The fermionic action

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ma per seguir virtute e canoscenza

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Outline

The fermionic action

- Consider a real even spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D}, J, \gamma)$ of KO-dimension 2. The *fermionic action* is defined as [Con06]

$$S_f := \frac{1}{2}\langle J\widetilde{\xi} \mid \mathcal{D}\widetilde{\xi}\rangle,$$

where $\widetilde{\xi}$ is a Grassmann variable corresponding to $\xi = \gamma\xi \in \mathcal{H}^0$.

- Two discrepancies:
  - signature is Riemannian instead of Lorentzian;
  - the definition involves the real structure ('charge conjugation').

- Solution [Bar07]: consider an action functional of the form $\langle\psi|\mathcal{D}\psi\rangle$, where $\langle\cdot|\cdot\rangle$ denotes the indefinite inner product on a Krein space, and where $\mathcal{D}$ is Krein-self-adjoint.

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Krein spaces

- A Krein space is a vector space $\mathcal{H}$ with a non-degenerate inner product $\langle\cdot|\cdot\rangle$ which admits a *fundamental decomposition* $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$ (i.e., an orthogonal direct sum decomposition into a positive-definite subspace $\mathcal{H}^+$ and a negative-definite subspace $\mathcal{H}^-$) such that the subspaces $\mathcal{H}^+$ and $\mathcal{H}^-$ are *intrinsically complete* (i.e., complete with respect to the norms $\|\psi\|_{\mathcal{H}^\pm} := |\langle\psi|\psi\rangle|^{1/2}$).
- Given a fundamental decomposition $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$, we obtain a corresponding fundamental symmetry $\mathcal{J} = P^+ - P^-$, where $P^\pm$ denotes the projection onto $\mathcal{H}^\pm$.
- Given a fundamental symmetry $\mathcal{J}$, we denote by $\mathcal{H}_\mathcal{J}$ the corresponding Hilbert space for the *positive-definite* inner product $\langle\cdot|\cdot\rangle_\mathcal{J} := \langle\mathcal{J}\cdot|\cdot\rangle$.
- A Krein space $\mathcal{H}$ with fundamental symmetry $\mathcal{J}$ is called $\mathbb{Z}_2$*-graded* if $\mathcal{H}_\mathcal{J}$ is $\mathbb{Z}_2$-graded and $\mathcal{J}$ is homogeneous. This means:
  - we have a decomposition $\mathcal{H}^0 \oplus \mathcal{H}^1$;
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Lorentzian manifolds

- Let (M,g) be an n-dimensional space- and time-oriented Lorentzian spin manifold with an orthogonal direct sum decomposition of the tangent bundle $TM = E_t \oplus E_s$ with $\dim E_t = 1$ (with basis vector e_0) and $\dim E_s = n-1$ (with basis vectors $e_1, \ldots, e_{n-1}$) such that the metric g is negative-definite on E_t and positive-definite on E_s.

- We have a *timelike projection* $T\colon TM \to E_t$ and a *spacelike reflection* $r = 1 - 2T = (-1) \oplus 1$ on $TM = E_t \oplus E_s$.

- We can define a 'Wick rotated' metric g_r on M by setting

$$g_r(v,w) := g(rv,w).$$

Then (M,g_r) is a Riemannian manifold.

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Lorentzian spinors

- Given a decomposition $TM = \mathrm{E}_t \oplus \mathrm{E}_s$ there exists a positive-definite hermitian structure [Baum81]

$$(\cdot|\cdot)_{\mathcal{J}_M} : \Gamma_c^\infty(\mathrm{S}) \times \Gamma_c^\infty(\mathrm{S}) \to C_c^\infty(M).$$

  which gives rise to the inner product $\langle\cdot|\cdot\rangle_{\mathcal{J}_M} := \int_M (\cdot|\cdot)_{\mathcal{J}_M} \mathrm{dvol}_g$. The completion of $\Gamma_c^\infty(\mathrm{S})$ with respect to this inner product is denoted $L^2(\mathrm{S})$.

- The operator $\mathcal{J}_M := \gamma(e_0)$ on $L^2(\mathrm{S})$ is self-adjoint and unitary, and is related to the spacelike reflection r via $\mathcal{J}_M \gamma(v) \mathcal{J}_M = -\gamma(rv)$. Then $L^2(\mathrm{S})$ is a Krein space with the indefinite inner product $\langle\cdot|\cdot\rangle := \langle \mathcal{J}_M \cdot |\cdot\rangle_{\mathcal{J}_M}$ and with fundamental symmetry $\mathcal{J}_M$. This indefinite inner product $\langle\cdot|\cdot\rangle$ is independent of the choice of decomposition $TM = \mathrm{E}_t \oplus \mathrm{E}_s$.

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The Dirac operator

- Define the Lorentzian Dirac operator

$$\not{D} := \sum_{j=0}^{n-1} \kappa(j)\gamma(e_j)\nabla^S_{e_j},$$

where ∇^S is the lift of the Levi-Civita connection corresponding to g, and $\kappa(0) = -1$ and $\kappa(j) = 1$ for $j = 1, \ldots, n-1$.

- **Theorem [Baum81]:** Suppose there exists a decomposition $TM = E_t \oplus E_s$ such that g_r is complete. Then $i\not{D}$ is *essentially Krein-self-adjoint*.

- We are going to consider the data

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Spectral triples

Definition: An even spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ consists of

- a $\mathbb{Z}_2$-graded Hilbert space $\mathcal{H}$;
- an even $*$-algebra representation $\pi \colon \mathcal{A} \to B^0(\mathcal{H})$;
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- a closed, odd operator $\mathcal{D} \colon \operatorname{Dom} \mathcal{D} \to \mathcal{H}$ such that:
  1. the linear subspace $\mathcal{E} := \operatorname{Dom} \mathcal{D}$ is dense in $\mathcal{H}$;
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  3. the commutator $[\mathcal{D}, \pi(a)]$ is bounded on $\mathcal{E}$ for each $a \in \mathcal{A}$;
  4. the map $\pi(a) \circ \iota \colon \mathcal{E} \hookrightarrow \mathcal{H} \to \mathcal{H}$ is compact for each $a \in \mathcal{A}$.

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Note: $\mathcal{E}$ is equipped with the norm $\|\psi\|_{\mathcal{E}} := \|\psi\| + \|\mathcal{J}\mathcal{D}\psi\| + \|\mathcal{D}\mathcal{J}\psi\|$.

We say an even Krein spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D}, \mathcal{J})$ is of Lorentz-type when $\mathcal{J}$ is *odd*.

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Almost-commutative manifolds

- Let (M, g) be an even-dimensional time- and space-oriented Lorentzian spin manifold. Suppose there exists a spacelike reflection r such that g_r is complete. Then

$$\big(C_c^\infty(M), L^2(\mathbb{S}), i\not{D}, \mathcal{J}_M = \gamma(e_0)\big)$$

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- A finite space F is an even Krein spectral triple $(\mathcal{A}_F, \mathcal{H}_F, \mathcal{D}_F, \mathcal{J}_F)$ such that $\dim \mathcal{H}_F < \infty$ and $\mathcal{J}_F$ is even.

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- We define the Krein action $S_{\mathcal{K}}\colon \mathcal{H}^0 \cap \operatorname{Dom}\mathcal{D} \to \mathbb{C}$ to be the functional

$$S_{\mathcal{K}}[\psi] := \langle\psi|\mathcal{D}\psi\rangle.$$

We note that $S_{\mathcal{K}}[\psi]$ is real-valued and (in general) non-zero.

- **Remark:** this action is *classical*. In particular, there are no Grassmann variables.

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The perturbation semi-group

- Let $\mathcal{A}$ be a unital $*$-algebra. Let $A = \sum_j a_j \otimes b_j^{\mathrm{op}} \in \mathcal{A} \odot \mathcal{A}^{\mathrm{op}}$. Define $\overline{A} := \sum b_j^* \otimes a_j^{*\mathrm{op}}$.
  - A is *real* if $\overline{A} = A$.
  - A is *normalised* if $\sum a_j b_j = 1 \in \mathcal{A}$.
- **Definition [CCvS13]:** The *perturbation semi-group* $\mathrm{Pert}(\mathcal{A})$ consists of the real normalised elements in $\mathcal{A} \odot \mathcal{A}^{\mathrm{op}}$.
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$$\eta_{\mathcal{D}}\Big(\sum_j a_j \otimes b_j^{\mathrm{op}}\Big) := \sum_{j,k} \big(a_j (a_k^*)^{\mathrm{op}}\big) \big[\mathcal{D}, b_j (b_k^*)^{\mathrm{op}}\big].$$

Fact: if $A \in \mathrm{Pert}(\mathcal{A})$ is real, then $\eta_{\mathcal{D}}(A)$ is Krein-self-adjoint.

The perturbation semi-group

- Let $\mathcal{A}$ be a unital $*$-algebra. Let $A = \sum_j a_j \otimes b_j^{\mathrm{op}} \in \mathcal{A} \odot \mathcal{A}^{\mathrm{op}}$. Define $\overline{A} := \sum b_j^* \otimes a_j^{*\mathrm{op}}$.
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Fluctuations

- If $(\mathcal{A} \odot \mathcal{A}^{\mathrm{op}}, \mathcal{H}, \mathcal{D}, \mathcal{J})$ satisfies the *order-one condition*

$$\big[a, [\mathcal{D}, b^{\mathrm{op}}]\big] = 0 \qquad \forall a, b \in \mathcal{A},$$

then

$$\eta_{\mathcal{D}}\Big(\sum_j a_j \otimes b_j^{\mathrm{op}}\Big) = \sum_j a_j[\mathcal{D}, b_j] + \sum_j a_j^{*\mathrm{op}}[\mathcal{D}, b_j^{*\mathrm{op}}].$$

- By the *fluctuation* of $\mathcal{D}$ by $A \in \mathrm{Pert}(\mathcal{A})$ we mean the map

$$\mathcal{D} \mapsto \mathcal{D}_A := \mathcal{D} + \eta_{\mathcal{D}}(A),$$

and we refer to $\mathcal{D}_A$ as the *fluctuated Dirac operator*.

- **Proposition [CCvS13]:** A fluctuation of a fluctuated Dirac operator is again a fluctuated Dirac operator. To be precise: $(\mathcal{D}_A)_{A'} = \mathcal{D}_{A'A}$ for all perturbations $A, A' \in \mathrm{Pert}(\mathcal{A})$.

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The gauge group

- The unitary group $\mathcal{U}(\mathcal{A})$ acts on $\mathrm{Pert}(\mathcal{A})$ via

$$\Delta(u)A := \big(u \otimes (u^*)^{\mathsf{op}}\big)\big(\sum a_j \otimes b_j^{\mathsf{op}}\big) = \sum u a_j \otimes (b_j u^*)^{\mathsf{op}}.$$

  We can compose Δ with the $*$-algebra representation $\pi\colon \mathcal{A} \odot \mathcal{A}^{\mathsf{op}} \to \mathcal{B}(\mathcal{H})$ to obtain a group representation

$$\rho := \pi \circ \Delta\colon \mathcal{U}(\mathcal{A}) \to \mathcal{B}(\mathcal{H}).$$

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$$\mathcal{G}(\mathcal{A}) := \{\rho(u) \mid u \in \mathcal{U}(\mathcal{A})\} \simeq \mathcal{U}(\mathcal{A}) / \operatorname{Ker} \rho.$$

- **Proposition:** The Krein action $S_{\mathcal{K}}[\psi, A] := \langle \psi | \mathcal{D}_A \psi \rangle$ of the fluctuated Dirac operator $\mathcal{D}_A$ is invariant under the action of the gauge group given by $\psi \mapsto \rho(u)\psi$ and $A \mapsto \Delta(u)A$.

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The finite space (1)

- Define $\mathcal{H}_F := \mathbb{C}^4$ with the basis $\{\nu_R, e_R, \nu_L, e_L\}$ and the $\mathbb{Z}_2$-grading

$$\mathcal{H}_F^0 = \mathcal{H}_L = \text{span}\{\nu_L, e_L\}, \qquad \mathcal{H}_F^1 = \mathcal{H}_R = \text{span}\{\nu_R, e_R\}$$

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$$\pi(\lambda, q) := q_\lambda \oplus q := \begin{pmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{pmatrix} \oplus \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}, \quad \pi^{\text{op}}((\lambda, q)^{\text{op}}) := \lambda \oplus \lambda.$$

- The representation $\tilde{\pi} := \pi \otimes \pi^{\text{op}}$ of $\mathcal{A}_F \odot \mathcal{A}_F^{\text{op}}$ on $\mathcal{H}_R \oplus \mathcal{H}_L$ is then given by

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The finite space (2)

- We define the *mass matrix* on the basis $\{\nu_R, e_R, \nu_L, e_L\}$ as

$$\mathcal{D}_F := \begin{pmatrix} 0 & 0 & -im_\nu & 0 \\ 0 & 0 & 0 & -im_e \\ im_\nu & 0 & 0 & 0 \\ 0 & im_e & 0 & 0 \end{pmatrix}.$$

- We then consider the even finite space $F_{EW} := (\mathcal{A}_F, \mathcal{H}_F, \mathcal{D}_F, \mathcal{J}_F = 1)$.
- The gauge group of F_{EW} equals

$$\mathcal{G}(F_{EW}) = \big(U(1) \times SU(2)\big)/\mathbb{Z}_2.$$

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Fluctuations

- We consider the almost-commutative manifold

$$F_{EW} \times M := \Big(C_c^\infty(M, \mathcal{A}_F \odot \mathcal{A}_F^{\mathrm{op}}), \mathcal{H}_F \hat{\otimes} L^2(\mathbb{S}), 1 \hat{\otimes} i\not{D} + i\mathcal{D}_F \hat{\otimes} 1, 1 \hat{\otimes} \mathcal{J}_M \Big) .$$

- **Proposition:** The fluctuation of $\mathcal{D} := 1 \hat{\otimes} i\not{D} + i\mathcal{D}_F \hat{\otimes} 1$ by $A \in \mathrm{Pert}(C_c^\infty(M, \mathcal{A}_F))$ is

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where the *gauge field* A_μ and the *Higgs field* ϕ are given by

$$A_\mu = \begin{pmatrix} 0 & 0 & & \\ 0 & -2\Lambda_\mu & & \\ & & \multicolumn{2}{c}{Q_\mu - \Lambda_\mu} \end{pmatrix}, \quad \phi = \begin{pmatrix} 0 & 0 & m_\nu \overline{\phi_1} & m_\nu \overline{\phi_2} \\ 0 & 0 & -m_e \phi_2 & m_e \phi_1 \\ -m_\nu \phi_1 & m_e \overline{\phi_2} & 0 & 0 \\ -m_\nu \phi_2 & -m_e \overline{\phi_1} & 0 & 0 \end{pmatrix},$$

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The Krein action

- Consider $\xi = \nu_R \hat{\otimes} \psi_R^\nu + e_R \hat{\otimes} \psi_R^e + \nu_L \hat{\otimes} \psi_L^\nu + e_L \hat{\otimes} \psi_L^e \in \mathcal{H}^0$, and define

$$\Psi_L := \begin{pmatrix} \psi_L^\nu \\ \psi_L^e \end{pmatrix} \in L^2(\mathcal{S})^0 \otimes \mathbb{C}^2, \qquad \Psi_R := \begin{pmatrix} \psi_R^\nu \\ \psi_R^e \end{pmatrix} \in L^2(\mathcal{S})^1 \otimes \mathbb{C}^2,$$

$$\Psi := \Psi_L + \Psi_R \in L^2(\mathcal{S}) \otimes \mathbb{C}^2.$$

- **Proposition:** The Krein action for $F_{EW} \times M$ is given by

$$S_{EW}[\Psi, A] = \langle \Psi \mid i\not{D}\Psi \rangle + \langle \psi_R^e \mid -2i\gamma^\mu \Lambda_\mu \psi_R^e \rangle + \langle \Psi_L \mid i\gamma^\mu (Q_\mu - \Lambda_\mu)\Psi_L \rangle + \langle \Psi_R \mid \Phi \Psi_L \rangle + \langle \Psi_L \mid \Phi^* \Psi_R \rangle,$$

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Majorana masses

- On $\hat{\mathcal{H}}_F := \mathcal{H}_F \oplus \mathcal{H}_{\overline{F}}$, we consider the operators

$$\hat{\mathcal{D}}_F := \begin{pmatrix} \mathcal{D}_F & -\mathcal{D}_M^* \\ \mathcal{D}_M & \overline{\mathcal{D}_F} \end{pmatrix}, \qquad \hat{\mathcal{J}}_F := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$\hat{\Gamma}_F := \begin{pmatrix} \Gamma_F & 0 \\ 0 & -\Gamma_F \end{pmatrix}, \qquad \hat{J}_F := \begin{pmatrix} 0 & \text{c. c.} \\ \text{c. c.} & 0 \end{pmatrix},$$

where $\mathcal{D}_M \nu_R := i m_R \overline{\nu_R}$ and $\mathcal{D}_M e_R = \mathcal{D}_M \nu_L = \mathcal{D}_M e_L = 0$.

- Define $\hat{\pi} \colon \mathcal{A}_F \to \mathcal{B}(\mathcal{H}_F \oplus \mathcal{H}_{\overline{F}})$ and $\hat{\pi}^{\mathrm{op}} \colon \mathcal{A}_F^{\mathrm{op}} \to \mathcal{B}(\mathcal{H}_F \oplus \mathcal{H}_{\overline{F}})$ by

$$\hat{\pi}(a) := \pi(a) \oplus \pi^{\mathrm{op}}(a^t), \qquad \hat{\pi}^{\mathrm{op}}(a) := \hat{J}_F \hat{\pi}(a^*) \hat{J}_F.$$

- We obtain a new finite space $\hat{F}_{EW} := (\mathcal{A}_F \odot \mathcal{A}_F^{\mathrm{op}}, \hat{\mathcal{H}}_F, \hat{\mathcal{D}}_F, \hat{\mathcal{J}}_F)$ with grading operator $\hat{\Gamma}_F$ and additionally with a real structure $\hat{J}_F$.

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$$\hat{\pi}(a) := \pi(a) \oplus \pi^{\text{op}}(a^t), \qquad \hat{\pi}^{\text{op}}(a) := \hat{J}_F \hat{\pi}(a^*) \hat{J}_F.$$

- We obtain a new finite space $\hat{F}_{EW} := (\mathcal{A}_F \odot \mathcal{A}_F^{\text{op}}, \hat{\mathcal{H}}_F, \hat{\mathcal{D}}_F, \hat{\mathcal{J}}_F)$ with grading operator $\hat{\Gamma}_F$ and additionally with a real structure $\hat{J}_F$.

Majorana masses

- On $\hat{\mathcal{H}}_F := \mathcal{H}_F \oplus \mathcal{H}_{\overline{F}}$, we consider the operators

$$\hat{\mathcal{D}}_F := \begin{pmatrix} \mathcal{D}_F & -\mathcal{D}_M^* \\ \mathcal{D}_M & \overline{\mathcal{D}_F} \end{pmatrix}, \qquad \hat{\mathcal{J}}_F := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$\hat{\Gamma}_F := \begin{pmatrix} \Gamma_F & 0 \\ 0 & -\Gamma_F \end{pmatrix}, \qquad \hat{J}_F := \begin{pmatrix} 0 & \mathrm{c.\,c.} \\ \mathrm{c.\,c.} & 0 \end{pmatrix},$$

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The Krein action + Majorana masses

- Consider $\hat{F}_{EW} \times M$ with the real structure $J := \hat{J}_F \hat{\otimes} J_M$.
- Following [Bar07], consider $\eta \in \mathcal{H}^0$ such that $J\eta = \eta$. Then $\eta = \xi + J\xi$, with $\xi \in (\mathcal{H}_F \hat{\otimes} L^2(S))^0$ as before. We have

$$\langle \eta \,|\, \mathcal{D}_A \eta \rangle = \langle \xi \,|\, \mathcal{D}_A \xi \rangle + \langle J\xi \,|\, \mathcal{D}_A \xi \rangle + \langle \xi \,|\, \mathcal{D}_A J\xi \rangle + \langle J\xi \,|\, \mathcal{D}_A J\xi \rangle.$$

- One finds that $\langle J\xi \,|\, \mathcal{D}_A J\xi \rangle = \langle \xi \,|\, \mathcal{D}_A \xi \rangle = S_{EW}[\Psi, A]$.
- The new contributions are

$$\langle J\xi | \mathcal{D}_A \xi \rangle = -m_R \langle J_M \psi_R^\nu | \psi_R^\nu \rangle, \qquad \langle \xi | \mathcal{D}_A J\xi \rangle = -m_R \langle \psi_R^\nu | J_M \psi_R^\nu \rangle.$$

- Summarising, we obtain the new action S_{EW+M} given by

$$S_{EW+M}[\Psi, A] = 2S_{EW}[\Psi, A] - m_R \langle \psi_R^\nu | J_M \psi_R^\nu \rangle - m_R \langle J_M \psi_R^\nu | \psi_R^\nu \rangle,$$

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Conclusion

- The fermionic action in Lorentzian signature (the Krein action) matches *exactly* with the physical Lagrangian.
- The action is purely classical; there are no anti-commuting variables.
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