Masterclass ESI 2018 Program
“Bivariant K-theory in geometry and physics”

Chris Bourne – Topological phases, index theory and Kasparov theory

Our aim is to show how the mathematics of topological states of matter fits within the framework of (bivariant) K-theory.

We first review of the K-theoretic classification of gapped free-fermionic Hamiltonians, including systems with including systems with crystallographic symmetry. We then outline how the physical invariants of interest can be linked to numerical phase labels that arise as pairings of these K-theory classes with elements in cyclic cohomology, K-homology and KK-theory. In particular, we show how these pairings extend to random Hamiltonians whose spectrum contains regions of almost surely dense pure point spectrum (dynamical localisation).

We also explain how the bulk-boundary correspondence of topological phases naturally fits within our K-theoretic framework.

Time permitting, we will discuss current limitations of our method and future directions in the mathematics and physics of topological phases (crytallographic bulk-boundary correspondence, interacting systems etc.).

Koen van den Dungen – Towards noncommutative Lorentzian geometry

One can think of spectral triples as a noncommutative generalisation of Riemannian manifolds. For applications in physics, one would also like to study noncommutative Lorentzian manifolds (i.e. spacetimes instead of spaces). In these lectures I will discuss several possible approaches to adapt the framework of noncommutative geometry and unbounded KK-theory in order to allow for the study of noncommutative spacetimes.

Jens Kaad – Recent advances in (unbounded) KK-theory

It is perhaps the deepest structural result in Kasparov's bivariant K-theory that these abelian groups admit a bivariant and associative pairing known as the interior Kasparov product. Stated a bit boldly one could even say that the existence of the interior Kasparov product is the main reason for studying KK-theory at all. Nonetheless, given two concrete classes in KK-theory it can be difficult to find a concrete representative for their interior Kasparov product. The idea behind unbounded KK-theory is to approach Kasparov's bivariant K-theory entirely from the unbounded point of view, thus where a cycle in the theory is determined by an unbounded operator instead of by a bounded operator. This approach is highly motivated by noncommutative differential geometry, where the prototypical example of a spectral triple is given by the Dirac operator of a spin manifold. In this unbounded setting, one also has a good candidate for the Kasparov product: given two unbounded operators D_1 and D_2 their unbounded Kasparov product can often be written as an explicit tensor sum:

$$D_1 \otimes 1 + 1 \otimes_\nabla D_2,$$

where ∇ is a hermitian connection used to make sense of the right leg in this formula. Another important feature of unbounded KK-theory is that it provides a machinery for constructing explicit new geometries out of existing geometric building blocks.

In these lectures I will give a gentle introduction to unbounded KK-theory roughly following the plan here below:

Lecture 1: I will go through the theory of unbounded operators on Hilbert C*-modules and present examples of unbounded KK-cycles coming from first order differential operators, smooth vector bundles and circle actions.

Lecture 2: I will present the unbounded Kasparov product of unbounded KK-cycles in a non-technical way. This includes an introduction to hermitian connections in the context of Hilbert C*-modules, in particular we shall look at Grassmann-connections.

Lecture 3: I will give concrete examples of the unbounded Kasparov product: twisting by vector bundles, twisting by finitely generated projective modules, and spectral triples over the noncommutative torus. If time permits, I will also give a proof of Bott-periodicity using techniques from unbounded KK-theory: in the end this amounts to computing the index of the harmonic oscillator $d/dt + t$.

Bram Mesland and Adam Rennie – Introduction to operator algebras and KK-theory

The aim of this course is to introduce the main basic notions to work with Kasparov's KK-theory. Our focus will be on technical corner stones that can be used in practice, ie in explicit calculations. Moreover the material introduced in these lectures serves as background for the other courses in the masterclass. We will discuss the following topics

- C*-algebras, Hilbert C*-modules and discuss their properties. The relation between the Kasparov stabilisation theorem and frames in Hilbert C*-modules.
- Fredholm operators, extensions of C*-algebras and the Kasparov-Stinespring theorem. In examples the passage of an extension to a Fredholm module can often be carried out explicitly.
- Two different pictures of the Kasparov KK-groups, Fredholm modules and extensions. Explicit passage from KK-theory to K-theory and K-homology, exact sequences and the index map. The main properties of the Kasparov product and its difficulties.

Emil Prodan – Physics and applications of topological phases: How KK-theory can help?

These seminars are driven by the following questions:

1. What are the advantages of C*-algebraic formulations of physical models?
2. What K-theory has to offer to physicists and engineers?
3. Why Index Theory?
4. How is KK-theory opening new directions in materials science?

My plan is to go through a list of interesting natural and synthetic materials (many already implemented and characterized in laboratories), whose C*-algebras can be explicitly computed. With these examples, I will cover C*-algebras such as the non-commutative tori, generic crossed product, graph and grupoid algebras. For several cases, I will show through explicit numerical computations how the structure of the spectrum can be read-off from the K-theory of the underlying C*-algebra and how various pairings provide bulk topological invariants with well defined physical meaning. I will demonstrate the bulk-boundary principle for several examples. I will then discuss the regime of strong disorder where the Fermi energy resides inside the essential spectrum and show how Index Theory becomes useful in such situations. The last part will be devoted entirely on KK-theory, and here I will cover applications which are well understood at this time as well as some speculative proposals on how we can move forward the program on interacting topological phases.