A Counterfactual History of the Hypoelliptic Laplacian

Nigel Higson

Department of Mathematics
Pennsylvania State University

Nijmegen NCG Seminar
October 6, 2020

First Introduction—Some Questions

The name *hypoelliptic Laplacian* is Jean-Michel Bismut's term for an operator constructed by him for use in carrying out very striking computations in spectral geometry.

I'll try to answer some of the obvious questions:

- *What is it?*
- *What does it do?*
- *Where does it come from?*
- *How does it do what it does?*

Hypoelliptic Laplacian on the Circle

There is a version of the hypoelliptic Laplacian for every (real reductive) Lie group. I'll mostly consider compact groups in this talk. In fact I'll mostly focus on the circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$.

Here is the answer to the first question in the case of the circle:

$$\mathsf{L}_b = \begin{bmatrix} \frac{1}{2b^2}(y^2 - \partial_y^2 - 1) + \frac{1}{b} y\partial_x & 0 \\ 0 & \frac{1}{2b^2}(y^2 - \partial_y^2 + 1) + \frac{1}{b} y\partial_x \end{bmatrix}$$

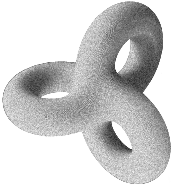
This operator acts on $\mathbb{T}\times\mathbb{R}$, with x coordinatizing the circle and y coordinatizing the line (which is the Lie algebra of $\mathbb{T}$).

As for b, it is a positive parameter, so actually the hypoelliptic Laplacian is a family of operators.

It is also evident from the formula what the hypoelliptic Laplacian is *not*: it is not elliptic and it is not even formally self-adjoint.

Selberg Trace Formula

As for what L_b *does* (in general, and not just for $\mathbb{T}$) it is designed to prove identities such as the Selberg trace formula.



This relates the eigenvalues λ of the Laplacian on a (closed) hyperbolic surface S, to the lengths and primitive lengths of the closed geodesics γ on S:

$$\sum_{\lambda} e^{-t\lambda} = \frac{\mathrm{Area}(S)}{4\pi t} \cdot \frac{e^{-t/4}}{\sqrt{4\pi t}} \int_0^{\infty} \frac{x e^{-x^2/4t}}{\sinh(x/2)}\, dx + \frac{e^{-t/4}}{\sqrt{4\pi t}} \sum_{\gamma} \frac{\ell_0(\gamma)/2}{\sinh\left(\ell(\gamma)/2\right)} e^{-\ell(\gamma)^2/4t}$$

Poisson Summation Formula

Returning to the circle (of circumference c), there is a counterpart of the Selberg formula:

$$\sum_{k\in\mathbb{Z}} e^{-4\pi^2 k^2 t/c^2} = \frac{c}{\sqrt{4\pi t}} \sum_{n\in\mathbb{Z}} e^{-n^2 c^2/t}$$

It's much simpler. But you can see both the eigenvalues of the Laplace operator and the lengths of the closed geodesics.

Of course, the formula is a special case of the Poisson summation formula, among other things. So easy harmonic analysis applies.

I shall explain how Bismut's method leads to a proof of the formula …but you'll see that the effort involved is considerable.

However, as the dimension increases, so can the complexity of the harmonic analysis, while the difficulty of Bismut's method remains more or less unchanged.

The Hypoelliptic Laplacian and Orbital Integrals

To return briefly to the Selberg formula, if S is a hyperbolic surface, then $S \cong \Gamma \backslash SL(2,\mathbb{R})/SO(2)$ where $\pi_1(S) \cong \Gamma$, and

$$\exp(-t\Delta_S)(p,p) = \sum_{\gamma\in\Gamma} \exp(-t\Delta_H)(P,\gamma P)$$

where $H = SL(2,\mathbb{R})/SO(2)$. It follows that

$$\mathrm{Tr}(\exp(-t\Delta_S)) = \sum_{\langle\gamma\rangle} \mathrm{vol}\big(Z_\Gamma(\gamma)\big\backslash Z_G(\gamma)\big) \cdot \mathrm{Tr}^{\langle\gamma\rangle}\big(\exp(-t\Delta_H/2)\big)$$

The sum is over representatives of conjugacy classes in Γ, and

$$\mathrm{Tr}^{\langle\gamma\rangle}\big(\exp(-t\Delta_H/2)\big) = \int_{Z_G(\gamma)\backslash G} \exp(-t\Delta_H)(gP,\gamma gP)\, dg.$$

Bismut uses L_b for $SL(2,\mathbb{R})$ to evaluate these orbital integrals.

Aside: Heat Kernels, Weyl's Law and McKean-Singer

I've already used the heat kernel $\exp(-t\Delta_S)(p,q)$ above. It is the integral kernel representing the heat operator $\exp(-t\Delta_S)$ arising from the heat equation:

$$\partial_t u_t + \Delta_S u_t = 0$$

Since the 1950's, the preferred method of attack in spectral geometry has been via the heat equation, not via resolvents, à la Weyl.

The first step is the formula

$$\exp(-t\Delta_S)(p,p) = \frac{1}{4\pi t} + \mathcal{O}(1)$$

as $t \to 0$, which already implies Weyl's asymptotic law.

This is the beginning of an asymptotic expansion that continues

$$\exp(-t\Delta_S)(p,p) = \frac{1}{4\pi t} + \frac{K(p)}{12\pi} + \mathcal{O}(t)$$

where $K(p)$ is the Gauss curvature at $p \in S$ [McKean & Singer, 1967].

Aside: McKean-Singer Versus Selberg

Do the local, McKean-Singer-type computations shed light on Selberg's formula? Unfortunately no, since

$$\sum_{\gamma} \frac{\ell_0(\gamma)/2}{\sinh\big(\ell(\gamma)/2\big)} e^{-\ell(\gamma)^2/4t} = \mathcal{O}(t^N)$$

(On the other hand the asymptotic formula

$$\frac{e^{-t/4}}{\sqrt{4\pi t}} \int_0^\infty \frac{xe^{-x^2/4t}}{\sinh(x/2)}\, dx = 1 - \tfrac{1}{3}t + \cdots$$

verifies the asymptotic expansion of McKean and Singer, using Selberg, for a hyperbolic surface.)

Similar remarks apply on the circle case.

It is remarkable that nevertheless the hypoelliptic Laplacian is a child of the heat equation method.

Second Introduction—A List of Ingredients

From now on I shall focus on the circle (of circumference one). But actually there would be few changes for compact groups, and not so many for the noncompact case.

1. I shall assemble a list of parts from which L_b is built:
   - The Dirac operator
   - The square root of the quantum harmonic oscillator
   - The (Kasparov) product

2. Then I'll explain the (counterfactual) steps that led to L_b:
   - Simplify the Kasparov product (incorrectly!)
   - Explore the consequences

3. Finally, with L_b to hand, I'll examine its geometric aspects, which lead to the Selberg-type formulas.

Square Roots of the Laplacian

On the circle, Bismut essentially uses the following Dirac operator

$$\mathrm{D} = \begin{bmatrix} 0 & -i\,\partial_x \\ -i\,\partial_x & 0 \end{bmatrix}$$

(which is more or less the de Rham operator). It acts as a self-adjoint operator.

This illustrates the general case. Bismut uses Kostant's cubic Dirac operator on a (reductive) group, acting not on spinors but on $\Lambda^\bullet(\mathfrak{g})$.

One has

$$\mathrm{D}^2 = \mathsf{Casimir} + \frac{1}{24}\,\mathsf{tr}(\mathsf{Casimir}_{\mathfrak{g}}) \cdot I$$

So the square is a Casimir, plus a scalar.

Disclosure: Bismut actually uses $i\,\mathrm{D}$, which will cause me to insert some square roots of minus one later.

The Supertrace and the McKean-Singer Formula

The supertrace is of course the functional

$$\mathsf{STr}\left(\begin{bmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{bmatrix}\right) = \mathsf{Tr}(A_{00}) - \mathsf{Tr}(A_{11}).$$

It vanishes on supercommutators.

McKean and Singer observed (in full generality, well beyond groups) that the quantity $\mathsf{STr}(\exp(-tD^2))$ is independent of $t>0$. In fact

$$\mathsf{STr}(\exp(-tD^2)) = \mathsf{Index}(D)$$

Proof. The derivative with respect to t is a supertrace of a supercommutator, and it is therefore zero:

$$\frac{d}{dt}\,\mathsf{STr}\big(\exp(-tD^2)\big) = -\,\mathsf{STr}\big(\{D, D\exp(-tD^2)\}\big) = 0$$

And then the large t limit is easy to compute.

Spectral Geometry on a Vector Space

The operator

$$H = -\partial_y^2 + y^2$$

on $L^2(\mathbb{R})$ is the well-known quantum harmonic oscillator, with simple spectrum $\{1, 3, 5, \dots\}$. The ground state is $\exp(-y^2/2)$.

There is an almost equally well-known "square root:"

$$Q = \begin{bmatrix} 0 & -\partial_y + y \\ \partial_y + y & 0 \end{bmatrix} : L^2(\mathbb{R}, \Lambda) \longrightarrow L^2(\mathbb{R}, \Lambda)$$

(with Λ the exterior algebra of $\mathbb{R}$) for which

$$Q^2 = \begin{bmatrix} -\partial_y^2 + y^2 - 1 & 0 \\ 0 & -\partial_y^2 + y^2 + 1 \end{bmatrix}$$

There is a counterpart on any euclidean vector space, with Q acting on functions valued in the exterior algebra, and

$$Q^2 = H + \dim(V)I - N$$

Products, Geometric and Operator-Theoretic

The Laplacian on a product of circles (or anything else) can be written as a sum of Laplacians acting on each factor:

$$\Delta_{\mathbb{T}\times\mathbb{T}} = \Delta_1 + \Delta_2 \colon L^2(\mathbb{T}\times\mathbb{T}) \longrightarrow L^2(\mathbb{T}\times\mathbb{T})$$

What about the square roots—the Dirac operators D?

Kasparov defines

$$\mathsf{D}_1 \# \mathsf{D}_2 \colon L^2(\mathbb{T}\times\mathbb{T}, \Lambda\otimes\Lambda) \longrightarrow L^2(\mathbb{T}\times\mathbb{T}, \Lambda\otimes\Lambda)$$

which is almost the sum of D_1 and D_2, acting on first and second factors. The difference: some $\pm$ signs are added strategically.

It has the fundamental properties that

$$(\mathsf{D}_1 \# \mathsf{D}_2)^2 = \mathsf{D}_1^2 + \mathsf{D}_2^2 \quad \text{and} \quad \mathrm{Ind}(\mathsf{D}_1 \# \mathsf{D}_2) = \mathrm{Ind}(\mathsf{D}_1)\cdot \mathrm{Ind}(\mathsf{D}_2)$$

Asymptotics (a Baby Case)

Now I shall combine D and Q into the product $\mathrm{D} \# \mathrm{Q}$.

Actually I shall introduce a positive parameter T, and study the family

$$\mathrm{D} \# T\mathrm{Q} \colon L^2\big(\mathbb{T} \times \mathbb{R}, \Lambda \otimes \Lambda\big) \longrightarrow L^2\big(\mathbb{T} \times \mathbb{R}, \Lambda \otimes \Lambda\big)$$

Here is why. Identify $L^2(\mathbb{T}, \Lambda)$ with the kernel of Q in $L^2(\mathbb{T}\times\mathbb{R}, \Lambda\otimes\Lambda)$ (consisting of the functions of degree zero in the second factor of Λ, and behaving like $e^{-y^2/2}$ in the y-direction).

Theorem

$$\lim_{T\to+\infty} \big(i\,\mathrm{I} \pm \mathrm{D} \# T\mathrm{Q}\big)^{-1} = \big(\mathrm{i}\,\mathrm{I} \pm \mathrm{D}\big)^{-1}$$

Proof. $(\mathrm{D} \# T\mathrm{Q})^2 = \mathrm{D}^2 + T^2\mathrm{Q}^2$. The rest is easy.

This refines the identity $\mathrm{Ind}(\mathrm{D} \# T\mathrm{Q}) = \mathrm{Ind}(\mathrm{D})$. Bismut and Lebeau developed this simple idea enormously . . .

Third Introduction—Counterfactual History

The discovery story, as I shall tell it, centers on spectral theory, although geometry plays a signicant role at the beginning.

I like to imagine that the discovery emerged from a sort of happy accident. I don't really believe it, but that is how I shall frame it.

A better—more complex—explanation is that the discovery was guided by Bismut's immense experience with the topics I've discussed up to now, and more of course.

Anyway, as I shall tell the story, the hypoelliptic Laplacian is automatically endowed with spectral significance, and the crucial problem is to add geometry back into the picture.

An Accidental Discovery?

Let me look again at the product

$$\mathrm{D} \# T\mathrm{Q} \colon L^2\big(\mathbb{T} \times \mathbb{R}, \Lambda \otimes \Lambda\big) \longrightarrow L^2\big(\mathbb{T} \times \mathbb{R}, \Lambda \otimes \Lambda\big)$$

Remember that in real life Λ is an exterior algebra . . .

. . . so it's tempting to multiply the tensor factors together, and build

$$\mathrm{D} \bar{\#} T\mathrm{Q} \colon L^2(\mathbb{T} \times \mathbb{R}, \Lambda) \longrightarrow L^2(\mathbb{T} \times \mathbb{R}, \Lambda)$$

What if one lazily writes

$$\mathrm{D} \bar{\#} T\mathrm{Q} = \mathrm{D} + T\mathrm{Q}?$$

Well, let's start with

$$(\mathrm{D} + T\mathrm{Q})^2 = \mathrm{D}^2 + T\{\mathrm{D}, \mathrm{Q}\} + T^2\mathrm{Q}^2$$

The cross-term, now non-zero, is

$$\{\mathrm{D}, \mathrm{Q}\} = 2y\partial_x.$$

From a geometrical point of view, $y\partial_x$ is the generator of the geodesic flow on (the tangent bundle of) $\mathbb{T}$ . . . which is interesting. Maybe.

Resolvent Convergence?

But does it amount to anything? For instance do we retain the formula

$$\lim_{T\to+\infty}\big(i\,\mathrm{I}\pm(\mathrm{D}+\mathrm{TQ})\big)^{-1}=\big(\mathrm{i\,I}\pm\mathrm{D}\big)^{-1}$$

so that D may be recovered from the new construction?

The initial indications are not promising, since D doesn't preserve the kernel of Q as it did before . . . quite the opposite: D exchanges the kernel and its orthogonal complement.

But D^2 *does* preserve the kernel of Q, so let's examine the matrix decomposition

$$(\mathrm{D}+T\mathrm{Q})^2=\begin{bmatrix}\mathrm{D}^2 & T\mathrm{DQ}\\ T\mathrm{QD} & \mathrm{D}^2+T\{\mathrm{D},\mathrm{Q}\}+T^2\mathrm{Q}^2\end{bmatrix}$$

with respect to

$$L^2(\mathbb{T}\times\mathbb{R},\Lambda)=\mathrm{Ker}(\mathrm{Q})\oplus\mathrm{Ker}(\mathrm{Q})^\perp\cong L^2(\mathbb{T})\oplus L^2(\mathbb{T})^\perp$$

with a view to computing resolvents.

Two by Two Block Matrix Calculations

If the bottom right entry d of a block matrix is invertible, then

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & bd^{-1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e & 0 \\ 0 & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ d^{-1}c & 1 \end{bmatrix}$$

where

$$e = a - bd^{-1}c$$

The $(1,1)$-entry of the inverse matrix is therefore e^{-1}.

But in our case

$$e = D^2 - D \cdot \frac{T^2Q^2}{D^2 + T\{D, Q\} + T^2Q^2} \cdot D$$

and (arguing informally, at least) $e \to 0$ as $T \to \infty$.

So D^2 disappears from the resolvent in the limit as $T \to \infty$.

Which is not good.

Two by Two Block Matrix Calculations

However after looking at these (rather informal) calculations a just a little bit more, a simple adjustment presents itself.

If one starts not with

$$(\mathsf{D} + T\mathsf{Q})^2 = \begin{bmatrix} \mathsf{D}^2 & T\mathsf{DQ} \\ T\mathsf{QD} & \mathsf{D}^2 + T\{\mathsf{D}, \mathsf{Q}\} + T^2\mathsf{Q}^2 \end{bmatrix}$$

but with

$$(\mathsf{D} + T\mathsf{Q})^2 - \mathsf{D}^2 = \begin{bmatrix} 0 & T\mathsf{DQ} \\ T\mathsf{QD} & T\{\mathsf{D}, \mathsf{Q}\} + T^2\mathsf{Q}^2 \end{bmatrix}$$

then it appears that $\mathsf{e} \to -\mathsf{D}^2|_{\mathsf{ker}(\mathsf{Q})}$ as $T \to \infty$.

To cope with the minus sign, make a small adjustment, and write

$$\mathsf{L}^T = (\sqrt{-1}\mathsf{D} + T\mathsf{Q})^2 + \mathsf{D}^2.$$

Then L^T converges in the resolvent sense to $\Delta_{\mathbb{T}}$ (it appears).

Block Matrix Calculations Summarized

Concerning the informal computations, let me state a simple, encouraging result.

Let D and Q be odd-graded self-adjoint operators on a finite-dimensional graded Hilbert space H.

Assume that D^2 commutes with Q, and that $\ker(\mathrm{Q})$ is entirely even-graded.

Theorem

The operator $\mathrm{L}^T = (\sqrt{-1}\mathrm{D} + T\mathrm{Q})^2 + \mathrm{D}^2$ *converges in the resolvent sense to the compression of* D^2 *to the kernel of* Q*. Moreover*

$$\lim_{T\to\infty} \mathrm{STr}\left(\exp\left(-t\,\mathrm{L}^T\right)\right) = \mathrm{Tr}\left(\exp\left(-t\,\mathrm{D}^2|_{\mathrm{kernel(Q)}}\right)\right)$$

for every $t > 0$*. In addition*

$$\frac{d}{dT}\,\mathrm{STr}\left(\exp\left(-t\,\mathrm{L}^T\right)\right) = 0$$

Definition of the Hypoelliptic Laplacian

Bismut uses b^{-1} instead of T and divides by 2. So he defines the hypoelliptic Laplacian (on the circle, or on any compact group) to be

$$\mathsf{L}_b = \tfrac{1}{2}(\sqrt{-1}\mathsf{D} + b^{-1}\mathsf{Q})^2 + \tfrac{1}{2}\mathsf{D}^2$$

This acts on $L^2(G\times\mathfrak{g}, \Lambda^\bullet(\mathfrak{g}))$. We get

$$\mathsf{L}_b = \begin{bmatrix} \frac{1}{2b^2}(y^2 - \partial_y^2 - 1) + \frac{1}{b}y\partial_x & 0 \\ 0 & \frac{1}{2b^2}(y^2 - \partial_y^2 + 1) + \frac{1}{b}y\partial_x \end{bmatrix}$$

in the case of the circle $\mathbb{T}$, as we saw before.

To summarize my story, L_b is designed with the formula

$$\lim_{b\to 0} \mathsf{STr}\big(\exp(-t\,\mathsf{L}_b)\big) = \mathsf{Tr}(\exp(-t\Delta_{\mathbb{T}}/2))$$

in mind.

Of course, quite a few difficult issues need to be resolved, now that we are looking at unbounded operators.

Fundamental Properties

Theorem

For each $b > 0$ *the hypoelliptic Laplacian operator* L_b *is*

- *hypoelliptic, and*
- *the generator of a one-parameter semigroup* $\exp(-t\,\mathsf{L}_b)$ *of trace-class operators.*

Theorem

$$\frac{d}{db}\,\mathrm{STr}\big(\exp(-t\,\mathsf{L}_b)\big) = 0$$

Theorem

$$\lim_{b\to 0}\mathrm{STr}\big(\exp(-t\,\mathsf{L}_b)\big) = \mathrm{Tr}(\exp(-t\Delta_{\mathbb{T}}/2))$$

Hypoellipticity and Subelliptic Estimates

Consider the toy model operator

$$P = -\partial_y^2 + \sin(2\pi y)\partial_x \tag{1}$$

on the 2-torus $\mathbb{T} \times \mathbb{T}$.

The following theorems involve the usual Sobolev norms. They may be proved using the pseudodifferential calculus and a somewhat intricate commutator method of J.J. Kohn.

Theorem

$$\|u\|_{s+1/4} \lessapprox \|Pu\|_s + \|u\|_s$$

Theorem

If σ_1 and σ_2 are smooth functions, and if $\sigma_2 \equiv 1$ on a neighborhood of $\mathrm{supp}(\sigma_1)$*, then*

$$\|\sigma_1 u\|_{s+1/8} \lessapprox \|\sigma_2 Pu\|_s + \|\sigma_2 u\|_s$$

Hypoellipticity and Subelliptic Estimates

The same commutator methods apply to L_b. But to obtain improved global estimates over the noncompact cylinder $\mathbb{T} \times \mathbb{R}$ one should work with the pseudodifferential calculus associated to the generalized Laplacian

$$\boldsymbol{\Theta} = H^2 + \Delta_{\mathbb{T}}$$

which is deemed to be of order 2.The methods of Connes and Moscovici are helpful here.

The same methods may be used to proved that $\exp(-t\mathsf{L}_b)$ is represented by a smooth integral kernel, Schwartz-class in the $\mathbb{R}$-direction, depending smoothly on both $t > 0$ and $b > 0$.

This is joint work with Nurulla Azamov, Ed MacDonald, Fedor Sukochev and Dima Zanin (of course the results are originally due to Bismut).

The Method of the Hypoelliptic Laplacian

As should now be clear, Bismut's approach to trace formulas using L_b is as follows:

1. Evaluate the limit of the supertrace of the heat kernel as b tends to zero:

$$\lim_{b\to 0} \mathrm{STr}\big(\exp(-t\mathsf{L}_b)\big) = \mathrm{Tr}\big(\exp(-t\Delta/2)\big)$$

2. Show that the b-derivative of the supertrace vanishes:

$$\frac{d}{db}\,\mathrm{STr}\big(\exp(-t\mathsf{L}_b)\big) = 0$$

3. Evaluate the limit

$$\lim_{b\to\infty} \mathrm{STr}\big(\exp(-t\mathsf{L}_b)\big)$$

I've already discussed the first two steps. The third requires still more new ideas, this time geometric, not spectral.

Geometry of the Hypoelliptic Laplacian

I shall work now with the *scalar* operator

$$L_b = \frac{1}{2b^2}(-\partial_y^2 + y^2) + \frac{1}{b} y\partial_x$$

for simplicity. Actually to begin with, I shall work with the even simpler operator

$$K = -\frac{1}{2}\partial_y^2 + y\partial_x$$

on the (x, y)-plane (this operator was initially studied by Kolmogorov).

I want to explain the influence of the term $y\partial_x$ on the behavior of solutions to the K-heat equation

$$\partial_t u_t + K u_t = 0$$

Since

$$u_t = \exp(-tK)u_0$$

this should also tell us something about the heat semigroup.

The Drift Term

Let u_t be a solution of the K-heat equation (it is a family of functions on the plane).

Define the *center of mass* of u_t to be

$$\begin{aligned} \mathrm{cm}(u_t) &= \big(\mathrm{cm}_x(u_t), \mathrm{cm}_y(u_t)\big) \\ &= \Big(\int\int u_t(x,y)\,x\,dx\,dy, \int\int u_t(x,y)\,y\,dx\,dy\Big) \end{aligned}$$

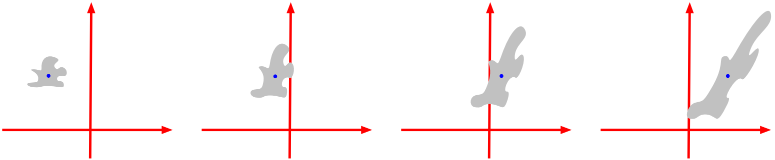
By differentiating under the integral sign, we find that

$$\frac{d}{dt}\,\mathrm{cm}(u_t) = (\mathrm{cm}_y(u_t), 0).$$

If the term $y\partial_x$ was removed from K, then the derivative would be zero. The drift is entirely attributable to $y\partial_x$.

The Drift Term

Here is a cartoon of what happens, showing where a solution of the K-heat equation is concentrated as t increases.



- If u_0 was concentrated higher, the drift would be faster.
- If u_0 was concentrated lower, the drift would be to the left.

Bismut undoubtedly understood this dynamical feature of K (shared by L_b) immediately, while first experimenting with $\mathrm{D} \# TQ \ldots$

The Concentration Property for the Heat Kernel

The heat operators for Laplacian (on the circle or elsewhere) have the following well-known property of concentration along the diagonal.

Proposition

Let σ_1 and σ_0 be smooth functions on $\mathbb{T}$ with disjoint supports. There is a positive constant k such that

$$\|\sigma_1 \exp(-t\Delta)\sigma_0\| = \mathcal{O}(e^{-kt^{-1}})$$

as $t \to 0$ [this is true for any reasonable norm on the left].

This may be proved in a variety of ways.

My favorite is an argument of Garding and Gaffney (originally used to study large distance behavior of heat kernels, not small time behavior).

It adapts very nicely to incorporate the drift phenomenon we've seen. If φ is a smooth function on the circle, define

$$\varphi_t(x, y) = \varphi(x - t^{-1}by)$$

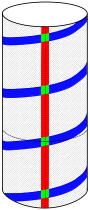
The Concentration Property for the Heat Kernel

Proposition

If φ and ψ are smooth functions on $\mathbb{T}$ with disjoint supports, then there is a positive constant k such that

$$\left\| \varphi_t \exp(-tL_b)\psi_0 \right\| \leq \mathcal{O}(e^{-kb^2})$$

for any fixed t as $b \to \infty$.



So the L_b-heat kernel concentrates on a drifted diagonal. This leads to

$$\operatorname{Tr}\big(\exp(-tL_b)\big) \approx \sum_{a \in t^{-1} b\mathbb{Z}} \int_{\mathbb{T}} dx \int_{a-b^{-1}C}^{a+b^{-1}C} dy \, \exp\big(-tL_b\big)\big((x,y),(x,y)\big)$$

for C and b large. The heat trace concentrates on geodesic bands.

More on the Concentration Phenomenon

Garding-Gaffney Identity: For a solution u_t of the heat equation

$$\partial_t u_t + \Delta u_t = 0$$

and any smooth function φ,

$$\partial_t \|\varphi u_t\|^2 = 2\|\nabla(\varphi) u_t\|^2 - 2\|\nabla(\varphi u_t)\|^2$$

Garding-Gaffney Inequality: $\partial_t \|\varphi u_t\|^2 \leq 2\|\nabla(\varphi) u_t\|^2$

Modified Garding-Gaffney Inequality: For a solution u_t of the hypoelliptic heat equation

$$\partial_t u_t + \mathsf{L}_b u_t = 0$$

and any smooth $\varphi_t(x, y) = \varphi(x - t^{-1} by)$,

$$\partial_t \|\varphi_t u_t\|^2 \leq \|\nabla_b(\varphi_t) u_t\|^2$$

where $\nabla_b = b^{-1}\partial_y$.

This is a variation on a standard finite-propagation argument.

Limit Argument

Returning to the computation of the large b limit ... there are two more steps. First, the following computation is proved by the change of variables

$$[a - b^{-1}C, a + b^{-1}C] \longrightarrow [-C, C]$$

Theorem

If $a = bn$, then

$$\lim_{b\to\infty} tb^{-3} \cdot \exp(-tL_b)\big((0, a+b^{-1}v), (0, a+b^{-1}v)\big) \\ = e^{-n^2/2}\exp(-tK)\big((0,v),(0,v)\big)$$

Corollary

$$\mathrm{STr}\big(\exp(-t\,\mathsf{L}_b)\big) = \sum_{n\in\mathbb{Z}} e^{-n^2/2t}\int_{-\infty}^{\infty} dv\, \exp(-tK)\big((0,v),(0,v)\big)$$

Explicit Formulas

Second, there is an explicit formula for the K-heat kernel, found by Kolmogorov:

$$\exp(-tK)\big((x_1, y_2), (x_2, y_2)\big)$$
$$= \frac{\sqrt{3}}{\pi t^2} \exp\Big(-\frac{1}{2t}\Big(y_1 - y_2\Big)^2 - \frac{6}{t^3}\Big(x_2 - x_1 - \frac{(y_1 + y_2)t}{2}\Big)^2\Big)$$

It illustrates the drift phenomenon clearly, and also the small t/large b concentration on the twisted diagonal.

However we only need the formula for $x_1 = x_2$ and $y_1 = y_2$.

We obtain

$$\lim_{b \to \infty} \mathrm{STr}(\exp(-t\mathsf{L}_b)) = \frac{1}{\sqrt{2\pi t}} \sum_{n \in \mathbb{Z}} \exp\Big(-\frac{n^2}{2t}\Big)$$

and from this we obtain the “Selberg trace formula on the circle."

Thank you!