Equilibrium states for C*-algebras of right LCM monoids

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Outline

- Isometries and equilibrium states: an informal match
- Right LCM (least common right-multiple) monoids
- Zappa-Szep monoids and Baumslag-Solitar monoids
- Natural scales on semigroups, time evolutions and equilibrium
- Some classification results on equilibrium states

Monoids and equilibrium states: motivation

$\mathcal{A}$ a C^*-algebra, $\sigma : \mathbb{R} \curvearrowright \mathcal{A}$ a one-parameter group. Suppose

$$V^*V = 1_{\mathcal{A}}, VV^* < 1_{\mathcal{A}},$$

and $V \mapsto N(V) \in (0, \infty)$ s.t. $\sigma_1(V) = N(V)^{it} V$.

An *equilibrium state* $\varphi : \mathcal{A} \to \mathbb{C}$ at parameter $\beta > 0$ is a trace "up to a factor", and in particular

$$\varphi(V^*V) = N(V)^{\beta} \varphi(VV^*) = 1.$$

P a left-cancellative monoid (use P as relic of positive cone), $\mathcal{A} = C^*(P)$ generated by isometries V_p with $V_p V_q = V_{pq}, p, q, \in P$ (and more relations), $N : P \to (0, \infty)$ homomorphism, $\sigma^N : \mathbb{R} \curvearrowright \mathcal{A}$ with $\sigma_t^N(V_p) = N_p^{it} V_p$. An equilibrium state φ_β must have *prescribed values* $\varphi_\beta(V_p V_p^*)$ *on projections* $V_p V_p^*$, $p \in P$. Do any exist? Why?

Monoids and equilibrium states: just a hype?

Recap: Given P a left-cancellative monoid, $\mathcal{A} = C^*(P)$ the universal C^*-algebra generated by isometries V_p for $p \in P$, $N : P \to (0, \infty)$ a homomorphism, $\sigma_t^N(V_p) = N_p^{it} V_p$ for $t \in \mathbb{R}$ a time evolution.

If $\varphi_\beta : C^*(P) \to \mathbb{C}$ is equilibrium state at β, then $\varphi(V_p V_p^*) = N_p^{-\beta}$ for all p.

1. Given β, do equilibrium states exist?
2. If so, where do they come from? A subalgebra?
3. If φ_β exists at some β, how many are there?
4. If φ_β exist at different β's, are they of the same type?

Equilibrium states (KMS states)

Rudolf Haag, Nico Hugenholtz and Marinus Winnink (1967): by analogy with finite systems $M_n(\mathbb{C})$ and their Gibbs states $\mathrm{Tr}(\cdot e^{-\beta H})/\mathrm{Tr}(e^{-\beta H})$, extend the notions of KMS_β state (for Kubo-Martin-Schwinger).

$\mathcal{A}$ a C^*-algebra, $\sigma : \mathbb{R} \to \mathrm{Aut}(\mathcal{A})$ time evolution, $\beta \in [0, \infty)$.

Definition: A state φ of $\mathcal{A}$ is a KMS_β state if

$$\varphi(ab) = \varphi(b\sigma_{i\beta}(a))$$

for all $a, b \in \mathcal{A}$ with *a analytic* element, meaning $t \mapsto \sigma_t(a)$ extends to an entire function, and similarly for b.
A state φ is a *ground state* if for all a, b analytic, the function $z \to \varphi(a\sigma_z(b))$ is bounded in the upper-half plane.

Remark. The analytic elements form a dense *-subalgebra in $\mathcal{A}$. For $C^*(P)$, we have analytic $V_pV_q^*$ for $p, q \in P$.

The monoid C^*-algebra of $\mathbb{N} \rtimes \mathbb{N}^\times$

Theorem (Marcelo Laca and Iain Raeburn (2010))

$C^*(\mathbb{N} \rtimes \mathbb{N}^\times)$ is the universal C^*-algebra generated by isometries s and v_p for each prime p, subject to the relations

1. $v_p s = s^p v_p$;
2. $v_p v_q = v_q v_p$,
3. $v_p^* v_q = v_q v_p^*$ *when* $p \neq q$,
4. $s^* v_p = s^{p-1} v_p s^*$, *and*
5. $v_p^* s^k v_p = 0$ *for* $1 \leq k < p$.

Dynamics: $\sigma_t(s) = s$ and $\sigma_t(v_p) = p^{it} v_p$, $t \in \mathbb{R}$. Then

(a) $\beta < 1$: no KMS_β states;
(b) $\beta \in [1, 2]$: unique KMS_β state;
(c) $\beta \in (2, \infty]$, KMS_β states parametrised by probability measures on $\mathbb{T}$;
(d) ground states: parametrised by states on Toeplitz alg. $\mathcal{T}$.

Type III_1 in (b): (Marcelo Laca and Sergey Neshveyev (2011))

Monoids and equilibrium states

$C^*(\mathbb{N} \rtimes \mathbb{N}^\times)$ seems to be a good example.

Are there other examples?

Baumslag-Solitar monoids with matching signs

Let $BS(c,d)^+ = \langle \mathfrak{a}, \mathfrak{b} \mid \mathfrak{a}\mathfrak{b}^c = \mathfrak{b}^d\mathfrak{a}\rangle^+$ with $c, d \in \mathbb{Z}$, $cd > 0$. Each $s \in BS(c,d)^+$ has a normal form

$$s = \mathfrak{b}^{i_1}\mathfrak{a}\mathfrak{b}^{i_2}\mathfrak{a}\ldots\mathfrak{a}\mathfrak{b}^{i_{\theta(s)}}\mathfrak{a}\mathfrak{b}^m,$$

with the height $\theta(s)$ equal the number of $\mathfrak{a}$'s in the decomposition, $m \in \mathbb{N}$. Define $\sigma_t(\mathfrak{a}) = d^{it}\mathfrak{a}$ when $|d| > 1$, $\sigma_t(\mathfrak{b}) = \mathfrak{b}$.

Theorem (Lisa Clark, Astrid an Huef, Iain Raeburn)

The equilibrium states for $(C^*(BS(c,d)^+), \sigma)$ are as follows:

1. *The ground states are parameterised by states on $C^*(\mathbb{N})$.*
2. *The KMS_β-states for $\beta \in (1, \infty]$ are parameterised by normalised traces on $C^*(\mathbb{Z})$.*
3. *There is a KMS_1 state.*

If $c \geq 1$ and $d \geq 2$, the KMS_1-state is unique iff $c \notin d\mathbb{Z}$. For $c \in d\mathbb{Z}$, there exist distinct φ_1 and φ_1'.

Self-similar actions

Nekrashevych: X a finite alphabet, X^* the free monoid in X, and G a group with $g(xv) = yh(v)$ for all $v \in X^*$, with $y \in X$ and $h = g|_v$ uniquely determined by g, x.

Theorem (Marcelo Laca, Jacqui Ramagge, Iain Raeburn and Mike Whittaker (2014))

Given a self-similar action (X, G)*, form the Toeplitz algebra* $\mathcal{T}_M$ *associated to* (X, G) *with its natural dynamics that scales a generator* T_v *with* $v \in X^*$ *by* $e^{it\ell(v)}$*. Then*

1. *For* $\beta > \log|X|$*, the simplex of* KMS_β *states is affinely homeomorphic to the simplex of normalised traces on* $C^*(G)$*.*
2. *There is a* KMS_β *state at* $\log|X|$*. It is the unique one if* (X, G) *is contracting.*

Contracting means, roughly, that the set of restrictions $g|_v$ of $g \in G$ to elements $v \in X^*$ is controlled by a finite set.

Right LCM (least common right-multiples) monoids

Let P be left cancellative monoid (or category). We say $r \in P$ is a *right multiple* of $p \in P$ if

$$r = ps \text{ for } s \in P.$$

Alternatively, p is a left-divisor of r. We say $r \in P$ is a *common right multiple* of p, q in P if

$$r = ps_1 = qs_2 \text{ for } s_1, s_2 \in P.$$

P is right LCM (or has conditional right LCM's) provided that each p, q with a common right multiple admit a least common right multiple.
There are lots of right LCM monoids.

Right LCM monoids and C^*-algebras

Xin Li (2012): $C^*(P)$ generated by isometries v_p, $p \in P$ with certain relations, where P is left-cancellative, .
Magnus Norling (2014) studied inverse semigroup C^*-algebras associated with left-cancellative monoids P satisfying Clifford's condition (cf. Mark Lawson):

$$pP \cap qP \neq \emptyset \Rightarrow pP \cap qP = rP.$$

Here pP is the set of right multiples of p and r is a least common right multiple of p, q. (Also rx for x non-trivial invertible works.)

Nathan Brownlowe, Jacqui Ramagge, David Robertson and Mike Whittaker (2014): studied $C^*(P)$ with P a Zappa-Szép product monoid $A \bowtie U$ (a generalisation of semi-direct product cf. Brin) and a right LCM monoid. Their examples: monoids $X^* \bowtie G$ that model self-similar group actions, Baumslag-Solitar monoids, the affine monoid $\mathbb{N} \rtimes \mathbb{N}^\times$.

Equilibrium states for right LCM monoids (I)

Theorem (Zahra Afsar, Nathan Brownlowe, NSL, Nicolai Stammeier (2017))

For P admissible right LCM with a scale $N : P \to \mathbb{N}$, $\sigma_t(v_p) = N_p^{it} v_p$, suppose $\beta_c \in \mathbb{R}$ is a critical point for partition function

$$\zeta(\beta) := \sum_{n \in Irr(N(P))} \sum_{f \in T_n} N_f^{-\beta},$$

so $\zeta(\beta)$ converges for $\beta > \beta_c$. On $C^*(P)$, we have

1. $\beta \in [0, 1)$*: no* KMS_β *state;*
2. $\beta \in [1, \beta_c]$*: at least one* KMS_β *state. Often unique!*
3. $\beta \in (\beta_c, \infty)$*:* KMS_β *states* $\xleftrightarrow{1-1}$ *normalised traces on* $C^*(P_c)$ *for* $P_c \subset P$ *the core submonoid;*
4. $KMS_\infty \xleftrightarrow{1-1}$ *normalised traces on* $C^*(P_c)$ *with* $P_c \subset P$*;*
5. *Ground states* $\xleftrightarrow{1-1}$ *states on* $C^*(P_c)$*.*

Admissible monoids

1. The affine monoid $\mathbb{N} \rtimes \mathbb{N}^{\times}$.
2. Monoids $\mathbb{N} \rtimes P \subseteq \mathbb{N} \rtimes \mathbb{N}^{\times}$ for P generated by a family of relatively prime numbers.
3. $X^* \bowtie G$ from self-similar actions.
4. Baumslag-Solitar monoids $BS(c,d)^+ = \langle \mathfrak{a}, \mathfrak{b} \mid \mathfrak{a}\mathfrak{b}^c = \mathfrak{b}^d\mathfrak{a} \rangle^+$, $c, d \geq 1$.

Admissible right LCM semigroups

Core subsemigroup: cf. Charles Starling (motivated by John Crisp and Marcelo Laca for quasi-lattice ordered (G, P)) is the submonoid $P_c \subset P$ of elements that admit a common right multiple, hence a right LCM, with all elements in P.

$$P_c = \{a \in P \mid aP \cap rP \neq \emptyset \text{ for all } r \in P\}.$$

Equivalence relation $s \sim t$ in P iff $sP_c = tP_c$.
Core irreducible elements: $s \in P_{ci}$ if

$$s = ta \text{ for } a \in P_c \Longrightarrow a \in P^*.$$

P is **admissible** (Afsar-Brownlowe-L-Stammeier) if

- $P = P_{ci}^1 P_c$;
- $P_{ci} \subset P$ is closed under taking right LCM's;
- $N : P \to \mathbb{N}$ scale s.t. for $n \in N(P)$, $\{p \in P \mid N_p = n\}/_\sim$ has size n and admits a system of representatives in P_{ci} with disjoint right ideals and forming a *foundation set*;
- $\mathrm{Irr}(N(P))$ generates the free abelian monoid on $\mathrm{Irr}(N(P))$.

Monoids and equilibrium states

Are we done yet?

Let $P_1 = BS(c,d)^+ = \langle \mathfrak{a}, \mathfrak{b} \mid \mathfrak{a}\mathfrak{b}^c = \mathfrak{b}^d\mathfrak{a}\rangle^+$, $c, d \geq 1$.

Define P_2, the Baumslag-Solitar monoid with opposite signs, as

$$BS(c,d)^+ = \langle \mathfrak{a}, \mathfrak{b} \mid \mathfrak{b}^{|d|}\mathfrak{a}\mathfrak{b}^{|c|} = \mathfrak{a}\rangle^+$$

for $cd < 0$. Jack Spielberg (2012): $BS(c,-d)^+$ with $c, d \geq 1$ is *very different* from $BS(c,d)^+$ with respect to taking least common right multiples *of elements in the group*.

- $\mathfrak{a}\mathfrak{b}\mathfrak{a}^{-1}P_1 \cap P_1 = pP_1$, for some $p \in P_1 \cap \mathfrak{a}\mathfrak{b}\mathfrak{a}^{-1}P_1$.
- $\mathfrak{a}\mathfrak{b}\mathfrak{a}^{-1}P_2 \cap P_2 = \bigcup_{m\in\mathbb{Z}} \mathfrak{a}\mathfrak{b}^m P_2$,

and the union is not absorbed by a finite one. For $m \in \mathbb{Z}$ and $n \in \mathbb{N}$ so that $nc + m \geq 0$, we have $\mathfrak{a}\mathfrak{b}^m = \mathfrak{b}^{nd}\mathfrak{a}\mathfrak{b}^{nc+m} \in P_2$ due to $\mathfrak{b}^d\mathfrak{a}\mathfrak{b}^c = \mathfrak{a}$. Consequence: P_2 is not admissible.

More about $BS(c, -d)^+$

$BS(c, -d)^+ = \langle \mathfrak{a}, \mathfrak{b} \mid \mathfrak{b}^d \mathfrak{a} \mathfrak{b}^c = \mathfrak{a} \rangle^+$, with $c, d \geq 1$.
Fact: $BS(c, -d)^+$ is group embeddable but does not admit any embedding into a group that fulfills the Toeplitz condition (X. Li). This has implications for crossed product structure by this monoid.

Theorem (Astrid an Huef, Brita Nucinkis, Camila Sehnem and Dilian Yang (2019))

$C^*(BS(c, -d)^+)$ is nuclear.

Crucial: $BS(c, -d)^+$ admits an infinite descending chain $\mathfrak{b}^{nd}\mathfrak{a}$, $n \geq 0$. In fact, more general characterisation of nuclearity of $C^*(P)$ for a class of P one-relator monoids.

Equilibrium states for right LCM monoids (II)

- P right LCM monoid.
- P_c, the submonoid of $a \in P$ s.t. a, p admit a common right multiple $\forall p \in P$.
- $N : P \to \mathbb{N}$ a generalised scale.
- Let $[p] = \{q \in P \mid pa = qb\}$ for $a, b \in P_c$, and define a partition function

$$\zeta(\beta) = \sum_{[p] \in P/P_c} N_p^{-\beta}.$$

Suppose $\beta_c \in \mathbb{R}$ so that $\zeta(\beta)$ converges for $\beta > \beta_c$.

Theorem (Nathan Brownlowe, NSL, Jacqui Ramagge and Nicolai Stammeier 2019)

For $(C^*(P), \sigma^N)$, there are no KMS_β states for $\beta < 1$. For $\beta > \beta_c$, there is an affine homeomorphism between KMS_β states on $C^*(P)$ and normalised traces on the subalgebra $C^*(P_c)$.

Uniqueness of equilibrium state

Let P, P_c, $P/\sim$, $N : P \to \mathbb{N}$, $\zeta(\beta)$ as before. Define the *absorbing elements* of $a, b \in P_c$, $a \neq b$ at $n \in N(P)$

$$A_n^{a,b} := \{[s] \in N^{-1}(n)/_\sim \mid asc = bsc \text{ for some } c \in P_c\}.$$

Theorem (Brownlowe-L-Ramagge-Stammeier)

(a) There is a KMS_1 state at $\beta = 1$ determined by

$$\psi_1(v_a v_b^*) = \lim_{n \in N(P)} n^{-1}|A_n^{a,b}| \quad \text{for } a, b \in P_c.$$

(b) The state ψ_1 is the unique KMS_1 state provided that for all $a, b \in P_c$, $n \in N(P)$

$$\lim_{n\to\infty} n^{-1}|F_n^{a,b} \setminus A_n^{a,b}| = 0, \tag{1}$$

where $F_n^{a,b} := \{[s] \in N^{-1}(n)/_\sim \mid asc = bsd, c, d \in P_c\}$

Uniqueness of KMS_1 states and G-regular points

Given (X, G) a faithful self-similar action with G countable, let $X^{\mathbb{N}}$ be the space of right-infinite words with product topology.

Definition (Volodymyr Nekrashevych (2009))

Given $g \in G$, a point $w \in X^{\mathbb{N}}$ is g-regular (or g-generic) if either w is not fixed by g or it lies in the interior of the set of fixed points for g.

Note: That w is in the interior of the set of fixed points for g means that there is a finite word $v \in X^*$ such that $w \in vX^{\mathbb{N}}$ and $v \in A_n^{g,1}$, where $n = |X|^{\ell(v)}$.

Proposition (Brownlowe-L-Ramagge-Stammeier (2019))

Let μ be the probab. measure on $X^{\mathbb{N}}$ given as the product of uniform distribution on X. Then μ-almost every point is G-regular, i.e. g-regular for every g, precisely when condition (1) *for uniqueness of KMS_1 state is satisfied for $P = X^* \bowtie G$.*

Measurable G-regularity and von Neumann algebras

Keisuke Yoshida (2019): von Neumann algebras associated with (X, G).

Theorem (Keisuke Yoshida)

For every self-similar action (X, G) with G countable, let $X^{\mathbb{N}}_{G-\text{reg}}$ denote the set of G-regular points.
(a) Then either $\mu(X^{\mathbb{N}}_{G-\text{reg}}) = 1$ or $\mu(X^{\mathbb{N}}_{G-\text{reg}}) = 0$.
(b) If $\mu(X^{\mathbb{N}}_{G-\text{reg}}) = 1$, there is a unique $KMS_{\log|X|}$ (scaled dynamics changes 1 to $\log|X|$) state on $\mathcal{O}_{X,G}$.
(c) The von Neumann algebra corresponding to the unique equilibrium state at $\log|X|$ is an AFD type $III_{|X|^{-1}}$ factor whenever G is amenable.

Main technique

To pass from states and normalised traces on $C^*(P_c)$ to KMS$_\beta$ states of $C^*(P)$ for $\beta > \beta_c$, we establish existence of a Fock-type module $\mathcal{F}$ with a right action of $C^*(P_c)$.

Then we can use induction of GNS representations associated to states of $C^*(P_c)$ to get states of $C^*(P)$. Suitable limits of averages over finite subsets of $N(P)$ will produce KMS$_\beta$ states. A reconstruction formula will show that they all arise in this manner.

Reminiscent of the construction of Gibbs type states on the Toeplitz algebra $\mathcal{T}_M$ of a Hilbert bimodule M due to Marcelo Laca and Sergey Neshveyev (2004).

Uniqueness: alternate approaches

Theorem (Sergey Neshveyev and Nicolai Stammeier (2020))

Let P be a countable right LCM monoid with a scale $N : P \to (0, \infty)$ and assume that there is a probability measure μ_N on the boundary space $\widehat{E(P)}$ of P with $\mu_N(Z_p) = N_p^{-1}$, $p \in P$, with Z_p cylinder set. Then there is a unique KMS_1 state iff for all distinct $a, b \in P$ with a common right multiple and $N(a) = N(b)$, the measure of the set of trivially fixed points coincides with the measure of the set of fixed points.

Thus, if N is a generalised scale (Brownlowe-L-Ramagge-Stammeier), the sufficient condition (1) for uniqueness of the KMS_1 state is also necessary.

Joan Claramunt and Aidan Sims (2018) established uniqueness of the KMS state for self-similar actions of groupoids on graphs using iterations of self-maps on the simplex of traces of $C^*(G)$.