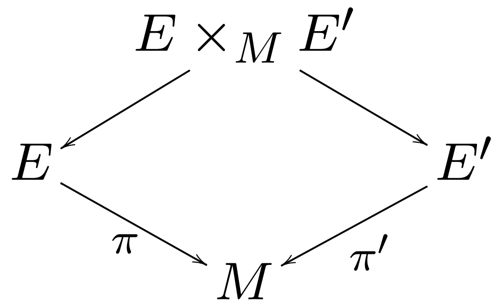
Long exact sequence in cohomology; for any sphere bundle

In particular, for circle bundles: $\mathrm{U}(1) \to E \xrightarrow{\pi} X$

$$\cdots \longrightarrow H^k(E) \xrightarrow{\pi_*} H^{k-1}(X) \xrightarrow{\cup c_1(E)} H^{k+1}(X) \xrightarrow{\pi^*} H^{k+1}(E) \longrightarrow \cdots$$

$$\cdots \longrightarrow H^3(E) \xrightarrow{\pi_*} H^2(X) \xrightarrow{\cup c_1(E)} H^4(X) \longrightarrow \cdots$$

$$H^3(E) \ni H \mapsto \pi_*(H) = F' = c_1(E')$$



$$\cdots \longrightarrow H^3(E') \xrightarrow{\pi_*} H^2(X) \xrightarrow{\cup c_1(E')} H^4(X) \longrightarrow \cdots$$

$$F' \cup F = 0 = F \cup F'$$

$$\Rightarrow \quad \exists \quad H^3(E') \ni H' \mapsto \pi_*(H) = F = c_1(E)$$

T-dual (E, H) and (E', H')

Bouwknegt, Evslin, Mathai, 2004

difficult to generalize to quantum spaces

rather go to K-theory ; a six term exact sequence (see later)

Projective spaces and lens spaces

$\mathbb{C}\mathsf{P}^n = \mathsf{S}^{2n+1}/\mathsf{U}(1)$ and $L^{(n,r)} = \mathsf{S}^{2n+1}/\mathbb{Z}_r$

assemble in principal bundles : $\mathsf{S}^{2n+1} \longrightarrow \mathsf{L}^{(n,r)} \xrightarrow{\pi} \mathbb{C}\mathsf{P}^n$

This leads to the Gysin sequence in topological K-theory:

$$0 \longrightarrow K^1(\mathsf{L}^{(n,r)}) \xrightarrow{\delta} K^0(\mathbb{C}\mathsf{P}^n) \xrightarrow{\alpha} K^0(\mathbb{C}\mathsf{P}^n) \xrightarrow{\pi^*} K^0(\mathsf{L}^{(n,r)}) \longrightarrow 0$$

δ is a 'connecting homomorphism'

α is multiplication by the Euler class $\chi(\mathcal{L}_{-r}) := 1 - [\mathcal{L}_{-r}]$

From this:

$$K^1(\mathsf{L}^{(n,r)}) \simeq \mathsf{ker}(\alpha) \quad \text{and} \quad K^0(\mathsf{L}^{(n,r)}) \simeq \mathsf{coker}(\alpha)$$

torsion groups

U(1)-principal bundles

The Hopf algebra

$$\mathcal{H} = \mathcal{O}(\mathrm{U}(1)) := \mathbb{C}[z, z^{-1}] / \left\langle 1 - zz^{-1} \right\rangle$$

$$\Delta : z^n \mapsto z^n \otimes z^n \quad ; \quad S : z^n \mapsto z^{-n} \quad ; \quad \epsilon : z^n \mapsto 1$$

Let $\mathcal{A}$ be a right comodule algebra over $\mathcal{H}$ with coaction

$$\Delta_R : \mathcal{A} \to \mathcal{A} \otimes \mathcal{H}$$

$\mathcal{B} := \{x \in \mathcal{A} \mid \Delta_R(x) = x \otimes 1\}$ be the subalgebra of coinvariants

Definition 1. *The datum* $\left(\mathcal{A}, \mathcal{H}, \mathcal{B}\right)$ *is a* quantum principal U(1)-bundle *when the* canonical map *is an isomorphism*

$$\mathrm{can} : \mathcal{A} \otimes_{\mathcal{B}} \mathcal{A} \to \mathcal{A} \otimes \mathcal{H}\,, \quad x \otimes y \mapsto x\, \Delta_R(y)\,.$$

$\mathbb{Z}$-graded algebras

$\mathcal{A} = \oplus_{n\in\mathbb{Z}}\mathcal{A}_n$ a $\mathbb{Z}$-graded algebra. A right $\mathcal{H}$-comodule algebra:

$$\Delta_R : \mathcal{A} \to \mathcal{A} \otimes \mathcal{H} \quad x \mapsto x \otimes z^{-n}, \text{ for } x \in \mathcal{A}_n ,$$

with the subalgebra of coinvariants given by $\mathcal{A}_0$.

Proposition 2. *The triple* $\left(\mathcal{A}, \mathcal{H}, \mathcal{A}_0\right)$ *is a quantum principal* U(1)-*bundle if and only if there exist finite sequences*

$$\{\xi_j\}_{j=1}^N , \ \{\beta_i\}_{i=1}^M \textit{ in } \mathcal{A}_1 \quad \text{and} \quad \{\eta_j\}_{j=1}^N , \ \{\alpha_i\}_{i=1}^M \textit{ in } \mathcal{A}_{-1}$$

such that:

$$\sum_{j=1}^N \xi_j \eta_j = 1_{\mathcal{A}} = \sum_{i=1}^M \alpha_i \beta_i .$$

Corollary 3. *Same conditions as above. The right-modules $\mathcal{A}_1$ and $\mathcal{A}_{-1}$ are finitely generated and projective over $\mathcal{A}_0$.*

Proof. For $\mathcal{A}_1$: define the module homomorphisms

$$\Phi_1 : \mathcal{A}_1 \to (\mathcal{A}_0)^N\,, \quad \Phi_1(\zeta) = \begin{pmatrix} \eta_1\,\zeta \\ \eta_2\,\zeta \\ \vdots \\ \eta_N\,\zeta \end{pmatrix} \quad \text{and}$$

$$\Psi_1 : (\mathcal{A}_0)^N \to \mathcal{A}_1\,, \quad \Psi_1 \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix} = \sum_j \xi_j\, x_j\,.$$

Then $\Psi_1\Phi_1 = \mathrm{Id}_{\mathcal{A}_1}$.

Thus $E_1 := \Phi_1\Psi_1$ is an idempotent in $M_N(\mathcal{A}_0)$. □

The above results show that $\left(\mathcal{A}, \mathcal{H}, \mathcal{A}_0\right)$ is a quantum principal U(1)-bundle if and only if $\mathcal{A}$ is *strongly* $\mathbb{Z}$-graded, that is

$$\mathcal{A}_n \mathcal{A}_{(m)} = \mathcal{A}_{(n+m)}$$

Equivalently, the right-modules $\mathcal{A}_{(\pm 1)}$ are finitely generated and projective over $\mathcal{A}_0$ if and only if $\mathcal{A}$ is *strongly* $\mathbb{Z}$-graded

C. Nastasescu, F. Van Oystaeyen, *Graded Ring Theory*

K.H. Ulbrich, 1981

More generally: G any group with unit e

An algebra $\mathcal{A}$ is *G-graded* if $\mathcal{A} = \oplus_{g \in G} \mathcal{A}_g$, and $\mathcal{A}_g \mathcal{A}_h \subseteq \mathcal{A}_{gh}$

If $\mathcal{H} := \mathbb{C}G$ the group algebra, then $\mathcal{A}$ is G-graded if and only if $\mathcal{A}$ is a right $\mathcal{H}$-comodule algebra for the coaction $\delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{H}$

$$\delta(a_g) = a_g \otimes g, \qquad a_g \in \mathcal{A}_g;$$

coinvariants given by $\mathcal{A}^{co\mathcal{H}} = \mathcal{A}_e$, the identity components.

Proposition 4. *The datum $\left(\mathcal{A}, \mathcal{H}, \mathcal{A}_e\right)$ is a noncommutative principal $\mathcal{H}$-bundle for the canonical map*

$$\text{can} : \mathcal{A} \otimes_{\mathcal{A}_e} \mathcal{A} \to \mathcal{A} \otimes \mathcal{H}\,, \quad a \otimes b \mapsto \sum\nolimits_g ab_g \otimes g\,,$$

if and only if $\mathcal{A}$ is strongly graded*, that is $\mathcal{A}_g \mathcal{A}_h = \mathcal{A}_{gh}$.*

When $G = \mathbb{Z} = \widehat{\mathrm{U}(1)}$, then $\mathbb{C}G = \mathcal{O}(\mathrm{U}(1))$ as before.

More general scheme: Pimsner algebras M.V. Pimsner '97

The right-modules $\mathcal{A}_1$ and $\mathcal{A}_{-1}$ before are 'line bundles' over $\mathcal{A}_0$

The slogan: a line bundle is a self-Morita equivalence bimodule

E a (right) Hilbert module over B

B-valued hermitian structure $\langle \cdot, \cdot \rangle$ on E

$\mathcal{L}(E)$ adjointable operators; $\mathcal{K}(E) \subseteq \mathcal{L}(E)$ compact operators

with $\xi, \eta \in E$, denote $\theta_{\xi,\eta} \in \mathcal{K}(E)$ defined by $\theta_{\xi,\eta}(\zeta) = \xi \langle \eta, \zeta \rangle$

There is an isomorphism $\phi : B \to \mathcal{K}(E)$ and E is a B-bimodule

Comparing with before:

$$\mathcal{A}_0 \rightsquigarrow B \qquad \text{and} \qquad \mathcal{A}_{-1} \rightsquigarrow E$$

Look for the analogue of $\mathcal{A} \quad \rightsquigarrow \mathcal{O}_E$ Pimsner algebra

Examples

$$B = \mathcal{O}(\mathbb{C}\mathsf{P}^n_q) \qquad \text{quantum (weighted) projective spaces}$$

$$E = \mathcal{L}_{-r} \simeq (\mathcal{L}_{-1})^r \qquad \text{(powers of) tautological line bundle}$$

$$\mathcal{O}_E = \mathcal{O}(\mathsf{L}^{(n,r)}_q) \qquad \text{quantum lens spaces}$$

Define the B-module

$$E_\infty := \bigoplus_{N\in\mathbb{Z}} E^{\widehat{\otimes}_\phi N}, \qquad E^0 = B$$

$E \otimes_\phi E$ the inner tensor product: a B-Hilbert module with B-valued hermitian structure

$$\langle \xi_1 \otimes \eta_1, \xi_2 \otimes \eta_2 \rangle = \langle \eta_1, \phi(\langle \xi_1, \xi_2 \rangle)\eta_2 \rangle$$

$E^{-1} = E^*$ the dual module;
its elements are written as λ_ξ for $\xi \in E$: $\lambda_\xi(\eta) = \langle \xi, \eta \rangle$

For each $\xi \in E$ a bounded adjointable operator

$$S_\xi : E_\infty \to E_\infty$$

generated by $S_\xi : E^{\widehat{\otimes}_\phi N} \to E^{\widehat{\otimes}_\phi (N+1)}$:

$$\begin{aligned} S_\xi(b) &:= \xi\, b\,, && b \in B\,, \\ S_\xi(\xi_1 \otimes \cdots \otimes \xi_N) &:= \xi \otimes \xi_1 \otimes \cdots \otimes \xi_N\,, && N > 0\,, \\ S_\xi(\lambda_{\xi_1} \otimes \cdots \otimes \lambda_{\xi_{-N}}) &:= \lambda_{\xi_2\, \phi^{-1}(\theta_{\xi_1,\xi})} \otimes \lambda_{\xi_3} \otimes \cdots \otimes \lambda_{\xi_{-N}}\,, && N < 0\,. \end{aligned}$$

Definition 5. *The Pimsner algebra $\mathcal{O}_E$ of the pair (ϕ, E) is the smallest subalgebra of $\mathcal{L}(E_\infty)$ which contains the operators $S_\xi : E_\infty \to E_\infty$ for all $\xi \in E$.*

Pimsner: universality of $\mathcal{O}_E$

There is a natural inclusion

$$B \hookrightarrow \mathcal{O}_E \qquad \text{a generalized principal circle bundle}$$

roughly: as a vector space $\mathcal{O}_E \simeq E_\infty$ and

$$E^{\widehat{\otimes}_\phi N} \ni \eta \mapsto \eta\lambda^{-N}, \qquad \lambda \in \mathsf{U}(1)$$

Two natural classes in KK-theory:

1. the class $[E] \in KK_0(B, B)$
of the even Kasparov module $(E, \phi, 0)$ (with trivial grading)

the map $1 - [E]$ has the role of the *Euler class* $\chi(E) := 1 - [E]$

of the line bundle E over the 'noncommutative space' B

2. the class $[\partial] \in KK_1(\mathcal{O}_E, B)$
of the odd Kasparov module $(E_\infty, \widetilde{\phi}, F)$:

$F := 2P - 1 \in \mathcal{L}(E_\infty)$ of the projection $P : E_\infty \to E_\infty$ with

$$\mathrm{Im}(P) = \left(\oplus_{N=0}^{\infty} E^{\widehat{\otimes}_\phi N} \right) \subseteq E_\infty$$

and inclusion $\widetilde{\phi} : \mathcal{O}_E \to \mathcal{L}(E_\infty)$.

The Kasparov product induces group homomorphisms

$$[E] : K_*(B) \to K_*(B)\,, \quad [E] : K^*(B) \to K^*(B)$$

and

$$[\partial] : K_*(\mathcal{O}_E) \to K_{*+1}(B)\,, \quad [\partial] : K^*(B) \to K^{*+1}(\mathcal{O}_E)\,,$$

Associated six-terms exact sequences Gysin sequences:
in K-theory:

$$\begin{array}{ccccc}
K_0(B) & \xrightarrow{1-[E]} & K_0(B) & \xrightarrow{i_*} & K_0(\mathcal{O}_E) \\
{\scriptstyle [\partial]}\uparrow & & & & \downarrow{\scriptstyle [\partial]} \\
K_1(\mathcal{O}_E) & \underset{i_*}{\longleftarrow} & K_1(B) & \underset{1-[E]}{\longleftarrow} & K_1(B)
\end{array} \quad ;$$

the corresponding one in K-homology:

$$\begin{array}{ccccc}
K^0(B) & \underset{1-[E]}{\longleftarrow} & K^0(B) & \underset{i^*}{\longleftarrow} & K^0(\mathcal{O}_E) \\
\downarrow{\scriptstyle [\partial]} & & & & {\scriptstyle [\partial]}\uparrow \\
K^1(\mathcal{O}_E) & \xrightarrow{i^*} & K^1(B) & \xrightarrow{1-[E]} & K^1(B)
\end{array} \quad .$$

In fact in KK-theory

The quantum spheres and the projective spaces

The coordinate algebra $\mathcal{O}(\mathsf{S}_q^{2n+1})$ of quantum sphere S_q^{2n+1}:
$*$-algebra generated by $2n+2$ elements $\{z_i, z_i^*\}_{i=0,\dots,n}$ s.t.:

$$
\begin{aligned}
z_i z_j &= q^{-1} z_j z_i & 0 \le i < j \le n ,\\
z_i^* z_j &= q z_j z_i^* & i \neq j ,\\
[z_n^*, z_n] = 0 , \quad [z_i^*, z_i] &= (1-q^2) \sum_{j=i+1}^{n} z_j z_j^* & i = 0, \dots, n-1 ,
\end{aligned}
$$

and a sphere relation:

$$1 = z_0 z_0^* + z_1 z_1^* + \ldots + z_n z_n^* .$$

L. Vaksman, Ya. Soibelman, 1991 ; M. Welk, 2000

The $*$-subalgebra of $\mathcal{O}(\mathsf{S}_q^{2n+1})$ generated by

$$p_{ij} := z_i^* z_j$$

coordinate algebra $\mathcal{O}(\mathbb{C}\mathsf{P}_q^n)$ of the quantum projective space $\mathbb{C}\mathsf{P}_q^n$

Invariant elements for the U(1)-action on the algebra $\mathcal{O}(\mathsf{S}_q^{2n+1})$:

$$(z_0, z_1, \ldots, z_n) \mapsto (\lambda z_0, \lambda z_1, \ldots, \lambda z_n), \qquad \lambda \in \mathsf{U}(1).$$

the fibration $\mathsf{S}_q^{2n+1} \to \mathbb{C}\mathsf{P}_q^n$ is a quantum U(1)-principal bundle:

$$\mathcal{O}(\mathbb{C}\mathsf{P}_q^n) = \mathcal{O}(\mathsf{S}_q^{2n+1})^{\mathsf{U}(1)} \hookrightarrow \mathcal{O}(\mathsf{S}_q^{2n+1})\,.$$

The C^*-algebras $C(\mathsf{S}_q^{2n+1})$ and $C(\mathbb{C}\mathsf{P}_q^n)$ of continuous functions: completions of $\mathcal{O}(\mathsf{S}_q^{2n+1})$ and $\mathcal{O}(\mathbb{C}\mathsf{P}_q^n)$ in the universal C^*-norms

these are graph algebras J.H. Hong, W. Szymański 2002

$$\Rightarrow \qquad K_0(\mathbb{C}\mathsf{P}_q^n) \simeq \mathbb{Z}^{n+1} \simeq K^0(C(\mathbb{C}\mathsf{P}_q^n))$$

F. D'Andrea, G. L. 2010

Generators of the homology group $K^0(C(\mathbb{C}\mathsf{P}_q^n))$ given explicitly as (classes of) even Fredholm modules

$$\mu_k = (\mathcal{O}(\mathbb{C}\mathsf{P}_q^n), \mathcal{H}_{(k)}, \pi^{(k)}, \gamma_{(k)}, F_{(k)}) , \quad \text{for} \quad 0 \leq k \leq n \,.$$

Generators of the K-theory $K_0(\mathbb{C}\mathsf{P}^n_q)$ also given explicitly as projections whose entries are polynomial functions:

line bundles & projections

For $N \in \mathbb{Z}$, vector-valued functions

$$\Psi_N := (\psi^N_{j_0,\dots,j_n}) \qquad \text{s.t.} \qquad \Psi_N^* \Psi_N = 1$$

$\Rightarrow \qquad P_N := \Psi_N \Psi_N^*$ is a projection:

$$P_N \in \mathsf{M}_{d_N}(\mathcal{O}(\mathbb{C}\mathsf{P}^n_q)), \qquad d_N := \binom{|N|+n}{n},$$

Entries of P_N are U(1)-invariant and so elements of $\mathcal{O}(\mathbb{C}\mathsf{P}^n_q)$

Proposition 6. *For all* $N \in \mathbb{N}$ *and for all* $0 \le k \le n$ *it holds that*

$$\langle [\mu_k], [P_{-N}] \rangle := \mathrm{Tr}_{\mathcal{H}_k}(\gamma_{(k)}(\pi^{(k)}(\mathrm{Tr}\, P_{-N})) = \binom{N}{k} \,,$$

$[\mu_0], \ldots, [\mu_n]$ are generators of $K^0(C(\mathbb{C}\mathrm{P}_q^n))$,

and $[P_0], \ldots, [P_{-n}]$ are generators of $K_0(\mathbb{C}\mathrm{P}_q^n)$

The matrix of couplings $M \in \mathrm{M}_{n+1}(\mathbb{Z})$ is invertible over $\mathbb{Z}$:

$M_{ij} := \big\langle [\mu_i], [P_{-j}] \big\rangle = \binom{j}{i}, \qquad (M^{-1})_{ij} = (-1)^{i+j}\binom{j}{i}.$

These are bases of $\mathbb{Z}^{n+1}$ as $\mathbb{Z}$-modules;

they generate $\mathbb{Z}^{n+1}$ as an Abelian group.

The inclusion $\mathcal{O}(\mathbb{C}\mathsf{P}^n_q) \hookrightarrow \mathcal{O}(\mathsf{S}^{2n+1}_q)$ is a U(1) q.p.b.

To a projection P_N there corresponds an associated line bundle

$$\mathcal{L}_N \simeq (\mathcal{O}(\mathbb{C}\mathsf{P}^n_q))^{d_N} P_N \simeq P_{-N}(\mathcal{O}(\mathbb{C}\mathsf{P}^n_q))^{d_N}$$

$\mathcal{L}_N$ made of elements of $\mathcal{O}(\mathsf{S}^{2n+1}_q)$ transforming under U(1) as

$$\varphi_N \mapsto \varphi_N \lambda^{-N}, \qquad \lambda \in \mathsf{U}(1)$$

Each $\mathcal{L}_N$ is indeed a bimodule over $\mathcal{L}_0 = \mathcal{O}(\mathbb{C}\mathsf{P}^n_q)$; – the bimodule of equivariant maps for the IRREP of U(1) with weight N. Also,

$$\mathcal{L}_N \otimes_{\mathcal{O}(\mathbb{C}\mathsf{P}^n_q)} \mathcal{L}_M \simeq \mathcal{L}_{N+M}$$

Denote $[P_N] = [\mathcal{L}_N]$ in the group $K_0(\mathbb{C}\mathrm{P}^n_q)$.

The module $\mathcal{L}_N$ is a line bundle, in the sense that its ‘rank’ (as computed by pairing with $[\mu_0]$) is equal to 1

Completely characterized by its ‘first Chern number’ (as computed by pairing with the class $[\mu_1]$):

Proposition 7. *For all $N \in \mathbb{Z}$ it holds that*

$$\langle [\mu_0], [\mathcal{L}_N] \rangle = 1 \quad \textit{and} \quad \langle [\mu_1], [\mathcal{L}_N] \rangle = -N\,.$$

The line bundle $\mathcal{L}_{-1}$ emerges as a central character: its only non-vanishing charges are

$$\langle [\mu_0], [\mathcal{L}_{-1}] \rangle = 1 \qquad\qquad \langle [\mu_1], [\mathcal{L}_{-1}] \rangle = 1$$

$\mathcal{L}_{-1}$ is the *tautological line bundle* for $\mathbb{C}\mathrm{P}^n_q$,

with Euler class

$$u = \chi([\mathcal{L}_{-1}]) := 1 - [\mathcal{L}_{-1}] \, .$$

Proposition 8. *It holds that*

$$K_0(\mathbb{C}\mathrm{P}^n_q) \simeq \mathbb{Z}[u]/u^{n+1} \simeq \mathbb{Z}^{n+1} \, .$$

$[\mu_k]$ and $(-u)^j$ are dual bases of K-homology and K-theory

The quantum lens spaces

Fix an integer $r \geq 2$ and define

$$\mathcal{O}(\mathsf{L}_q^{(n,r)}) := \bigoplus_{N \in \mathbb{Z}} \mathcal{L}_{rN} \,.$$

Proposition 9.
$\mathcal{O}(\mathsf{L}_q^{(n,r)})$ is a $*$-algebra; all elements of $\mathcal{O}(\mathsf{S}_q^{2n+1})$ invariant under the action $\alpha_r : \mathbb{Z}_r \to \mathrm{Aut}(\mathcal{O}(\mathsf{S}_q^{2n+1}))$ of the cyclic group $\mathbb{Z}_r$:

$$(z_0, z_1, \ldots, z_n) \mapsto (e^{2\pi \mathrm{i}/r} z_0, e^{2\pi \mathrm{i}/r} z_1, \ldots, e^{2\pi \mathrm{i}/r} z_n) \,.$$

The 'dual' $\mathsf{L}_q^{(n,r)}$:
the *quantum lens space* of dimension $2n+1$ (and index r)

There are algebra inclusions

$$j : \mathcal{O}(\mathbb{C}\mathsf{P}_q^n) \hookrightarrow \mathcal{O}(\mathsf{L}_q^{(n,r)}) \hookrightarrow \mathcal{O}(\mathsf{S}_q^{2n+1}) \,.$$

Pulling back line bundles

Proposition 10. *The algebra inclusion* $j : \mathcal{O}(\mathbb{C}\mathsf{P}^n_q) \hookrightarrow \mathcal{O}(\mathsf{L}^{(n,r)}_q)$ *is a quantum principal bundle with structure group* $\widetilde{\mathsf{U}}(1) := \mathsf{U}(1)/\mathbb{Z}_r$*:*

$$\mathcal{O}(\mathbb{C}\mathsf{P}^n_q) = \mathcal{O}(\mathsf{L}^{(n,r)}_q)^{\widetilde{\mathsf{U}}(1)}\,.$$

Then one can 'pull-back' line bundles from $\mathbb{C}\mathsf{P}^n_q$ to $\mathsf{L}^{(n,r)}_q$.

$$\begin{array}{ccc}
\widetilde{\mathcal{L}}_N & \xleftarrow{\quad j_* \quad} & \mathcal{L}_N \\
\vdots & & \vdots \\
\downarrow & & \downarrow \\
\mathcal{O}(\mathsf{L}^{(n,r)_q}) & \xleftarrow[j]{} & \mathcal{O}(\mathbb{C}\mathsf{P}^n_q)\,.
\end{array}$$

Definition 11. *For each* $\mathcal{L}_N$ *an* $\mathcal{O}(\mathbb{C}\mathsf{P}^n_q)$*-bimodule (a line bundle over* $\mathbb{C}\mathsf{P}^n_q$*), its 'pull-back' to* $\mathsf{L}_q^{(n,r)}$ *is the* $\mathcal{O}(\mathsf{L}_q^{(n,r)})$*-bimodule*

$$\widetilde{\mathcal{L}}_N = j_*(\mathcal{L}_N) := \mathcal{O}(\mathsf{L}_q^{(n,r)}) \otimes_{\mathcal{O}(\mathbb{C}\mathsf{P}^n_q)} \mathcal{L}_N .$$

The algebra inclusion $j : \mathcal{O}(\mathbb{C}\mathsf{P}^n_q) \to \mathcal{O}(\mathsf{L}_q^{(n,r)})$ induces a map

$$j_* : K_0(\mathbb{C}\mathsf{P}^n_q) \to K_0(\mathsf{L}_q^{(n,r)})$$

Each $\mathcal{L}_N$ over $\mathbb{C}\mathsf{P}^n_q$ is not free when $N \neq 0$,

this need not be the case for $\widetilde{\mathcal{L}}_N$ over $\mathsf{L}^{(n,r)}_q$:

the pull-back $\widetilde{\mathcal{L}}_{-r}$ of $\mathcal{L}_{-r}$ is tautologically free :

$$\widetilde{\mathcal{L}}_{-r} = \mathcal{O}(\mathsf{L}^{(n,r)}_q) \otimes_{\mathcal{L}_0} \mathcal{L}_{-r} \simeq \mathcal{O}(\mathsf{L}^{(n,r)}_q) = \widetilde{\mathcal{L}}_0 \,.$$

$\Rightarrow (\widetilde{\mathcal{L}}_{-N})^{\otimes r} \simeq \widetilde{\mathcal{L}}_{-rN}$ also has trivial class for any $N \in \mathbb{Z}$

$\widetilde{\mathcal{L}}_{-N}$ define *torsion classes*; they generate the group $K_0(\mathsf{L}^{(n,r)}_q)$

Multiplying by the Euler class

A second crucial ingredient

$$\alpha : K_0(\mathbb{C}\mathsf{P}^n_q) \to K_0(\mathbb{C}\mathsf{P}^n_q),$$

α is multiplication by $\quad \chi(\mathcal{L}_{-r}) := 1 - [\mathcal{L}_{-r}]$

the Euler class of the line bundle $\mathcal{L}_{-r}$

Assembly these into an exact sequence, the *Gysin sequence*

$$0 \to K_1(\mathsf{L}_q^{(n,r)}) \xrightarrow{\partial} K_0(\mathbb{C}\mathsf{P}_q^n) \xrightarrow{\alpha} K_0(\mathbb{C}\mathsf{P}_q^n) \longrightarrow K_0(\mathsf{L}_q^{(n,r)}) \longrightarrow 0$$

$$0 \to K_1(\mathsf{L}_q^{(n,r)}) \xrightarrow{\mathrm{Ind}_{\mathfrak{D}}} K_0(\mathbb{C}\mathsf{P}_q^n) \longrightarrow \;.....$$

and

$$..... \longrightarrow K_0(\mathsf{L}_q^{(n,r)}) \xrightarrow{\mathrm{Ind}_{\mathfrak{D}}} 0$$

$\mathrm{Ind}_{\mathfrak{D}}$ comes from Kasparov theory

Some practical and important applications, notably, the computation of the K-theory of the quantum lens spaces $\mathsf{L}_q^{(n,r)}$.

Thus

$$K_1(\mathsf{L}_q^{(n,r)}) \simeq \ker(\alpha), \qquad K_0(\mathsf{L}_q^{(n,r)}) \simeq \operatorname{coker}(\alpha)\,.$$

Moreover, *geometric* generators of the groups

$$K_1(\mathsf{L}_q^{(n,r)}) \qquad\qquad K_0(\mathsf{L}_q^{(n,r)})$$

for the latter as pulled-back line bundles from $\mathbb{C}\mathsf{P}_q^n$ to $\mathsf{L}_q^{(n,r)}$

Explicit generators as integral combinations of powers of the pull-back to the lens space $\mathsf{L}_q^{(n,r)}$ of the generator

$$u := 1 - [\mathcal{L}_{-1}]$$

The K-theory of quantum lens spaces

Proposition 12. *The* $(n+1) \times (n+1)$ *matrix* α *has rank* n*:*

$$K_1(C(\mathsf{L}_q^{(n,r)})) \simeq \mathbb{Z}\,.$$

On the other hand, the structure of the cokernel of the matrix A depends on the divisibility properties of the integer r.

This leads to

$$K_0(\mathsf{L}_q^{(n,r)}) = \mathbb{Z} \oplus \mathbb{Z}/\alpha_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/\alpha_n\mathbb{Z}\,.$$

for suitable integers $\alpha_1, \ldots, \alpha_n$.

Example 13. For $n = 1$

$$K_0(C(\mathsf{L}_q^{(1,r)})) = \mathbb{Z} \oplus \mathbb{Z}_r \,.$$

From definition $[\widetilde{\mathcal{L}}_{-r}] = 1$, thus $\widetilde{\mathcal{L}}_{-1}$ generates the torsion part.

Alternatively, from $u^2 = 0$ it follows that $\mathcal{L}_{-j} = -(j-1) + j\mathcal{L}_{-1}$; upon lifting to $\mathsf{L}_q^{(1,r)}$, for $j = r$ this yields

$$r(1 - [\widetilde{\mathcal{L}}_{-1}]) = 0$$

or $1 - [\widetilde{\mathcal{L}}_{-1}]$ is cyclic of order r.

Example 14. If $r = 2$ $\mathsf{L}_q^{(n,2)} = \mathsf{S}_q^{2n+1}/\mathbb{Z}_2 = \mathbb{R}P_q^{2n+1}$,
the quantum real projective space, we get

$$K_0(C(\mathbb{R}P_q^{2n+1})) = \mathbb{Z} \oplus \mathbb{Z}_{2^n}$$

the generator $1 - [\widetilde{\mathcal{L}}_{-1}]$ is cyclic with the correct order 2^n.

Example 15. For $n = 2$ there are two cases.

When $r = 2k + 1$:

$$r\,\tilde{u} = 0, \quad r\,\tilde{u}^2 = 0, \qquad K_0(\mathsf{L}_q^{(2,r)}) = \mathbb{Z} \oplus \mathbb{Z}_r \oplus \mathbb{Z}_r$$

When $r = 2k$:

$$\tfrac{1}{2}r\,(\tilde{u}^2 + 2\,\tilde{u}) = 0, \quad 2r\,\tilde{u} = 0, \quad K_0(C(\mathsf{L}_q^{(2,r)})) = \mathbb{Z} \oplus \mathbb{Z}_{\frac{r}{2}} \oplus \mathbb{Z}_{2r}$$

T-dual Pimsner algebras: a simple example

$$0 \to K_1(\mathsf{L}_q^{(1,r)}) \xrightarrow{\partial} K_0(\mathbb{C}\mathsf{P}_q^1) \xrightarrow{1-[\mathcal{L}_{-r}]} K_0(\mathbb{C}\mathsf{P}_q^1) \longrightarrow K_0(\mathsf{L}_q^{(1,r)}) \longrightarrow 0$$

$$\ker(1-[\mathcal{L}_{-r}]) =< u >=< 1-[\mathcal{L}_{-1}] >$$

$\Rightarrow$

$$K_1(\mathsf{L}_q^{(1,r)}) \ni h \mapsto \partial(h) = h(1-[\mathcal{L}_{-1}]) \simeq 1-[\mathcal{L}_{-h}]$$

and

$$(1-[\mathcal{L}_{-r}])(1-[\mathcal{L}_{-h}]) = 0 = (1-[\mathcal{L}_{-h}])(1-[\mathcal{L}_{-r}])$$

The exactness of the dual sequence for

$$0 \to K_1(\mathsf{L}_q^{(1,h)}) \xrightarrow{\partial} K_0(\mathbb{C}\mathsf{P}_q^1) \xrightarrow{1-[\mathcal{L}_{-h}]} K_0(\mathbb{C}\mathsf{P}_q^1) \longrightarrow K_0(\mathsf{L}_q^{(1,h)}) \longrightarrow 0$$

implies there exists a $r \in K_1(\mathsf{L}_q^{(1,r)})$ such that

$$K_1(\mathsf{L}_q^{(1,h)}) \ni r \mapsto \partial(r) = r(1 - [\mathcal{L}_{-1}]) \simeq 1 - [\mathcal{L}_{-r}]$$

The couples

$$\left(\mathsf{L}_q^{(1,r)}, h \in K_1(\mathsf{L}_q^{(1,r)})\right) \text{ and } \left(\mathsf{L}_q^{(1,h)}, r \in K_1(\mathsf{L}_q^{(1,h)})\right)$$

are ‘T-dual’

More generally : Quantum w. projective lines and lens spaces:

$B = \mathcal{O}(W_q(k,l))$ = quantum weighted projective line
the fixed point algebra for a weighted circle action on $\mathcal{O}(S_q^3)$

$$z_0 \mapsto \lambda^k z_0 \,, \quad z_1 \mapsto \lambda^l z_1 \,, \quad \lambda \in \mathrm{U}(1)$$

The corresponding universal enveloping C^*-algebra $C(W_q(k,l))$ does not in fact depend on the label k: isomorphic to the unitalization of l copies of $\mathcal{K}$ = compact operators on $l^2(\mathbb{N}_0)$

$$C(W_q(k,l)) = \widetilde{\oplus_{s=0}^{l} \mathcal{K}}$$

Then: $$K_0(C(W_q(k,l))) = \mathbb{Z}^{l+1} \,, \quad K_1(C(W_q(k,l))) = 0$$

a partial resolution of singularity, since classically

$$K_0(C(W(k,l))) = \mathbb{Z}^2 \,.$$

$\mathcal{O}_E = \mathcal{O}(L_q(lk;k,l))$ = quantum lens space

Indeed, a vector space decomposition

$$\mathcal{O}(L_q(lk;k,l)) = \oplus_{N\in\mathbb{Z}} \mathcal{L}_n(k,l)\,,$$

with $E = \mathcal{L}_1(k,l)$ a right finitely projective module

$$\mathcal{L}_1(k,l) := (z_1^*)^k \cdot \mathcal{O}(W_q(k,l)) + (z_0^*)^l \cdot \mathcal{O}(W_q(k,l))$$

Also, $\mathcal{O}(L_q(lk;k,l))$ the fixes point algebra of a cyclic action

$$\mathbb{Z}/(lk)\mathbb{Z} \times S_q^3 \to S_q^3$$

$$z_0 \mapsto \exp(\frac{2\pi \mathrm{i}}{l})\ z_0\,, \quad z_1 \mapsto \exp(\frac{2\pi \mathrm{i}}{k})\ z_1\,.$$

K-theory and K-homology of quantum lens space

Denote the diagonal inclusion by $\iota : \mathbb{Z} \to \mathbb{Z}^l$, $1 \mapsto (1, \ldots, 1)$ with transpose $\iota^t : \mathbb{Z}^l \to \mathbb{Z}$, $\iota^t(m_1, \ldots, m_l) = m_1 + \ldots + m_l$.

Proposition 16. *(Arici, Kaad, L.) With $k, l \in \mathbb{N}$ coprime:*

$$K_0\Big(L_q(lk; k, l)\Big) \simeq \mathrm{coker}(1 - E) \simeq \mathbb{Z} \oplus \Big(\mathbb{Z}^l / \mathrm{Im}(\iota)\Big)$$
$$K_1\Big(L_q(lk; k, l)\Big) \simeq \ker(1 - E) \simeq \mathbb{Z}^l$$

as well as

$$K^0\Big(L_q(lk; k, l)\Big) \simeq \ker(1 - E^t) \simeq \mathbb{Z} \oplus \Big(\ker(\iota^t)\Big)$$
$$K^1\Big(L_q(lk; k, l)\Big) \simeq \mathrm{coker}(1 - E^t) \simeq \mathbb{Z}^l \, .$$

Again there is no dependence on the label k.

‘grand motivations / applications’ :

Gauge fields on noncommutative spaces

T-duality for noncommutative spaces

Chern-Simons theory

A Gysin sequence for U(1)-bundles

relates H-flux (three-forms on the total space E) to line bundles (two-forms on the base space M) also giving an isomorphism between Dixmier-Douady classes on E and line bundles on M

Summing up:

many nice and elegant and useful geometry structures

hope you enjoyed it

Thank you !!