Quantized calculus and quasi-inner functions

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Quantized Calculus

$$đf := [F, f], \quad F = F^*, \; F^2 = 1$$

$$\Omega^k := \{\omega = \sum f_0 \, đf_1 \ldots đf_k\}$$

$$\mathrm{Tr}(u^* \, đu), \; \mathrm{Tr}(\gamma\omega)$$


$\sigma_t^\varphi(x)$
$dZ = [F, Z]$
$\int |dZ|^p$
$\mathbb{C}$
$\mathbb{H}$

$F =$ Hilbert transform

$$k(s,t) = \left(\frac{i}{\pi}\right) \frac{f(s) - f(t)}{s - t}$$

$f \in \mathcal{S}(\mathbb{R}) \Rightarrow \text{đ}f =$ ∞ order and conformal invariance

Triangular unitary

$$U = \begin{pmatrix} u_{1,1} & u_{1,2} \\ 0 & u_{2,2} \end{pmatrix} \text{ unitary}$$

$$\Updownarrow$$

1. $u_{1,1}$ is an isometry.
2. $u_{2,2}$ is a coisometry.

3. $u_{1,2}$ is a partial isometry from the kernel of $u_{2,2}$ to the cokernel of $u_{1,1}$.

Positivity

U triangular in $(\mathcal{P}, P)$, $F = 2P - 1$, $f \geq 0$ then

$$\mathrm{Tr}(fU^*[F,U]) \leq 0$$

(since $U\mathcal{P}U^* \leq \mathcal{P}$)

Weil positivity

$$RH \iff \sum_v W_v(f * f^*) \leq 0,$$

Explicit Formulas

$$W_v(h) = \int_{\mathbb{Q}_v^*}' \frac{|w|^{1/2}}{|1-w|} h(w) d^* w$$

$$\tilde{f}(0) - \sum_{\rho \in Z} \tilde{f}(\rho) + \tilde{f}(1) = \sum_v \mathcal{W}_v(f)$$

Semi-local adele class space

$$X_S := \left(\prod_{v \in S} \mathbb{Q}_v\right) / \mathbb{Z}_S^*,$$

$$\mathbb{Z}_S = \{q \in \mathbb{Q} \mid |q|_v \leq 1\,,\ \forall v \notin S\}$$

1. The Hilbert space $L^2(X_S)$ of functions on the quotient $X_S := \mathbb{A}_S/\mathbb{Z}_S^*$ of the semi-local adeles $\mathbb{A}_S$ by the group of units of the ring $\mathbb{Z}_S$.
2. The Hilbert space $L^2(C_S)$ of functions on the semi-local idele class group $C_S := \mathbb{A}_S^*/\mathbb{Z}_S^*$.
3. The Hilbert space $L^2(\widehat{C_S})$ of functions on the Pontrjagin dual $\widehat{C_S}$ of the locally compact group C_S.

Fourier $\rightarrow$ unitary in $L^2(X_S)$

In $L^2(\widehat{C_S})$ Fourier is inversion composed with multiplication by

$$u = \prod u_v$$

$u_v =$ ratio of local factors

Trace of scaling in $L^2(X_S)$

$$\mathrm{Tr}(\hat{h}\left(\frac{1}{2}u^*\not{d}\,u\right)) =$$

$$\sum_{v\in S}\int'_{\mathbb{Q}_v^*}\frac{|w|^{1/2}}{|1-w|}h(w)d^*w$$

Inner function

$U \in L^\infty(S^1)$ is inner

$$\Updownarrow$$

$U =$ triangular unitary in

$$L^2(S^1) = H^2(D) \oplus (H^2(D))^\perp$$

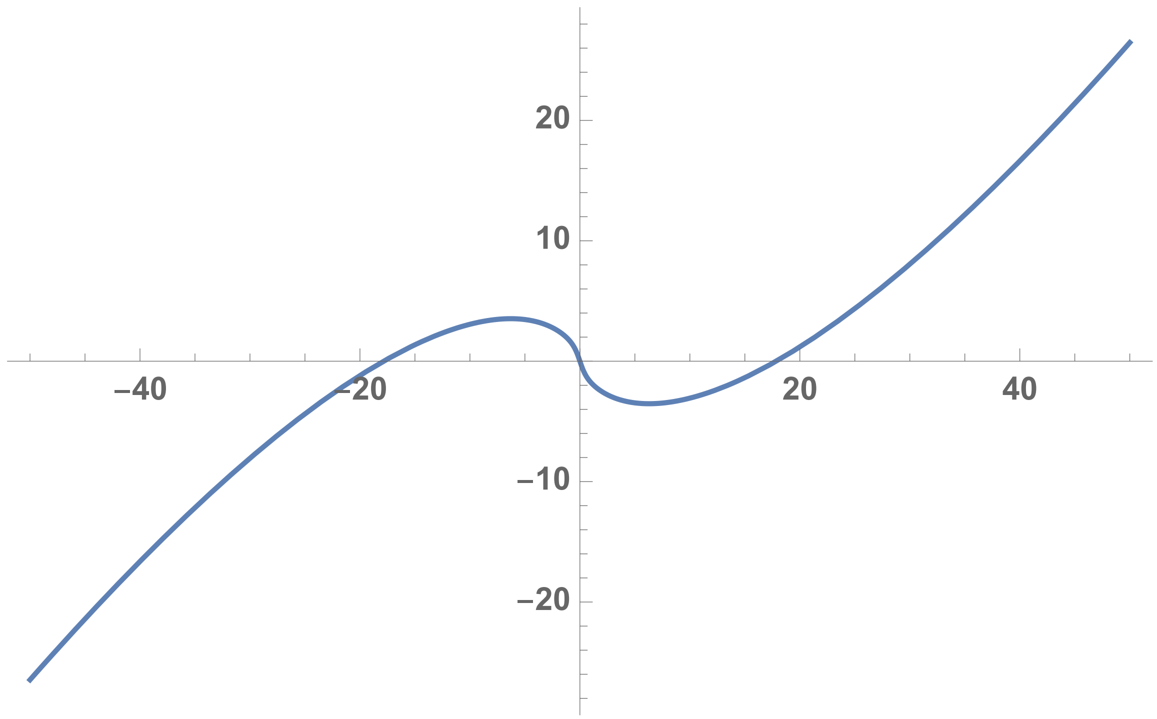
Archimedean place

u_∞ = ratio of archimedean local factors on the critical line : $z = 1/2 + is$

$$u(s) = \frac{\pi^{-z/2}\Gamma(z/2)}{\pi^{-(1-z)/2}\Gamma((1-z)/2)}$$

$\theta(s)$ = Riemann-Siegel angular function,

$$u(s) = e^{2i\theta(s)}.$$



20
10
−10
−20
−40
−20
20
40

Schwartz kernel

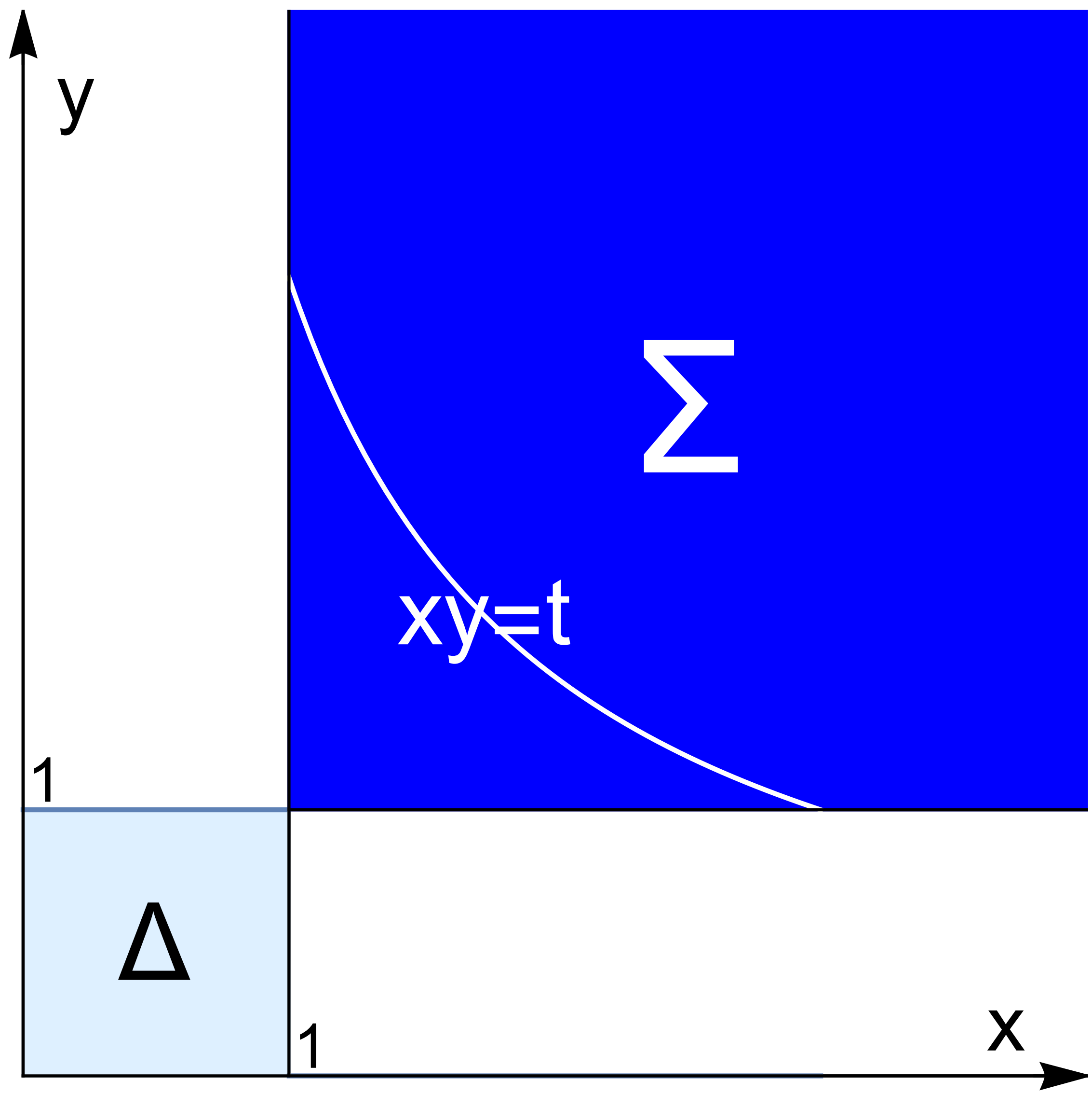
For $\vartheta_{\mathrm{m}}(\rho^{-1})(\frac{1}{2}u_\infty^* \eth u_\infty)^g$ the Schwartz kernel is

$$\ell_\rho(\nu,\mu) = 4\rho^{\frac{1}{2}}\mu^{\frac{1}{2}}\nu^{\frac{1}{2}}\int \cos(2\pi\rho\nu y)\cos(2\pi\mu y)(P(1/y) - P(\mu))dy$$

Distribution $W_{\mathbb{R}}$ represented by $\tau(\rho)$

$$\tau(\rho) = \mathrm{Tr}(\vartheta_{\mathrm{m}}(\rho^{-1})(\frac{1}{2}u_\infty^* \eth u_\infty)^g) =$$

$$4\rho^{\frac{1}{2}}\int_{x>0,y>0}\cos(2\pi\rho x y)\cos(2\pi x y)(P(1/y) - P(x))dydx$$


y
Σ
xy=t
1
Δ
1
x

Discrepancy $\delta(\rho)$

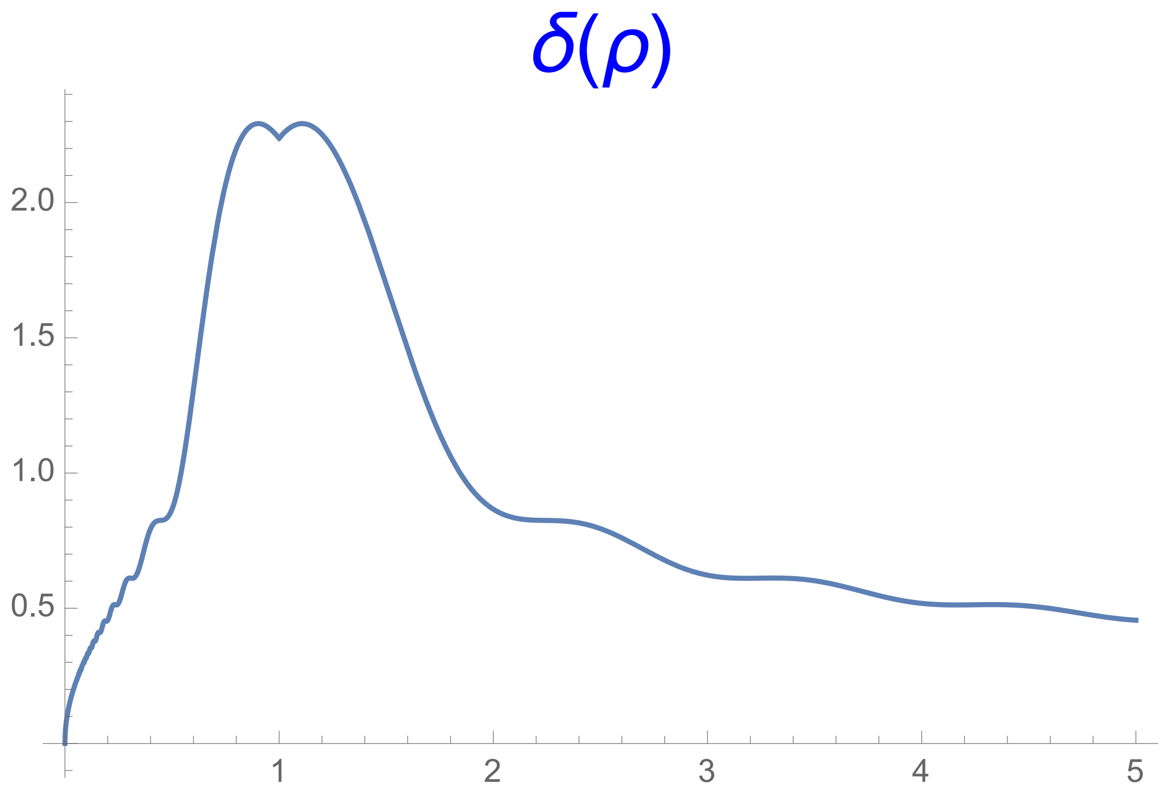
$$W_\infty := -W_{\mathbb{R}}$$

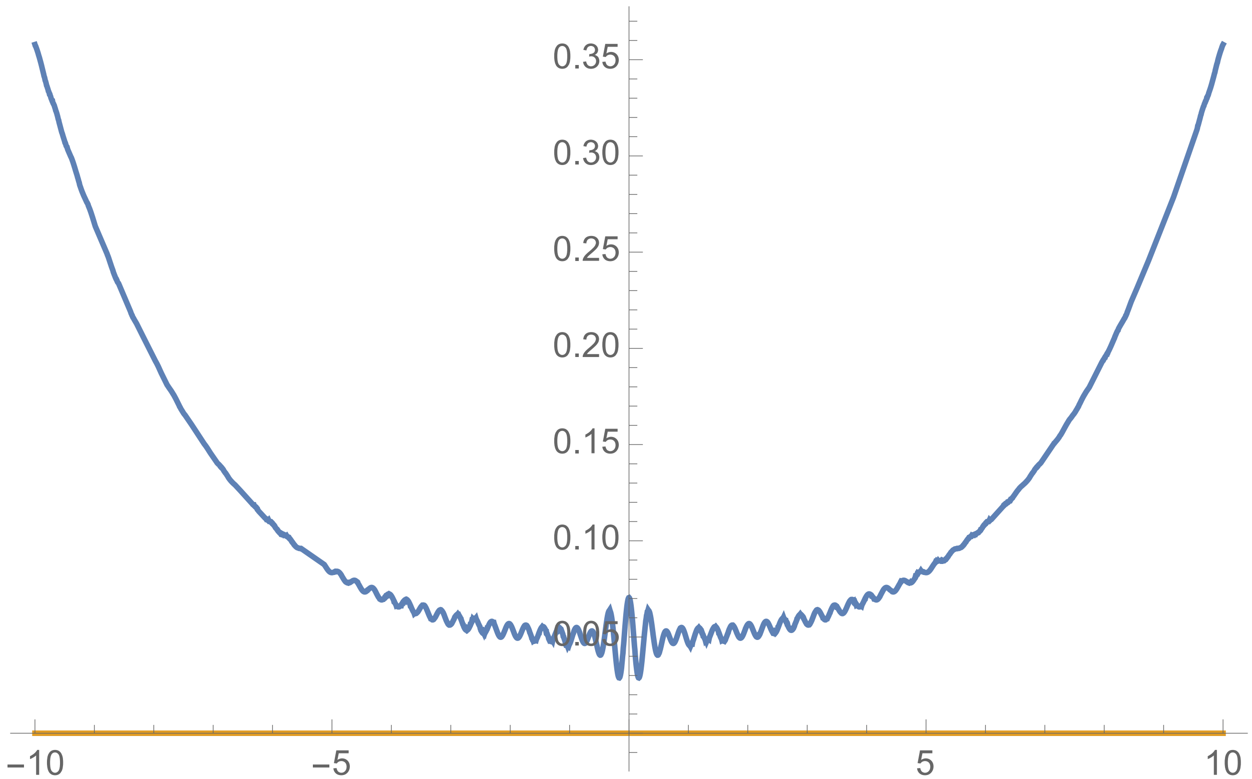
$$\delta(\rho) := \operatorname{tr}\left(\left(\vartheta_{\mathrm{m}}(\rho^{-1}) - P\vartheta_{\mathrm{m}}(\rho^{-1})P\right)\frac{1}{2}\,(u_\infty^* đu_\infty)^g\right)$$

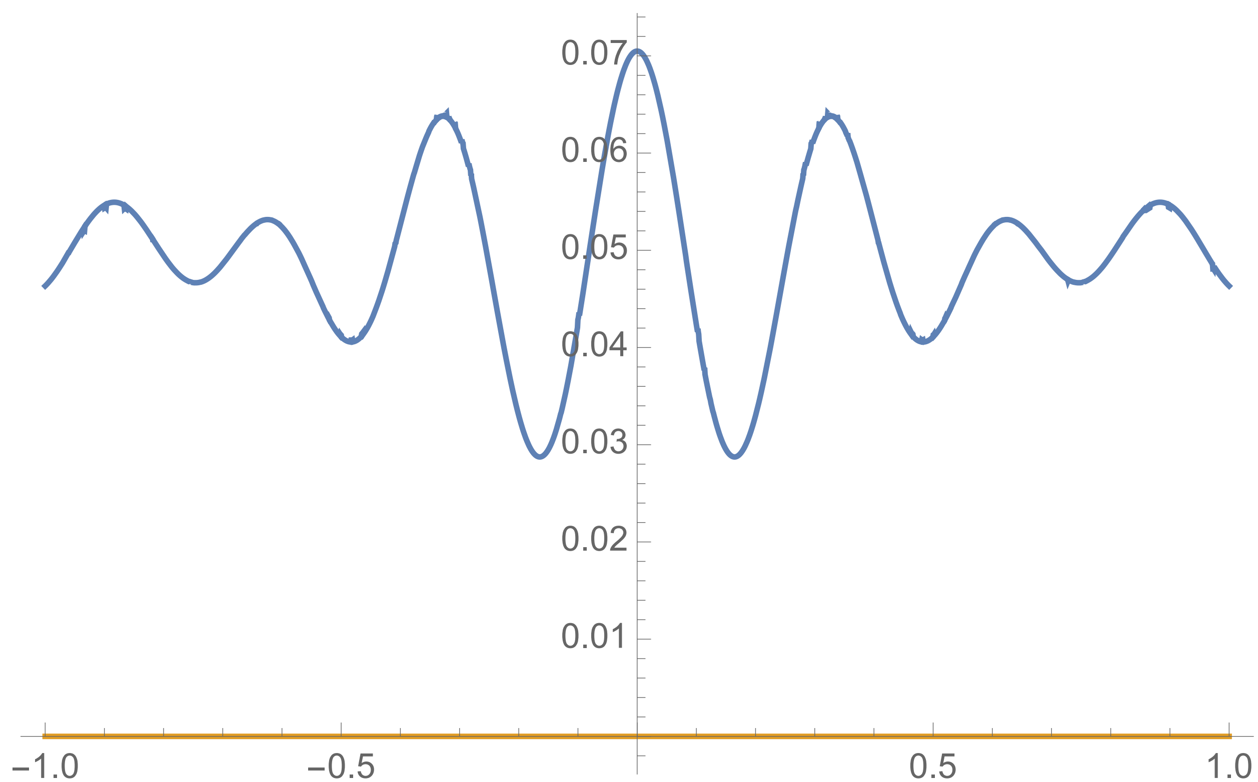
$$L(f) = D(f) + W_\infty(f),$$

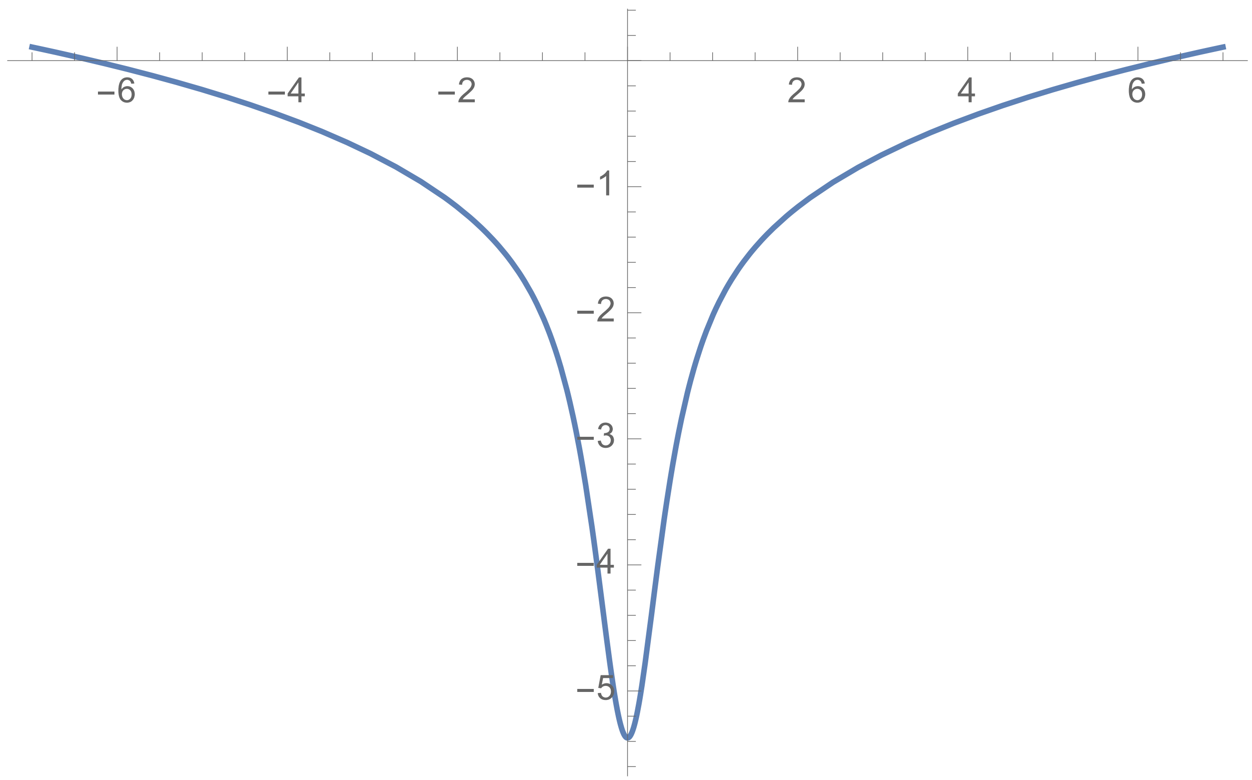
$$D(f) := \int f(\rho^{-1})\delta(\rho)d^*\rho$$

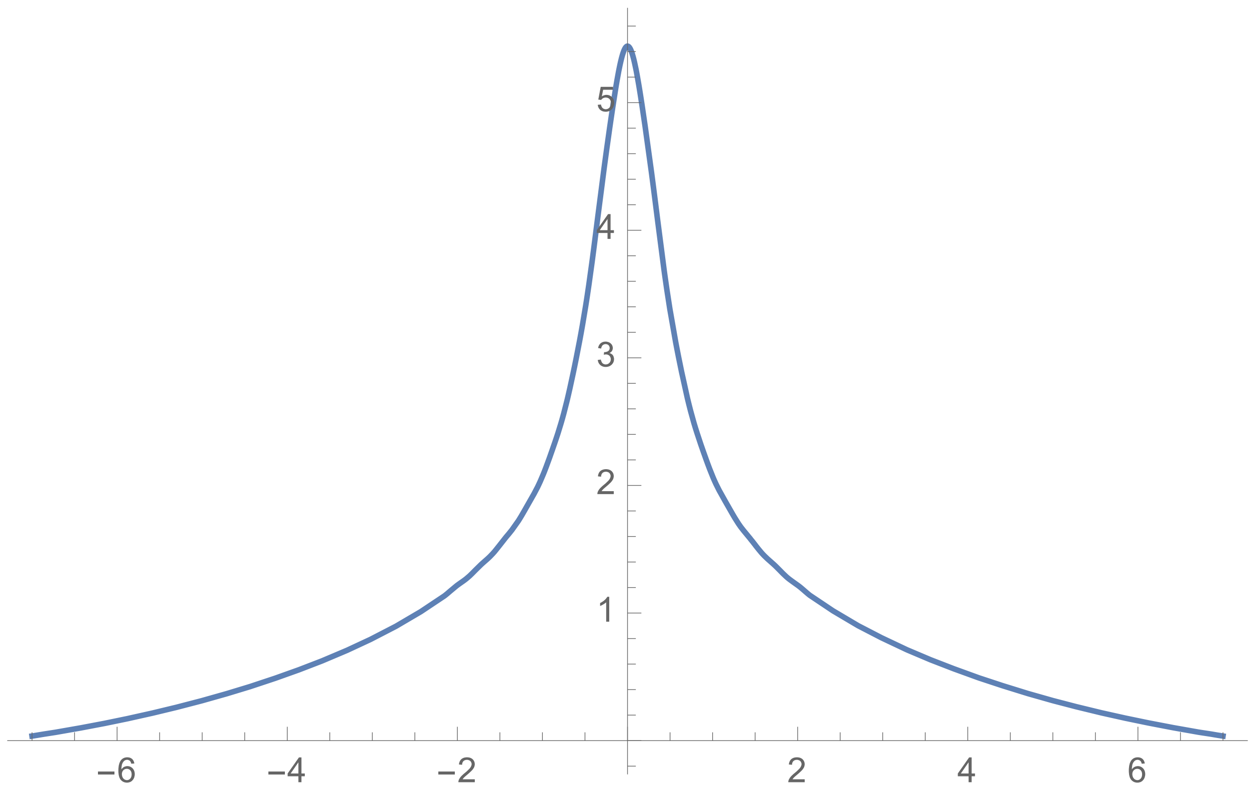
$$\rho > 1 \Rightarrow \delta(\rho) = 2\rho^{\frac{1}{2}}\left(\frac{\mathrm{Si}(2\pi(1+\rho))}{2\pi(1+\rho)} + \frac{\mathrm{Si}(2\pi(\rho-1))}{2\pi(\rho-1)}\right)$$


δ(ρ)
2.0
1.5
1.0
0.5
1
2
3
4
5


0.35
0.30
0.25
0.20
0.15
0.10
0.05
−10
−5
5
10


0.07
0.06
0.05
0.04
0.03
0.02
0.01
−1.0
−0.5
0.5
1.0


−6
−4
−2
2
4
6
−1
−2
−3
−4
−5


5
4
3
2
1
−6
−4
−2
2
4
6

$\delta(\rho)$ and quantum cell

$\mathcal{P}$ cutoff projection $\Lambda = 1$

$$\rho \geq 1 \Rightarrow \delta(\rho) =$$

$$= \mathbf{tr}\left(\vartheta_m(\rho^{-1})\widehat{\mathcal{P}}\,\mathcal{P}\right)$$

Moving little square Δ inside Σ

Cutoff projections, (Selecta 1998)

Representation of dihedral group

$S =$ **projection on Sonin's space**

Sonin space = kernel $(U_\infty)_{22}$

$S(1,1) \subset L^2(\mathbb{R})_{\mathrm{ev}}$ be Sonin's space of even functions, which, together with their Fourier transform, vanish identically in the interval $[-1,1]$.

The image $(\mathbb{F}_\mu \circ w)(S(1,1))$ of Sonin's space is the kernel of the operator $(1-\mathcal{P})U_\infty(1-\mathcal{P}) = (U_\infty)_{22}$.

Strong form of Weil positivity

Let $g \in C_c^\infty(\mathbb{R}_+^*)$ have support in the interval $[2^{-1/2}, 2^{1/2}]$

and Fourier transform vanishing at $\frac{i}{2}$ and 0. Then, with

$W_\infty := -W_\mathbb{R}$ one has

$$W_\infty(g * g^*) \geq \mathrm{Tr}(\vartheta(g)\, S\, \vartheta(g)^*).$$

1. Discretize the group $\mathbb{R}_+^*$ by approximating it with $q^{\mathbb{Z}}$, where $q \to 1^+$.
2. Identify the approximating operator K_q used in 1. as a Toeplitz matrix and compute numerically its eigenvalues.
3. Apply the general theory of Toeplitz matrices to rewrite K_q in canonical form.
4. Guess a formula for the operator K independently of q, by comparing its approximate behavior for different values of $q \to 1^+$.
5. Construct a finite rank operator T that provides a good approximation of K on $L^2(\sqrt{I}, d^*\rho)$ for $I = [\frac{1}{2}, 2]$.
6. Compute the spectrum of T and identify the single eigenvector which, after conditioning, makes the functional $E \circ Q$ negative.

Theory of Toeplitz matrices

T self-adjoint with simple largest eigenvalue $\lambda_{\max}$, then all zeros of polynomial with coeffs from eigenvector are of modulus 1 and

$$\lambda_{\max} \mathrm{Id} - T = \lambda_{\max} \sum d(j) e(j)$$

Quasi-Inner function

$U \in L^\infty(S^1)$ is quasi-inner

$$\Updownarrow$$

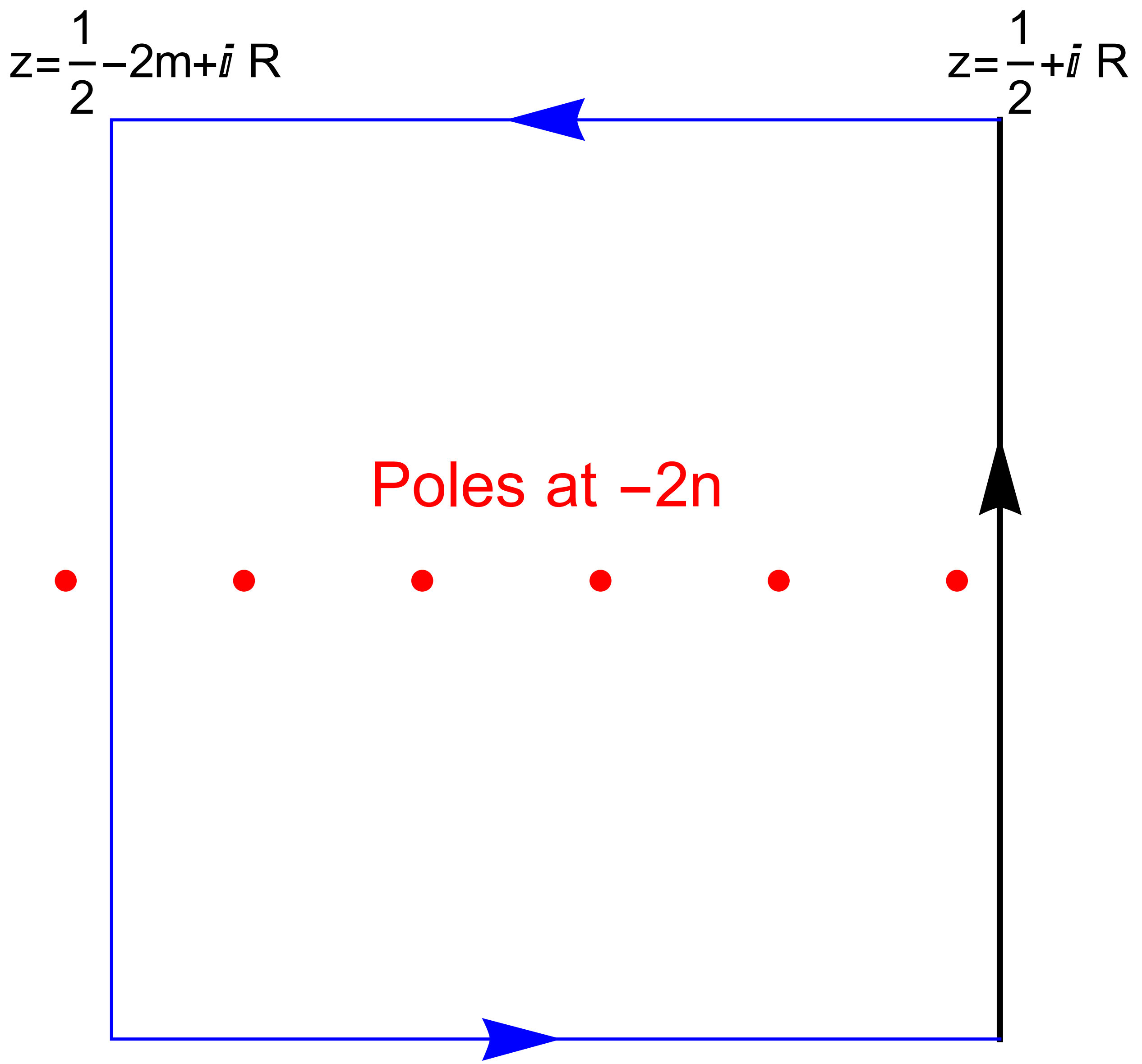
U = triangular unitary in the Calkin algebra for

$$L^2(S^1) = H^2(D) \oplus (H^2(D))^\perp$$

Theorem

$$\rho_\infty(z) := \frac{\pi^{-z/2}\Gamma(z/2)}{\pi^{-(1-z)/2}\Gamma((1-z)/2)}$$

is quasi-inner relative to the left half-plane $\mathbb{C}_- = \{z \in \mathbb{C} \mid \Re(z) \leq \frac{1}{2}\}$ with boundary the critical line $\Re(z) = \frac{1}{2}$.


$z=\frac{1}{2}-2m+i\,R$
$z=\frac{1}{2}+i\,R$
Poles at −2n

The off diagonal part $(1-\mathcal{P})\rho_\infty\mathcal{P}$ for the function ρ_∞ is the infinitesimal in $L^2(S^1)$

$$(1-\mathcal{P})\kappa\mathcal{P} = \sum_{\mathbb{N}}(-1)^{n+1}\frac{2\pi^{2n+\frac{1}{2}}}{(4n+1)\Gamma(n+1)\Gamma\left(n+\frac{1}{2}\right)}|\xi_n\rangle\langle\eta_n|$$

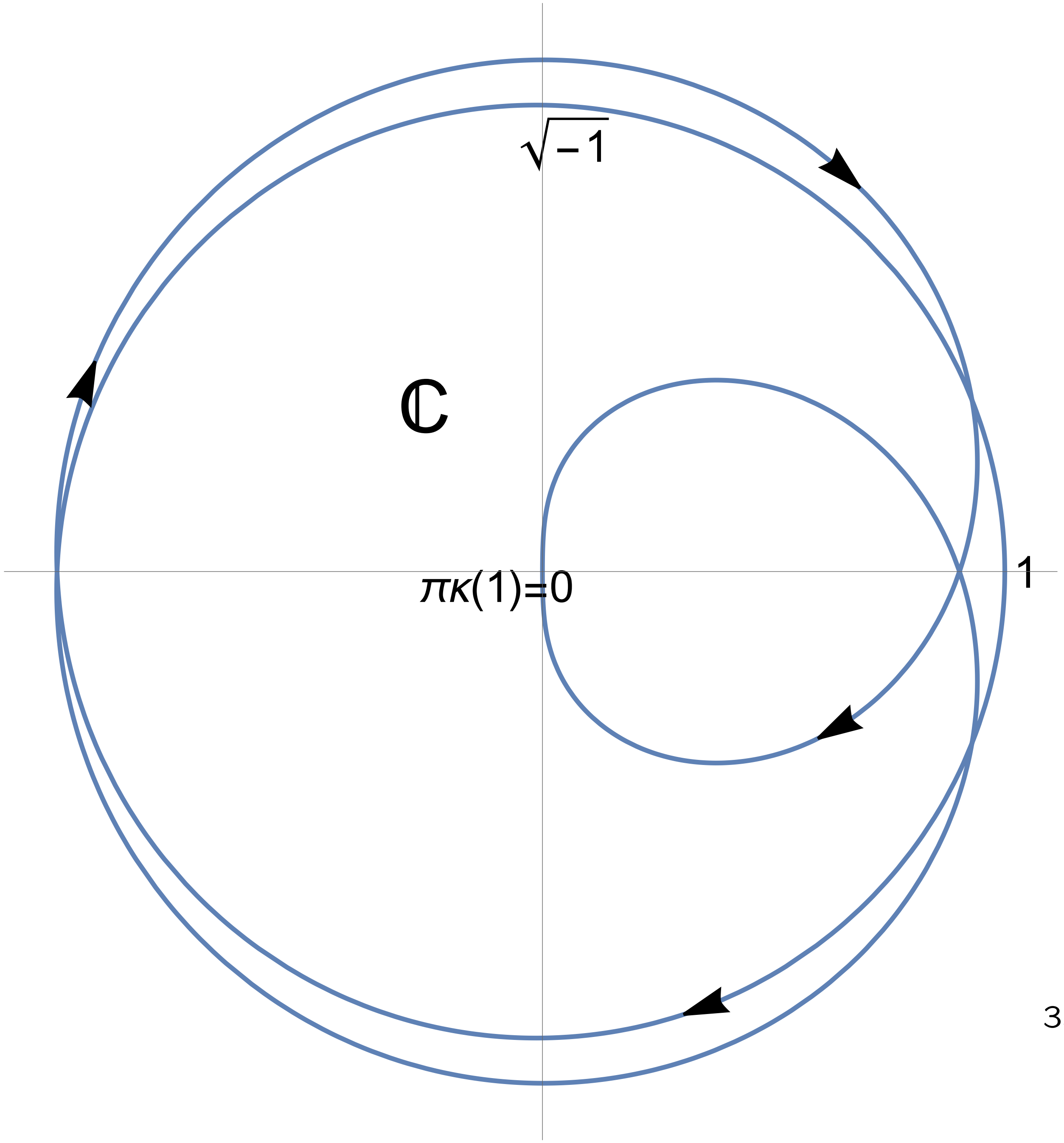
using the unit vectors $\xi_n := \xi_{x_n}/\|\xi_{x_n}\|$ and $\eta_n := \eta_{x_n}/\|\eta_{x_n}\|$, for $x_n := 1-\frac{4}{4n+3}$.

Theorem

The function

$$\kappa(v) := \rho_\infty \left(\frac{1}{2} + \frac{v+1}{v-1} \right)$$

belongs to $C^\infty(S^1) + H^\infty(\mathcal{U})$.



$\sqrt{-1}$
$\mathbb{C}$
$\pi\kappa(1)=0$
1

Semi-local case

Is the angle operator between the projections $\mathcal{P}$ and $\widehat{\mathcal{P}}$ compact ?

Is the analogue of Sonin's space

$$\{f \in L^2(X_S) \mid f(x) = 0 \;\&\; \mathbb{F}_\alpha f(x) = 0 \quad \forall x, |x| < 1\}$$

infinite dimensional ?

Ratio of local factors at p

$$\rho_p(z) := \frac{1 - p^{z-1}}{1 - p^{-z}}.$$

Theorem

The product of ratios of local factors

$$\rho_S = \rho_\infty \prod \rho_p$$

is a quasi-inner function relative to

$$\mathbb{C}_- = \{z \in \mathbb{C} \mid \Re(z) \leq \frac{1}{2}\}$$

Gauss multiplication

$$\prod_{k=0}^{m-1} \Gamma\left(z + \frac{k}{m}\right) = (2\pi)^{\frac{m-1}{2}} m^{\frac{1}{2}-mz} \Gamma(mz).$$

we use this to factor the archimedean ratio ρ_∞ into a product of m quasi-inner functions whose product with each ρ_v retains the property to be quasi-inner.

Theorem

Semi-local Sonin's space is simply the kernel of the diagonal part $u(S)_{22}$ for the quasi-inner function

$$u(S) = \prod_{v \in S} \rho_v$$

Inductive system of infinite dimensional spaces

The kernels of the $u(S)_{22}$ form an inductive system of infinite dimensional spaces.

Injective linear map $S(u(S)) \to S(u(S'))$

$$D(S,S') = \prod_{p \in S' \setminus S} (1 - p^{-z})$$

(i) Let $f \in S(\mathbb{R}, e_{\mathbb{R}})$ then the class of $\sigma_p \otimes f$ in $L^2(X_S)$ belongs to $S(X_S, \alpha)$ where $\alpha = (e_p, e_{\mathbb{R}})$.

(ii) One has

$$\mathbb{F}_\mu(w(\sigma_p \otimes f))(s) = \mathbb{F}_\mu(w(f))(s) \times \left(1 - p^{-\frac{1}{2} - is}\right)$$