Hecke operators and K-homology of arithmetic groups

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Gauge Theory and Noncommutative Geometry, Nijmegen 2016

Geometry and number theory

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- $H^*(M, \mathbb{Z}) = H^*(\Gamma, \mathbb{Z})$
- $H_*(M, \mathbb{Z}) = H_*(\Gamma, \mathbb{Z})$

Hecke operators on (co)homology

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Hecke operator

$$\begin{array}{ccc} H^*(\Gamma, \mathbb{Z}) & \xrightarrow{T_g} & H^*(\Gamma, \mathbb{Z}) \\ \mathrm{res} \downarrow & & \uparrow \mathrm{cores} \\ H^*(\Gamma_g, \mathbb{Z}) & \xrightarrow{\mathrm{Ad} g} & H^*(\Gamma_{g^{-1}}, \mathbb{Z}) \end{array} \tag{1}$$

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$\text{cores}(c)(\gamma) = \sum_{i=1}^{d} c(t_i(\gamma))$ for 1-cocycles $c : \Gamma \to \mathbb{Z}$

Hecke operator $T_g(c)(\gamma) = \sum_{i=1}^{d} c(g^{-1} t_i(\gamma) g)$.

Geometric picture

For $g \in \mathrm{Comm}(\Gamma, G)$, set $M_g := \mathbb{H}/\Gamma_g$ and $M_{g^{-1}} := \mathbb{H}/\Gamma_{g^{-1}}$

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$$\begin{array}{ccc} \mathbb{H} & \xrightarrow{g} & \mathbb{H} \\ \downarrow & & \downarrow \\ M_g & \xrightarrow{g} & M_{g^{-1}} \\ {\scriptstyle \pi_g}\downarrow & & \downarrow{\scriptstyle \pi_{g^{-1}}} \\ M & & M \end{array} \tag{2}$$

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Hecke correspondence $M \xleftarrow{\pi_{g^{-1}} \circ g} M_g \xrightarrow{\pi_g} M$

$$T_g := (\pi_{g^{-1}} \circ g)^* \circ \pi_{g!} : H^*(M, \mathbb{Z}) \to H^*(M, \mathbb{Z})$$

Motivation

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- the manifold model $H^1(M, \mathbb{Z})$

There is a third picture in which modular forms appear as distributions on the boundary $\mathbb{P}^1(\mathbb{C}) = \partial\mathbb{H}^3$.

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C^*-algebras and K-homology

K-homology of C^*-algebras provides a unifying framework to study the Hecke module from all these perspectives at once:

There is an exact sequence of G-C^*-algebras

$$0 \to C_0(\mathbb{H}) \to C(\overline{\mathbb{H}}) \to C(\partial\mathbb{H}) \to 0,$$

inducing an exact sequence of the crossed products

$$0 \to C_0(\mathbb{H}) \rtimes \Gamma \to C(\overline{\mathbb{H}}) \rtimes \Gamma \to C(\partial\mathbb{H}) \rtimes \Gamma \to 0.$$

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we obtain the exact hexagon

$$\begin{array}{ccccc} K^0(C_0(M)) & \xrightarrow{\partial} & K^1(C(\partial\mathbb{H}) \rtimes \Gamma) & \longrightarrow & K^1(C_r^*(\Gamma)) \\ \uparrow & & & & \downarrow \\ K^0(C_r^*(\Gamma)) & \longleftarrow & K^0(C(\partial\mathbb{H}) \rtimes \Gamma) & \xleftarrow[\partial]{} & K^1(C_0(M)) \end{array} \qquad (3)$$

Hecke operators in KK-theory

Lemma

Let B be a Γ-C^*-algebra and $g \in \mathrm{Comm}(\Gamma, G)$. For $d := [\Gamma : \Gamma_{g^{-1}}]$, the right C^*-$B \rtimes_r \Gamma$-module $(B \rtimes_r \Gamma_g)^d$ admits a left $B \rtimes_r \Gamma$ action by compact operators.

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By using the embedding $(B \rtimes_r \Gamma_g)^d \to (B \rtimes_r \Gamma)^d$ we obtain a $B \rtimes_r \Gamma$-bimodule T_g^Γ and an element $[T_g^\Gamma] \in KK_0(B \rtimes_r \Gamma, B \rtimes_r \Gamma)$.

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Lemma

Let $g \in \mathrm{Comm}(\Gamma, G)$ and $M \xleftarrow{\pi_{g^{-1}} \circ g} M_g \xrightarrow{\pi_g} M$ the associated Hecke correspondence. The C^*-algebra $C_0(M_g)$ can be made into a C^*-$C_0(M)$-bimodule, whose left action is by compact operators.

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The above bimodule is denoted T_g^M and defines an element $[T_g^M] \in KK_0(C_0(M), C_0(M))$.

Morita compatibilty

Let ${}_{C_0(\mathbb{H})\rtimes\Gamma}L^2(\mathbb{H})_{C_0(M)}$ denote the natural $(C_0(\mathbb{H})\rtimes\Gamma, C_0(M))$ Morita equivalence bimodule.

Proposition

There is a unitary isomorphism of $(C_0(\mathbb{H})\rtimes\Gamma, C_0(M))$-bimodules

$$T_g^\Gamma \otimes_{C_0(\mathbb{H})\rtimes\Gamma} L^2(\mathbb{H})_{C_0(M)} \xrightarrow{\sim} L^2(\mathbb{H}) \otimes_{C_0(M)} T_g^M.$$

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In particular

$$[T_g^{\Gamma}]\otimes[L^2(\mathbb{H})_{C_0(M)}] = [L^2(\mathbb{H})_{C_0(M)}]\otimes[T_g^M]\in KK_0(C_0(\mathbb{H})\rtimes\Gamma, C_0(M))),$$

and the action of the Hecke operators is compatible with the isomorphism $K^*(C_0(\mathbb{H})\rtimes\Gamma)\xrightarrow{\sim} K^*(C_0(M))$.

The boundary map

In KK-theory, the boundary map $\partial : K^*(A) \to K^*(C)$ associated to an extension

$$0 \to A \to B \to C \to 0,$$

of C^*-algebras is given by the Kasparov product with a class $[\partial] \in KK_1(C, A)$ representing the extension.

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We describe this class for the G-equivariant extension

$$0 \to C_0(\mathbb{H}) \to C(\overline{\mathbb{H}}) \to C(\partial\mathbb{H}) \to 0.$$

Harmonic measures

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Write $T_1(\mathbb{H}) = \partial\mathbb{H} \times \mathbb{H}$ and define a $C_0(\mathbb{H})$-valued inner product on $C_c(T_1\mathbb{H})$ via

$$\langle \Phi, \Psi \rangle(x) := \int \overline{\Phi(\xi, x)}\Psi(\xi, x) d\nu_x(\xi),$$

and denote the C^*-module completion by $L^2(T_1(\mathbb{H}), \nu_x)_{C_0(\mathbb{H})}$. It is a left module over $C(\partial\mathbb{H})$.

The equivariant extension class

The expectation operator $P : L^2(T_1(\mathbb{H}, \nu_x)) \to L^2(T_1(\mathbb{H}, \nu_x))$ is defined through

$$P\Phi(\xi, x) := \int \Phi(\xi, x) d\nu_x \xi,$$

and defines a projection operator.

Proposition

The pair $(L^2(T_1(\mathbb{H}), \nu_x), 2P - 1)$ is a G-equivariant Kasparov module for $(C(\partial\mathbb{H}), C_0(\mathbb{H}))$ that represents the class of the equivairant extension

$$0 \to C_0(\mathbb{H}) \to C(\overline{\mathbb{H}}) \to C(\partial\mathbb{H}) \to 0,$$

in $KK_1^G(C(\partial\mathbb{H}), C_0(\mathbb{H}))$.

Boundary compatibilty

By Kasparov descent, we obtain the Kasparov module $(L^2(T_1\mathbb{H}) \rtimes \Gamma, \nu_x), 2P - 1)$ representing the extension

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in $KK_1(C(\partial\mathbb{H}) \rtimes \Gamma, C_0(\mathbb{H}) \rtimes \Gamma)$.

Proposition

There is a unitary isomorphism

$$T_g^\Gamma \otimes_{C(\partial\mathbb{H})\rtimes\Gamma} L^2(T_1\mathbb{H}) \rtimes \Gamma, \nu_x) \xrightarrow{\sim} L^2(T_1\mathbb{H}) \rtimes \Gamma, \nu_x) \otimes_{C_0(\mathbb{H})\rtimes\Gamma} T_g^\Gamma,$$

of $(C(\partial\mathbb{H}) \rtimes \Gamma, C_0(\mathbb{H}) \rtimes \Gamma)$-bimodules intertwining the operators $1 \otimes P$ and $P \otimes 1$.

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of $(C(\partial\mathbb{H}) \rtimes \Gamma, C_0(\mathbb{H}) \rtimes \Gamma)$*-bimodules intertwining the operators* $1 \otimes P$ *and* $P \otimes 1$.

In particular $[T_g^\Gamma] \otimes [\partial] = [\partial] \otimes [T_g^\Gamma] \in KK_1(C(\partial\mathbb{H}) \rtimes \Gamma, C_0(\mathbb{H}) \rtimes \Gamma)$.

The Gysin sequence

Theorem (Sengun-M.)

The exact sequence

$$\begin{array}{ccccc} K^0(C_0(M)) & \xrightarrow{\partial} & K^1(C(\partial\mathbb{H}) \rtimes \Gamma) & \longrightarrow & K^1(C^*_r(\Gamma)) \\ \uparrow & & & & \downarrow \\ K^0(C^*_r(\Gamma)) & \longleftarrow & K^0(C(\partial\mathbb{H}) \rtimes \Gamma) & \xleftarrow[\partial]{} & K^1(C_0(M)) \end{array} \tag{4}$$

is Hecke equivariant.

Proposition

Let Γ be a discrete torsion free noncocompact subgroup of $PSL_2(\mathbb{C})$. Then

- $K^1(C^*_r(\Gamma)) \cong H^1(\Gamma, \mathbb{Z})$
- $K^1(\mathbb{P}^1(\mathbb{C}) \rtimes \Gamma) \cong H^1(\Gamma, \mathbb{Z}^2)$
- $K^0(C_0(M)) \cong H_2(\overline{M}, \partial\overline{M}) \cong H^1(\Gamma, \mathbb{Z})$

These isomorphisms follow from an application of the Kasparov spectral sequence and work by M. Matthey. They are non-explicit.

Back to dimension 3

Proposition

Let Γ *be a discrete torsion free noncocompact subgroup of* $PSL_2(\mathbb{C})$*. Then*

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These isomorphisms follow from an application of the Kasparov spectral sequence and work by M. Matthey. They are non-explicit.

Here $\overline{M}$ denotes the Borel-Serre compactification. In general $\overline{M}$ is a manifold with corners.

Back to dimension 3

Theorem (Sengun-M.)

Let Γ be a discrete torsion free noncocompact subgroup of $PSL_2(\mathbb{C})$. There are explicit Hecke equivariant isomorphisms

$$H^1(\Gamma, \mathbb{Z}) \to K^1(C_r^*(\Gamma)), \quad H_2(\overline{M}, \partial\overline{M}) \to K^0(C_0(M)).$$

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Under these isomorphisms, the cohomology pairing $H_* \times H^* \to \mathbb{Z}$ corresponds to the index pairing $K_* \times K^* \to \mathbb{Z}$.

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Under these isomorphisms, the cohomology pairing $H_* \times H^* \to \mathbb{Z}$ corresponds to the index pairing $K_* \times K^* \to \mathbb{Z}$.

The isomorphisms are obtained by constructing explicit spectral triples associated to a cohomology class.

Further results

The K-homology hexagon simplifies to

$$0 \to K^0(C_0(M)) \to K^1(C(\partial\mathbb{H}) \rtimes \Gamma) \to K^1(C_r^*(\Gamma)) \to 0,$$

and the isomorphism $H^1(\Gamma, \mathbb{Z}) \to K^1(C_r^*(\Gamma))$ extends to a map $H^1(\Gamma, \mathbb{Z}) \to K^1(C(\partial\mathbb{H}) \rtimes \Gamma)$ compatible with the restriction map.

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By constructing an explicit unbounded representative of the extension class, we can explicitly compute the map $\partial : K^0(C_0(M)) \to K^1(C(\partial\mathbb{H}) \rtimes \Gamma)$ on the level of spectral triples.

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By constructing an explicit unbounded representative of the extension class, we can explicitly compute the map $\partial : K^0(C_0(M)) \to K^1(C(\partial\mathbb{H}) \rtimes \Gamma)$ on the level of spectral triples.

The unbounded extension class involves operators constructed from the inverse of the Riesz potential operator

$$I_\varepsilon \Psi(\xi, x) = \int \frac{\Psi(\eta, x)}{d_x(\xi, \eta)^{n-\varepsilon}} d\nu_x \eta.$$

The main theorem

Theorem (Sengun-M.)

There is an explicit Hecke equivariant isomorphism of exact sequences

$$\begin{array}{ccccc} K^0(C_0(M)) & \longrightarrow & K^1(C(\partial\mathbb{H}) \rtimes \Gamma) & \longrightarrow & K^1(C_r^*(\Gamma)) \\ \uparrow & & \uparrow & & \uparrow \\ H_2(\overline{M}, \partial\overline{M}) & \longrightarrow & H^1(\Gamma, \mathbb{Z}^2) & \longrightarrow & H^1(\Gamma, \mathbb{Z}) \end{array} \tag{5}$$

defined on the level of spectral triples.

Some words on the construction

A cocycle $c : \Gamma \to \mathbb{Z}$ gives an spectral triple:
$(C(\partial\mathbb{H}) \rtimes \Gamma, \ell^2(\mathbb{Z}) \otimes L^2(\mathbb{H}, \mathcal{S}), N \otimes 1 + 1 \otimes \gamma_{\mathbb{H}})$.

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Here N is the "number operator" and $\gamma_{\mathbb{H}}$ is the "gamma-element" obtained from Kasparov's Dirac-dual-Dirac construction.

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