Asymptotic equivalence of two strict deformation quantizations and applications to the classical limit

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Plan of the talk

- Introduction
- Basics on strict deformation quantization
- Examples
- Bulk-boundary asymptotic equivalence
- Applications

- K. Landsman, Valter Moretti, C.J.F. van de Ven, Strict Deformation Quantization Map on the state space of $M_k(\mathbb{C})$ and the Classical Limit of the Curie-Weiss model. Rev. Math. Phys. Vol. 32 (2020).

- Valter Moretti, C. J. F. van de Ven, Bulk-boundary asymptotic equivalence of two strict deformation quantizations. Letters. Math. Phys. (2020).

- C. J. F. van de Ven, The classical limit of mean-field theories. Arxiv: 2007.03390.

Introduction

- Study the transition between quantum and classical theories. Main topics we shall discuss:
  (1) Existence of the classical limits of quantum (spin) systems.
  (2) Spontaneous Symmetry Breaking (SSB).
- A useful instrument to study (1) and (2) is based on the concept of strict deformation quantization $\rightarrow$ a modern mathematical and rigorous theory to connect quantum with classical theories.[1]

[1] Only a few pairs of quantum and classical C^*-algebras are known to relate in this way.

• Ingredients: sequence of C^*-algebras $(A_\hbar)_{\hbar\in I}$ over locally compact Hausdorff space I, $A_0 = C_0(X)$ where X a smooth Poisson manifold (possibly with boundary).

• Consider class of elements $a := \{a_0, a_\hbar\}_\hbar$ that is closed w.r.t. pointwise sums, products, the adjoint, and such that

$$||a|| := \sup_{\hbar\in I}\{||a_\hbar||_\hbar\} < \infty, \tag{1}$$

$$||aa^*|| = ||a||^2. \tag{2}$$

• By construction the set

$$A = \left\{ a = \{a_0, a_\hbar\}_\hbar \;\middle|\; \text{all conditions above are satisfied} \right\}, \tag{3}$$

is a C^*- algebra with norm (1).

• A continuous bundle of C^*-algebras over I consists of a C^*- algebra A (constructed by (3)), a collection of C^*-algebras $(A_\hbar)_{\hbar \in I}$ and surjective homomorphisms $\phi_\hbar : A \to A_\hbar$, such that $A \ni a := \{a_0, a_\hbar\}_\hbar$ satisfies

$$\phi_\hbar(a) = a_\hbar. \tag{4}$$

• Moreover, we require that for any $f \in C_0(I)$ one has $\{f(\hbar)a_\hbar\}_\hbar \in A$.

• We furthermore demand the continuity property for the norm, in that for each $a \in A$ one has

$$I \ni \hbar \mapsto ||a_\hbar||_\hbar \in C_0(I), \tag{5}$$

• If all these conditions are satisfied, the continuous cross-sections are then maps $I \ni \hbar \mapsto a_\hbar \in A_\hbar$, i.e., elements of A.

Definition (Strict deformation quantization)

- Continuous bundle of C^*-algebras $(A_\hbar)_{\hbar\in I}$ over I with $A_0 = C_0(X)$;

- A dense Poisson subalgebra $\tilde{A}_0 \subset C^\infty(X) \subset A_0$

- Quantization maps $Q_\hbar : \tilde{A}_0 \to A_\hbar$ such that Q_0 is the inclusion map $\tilde{A}_0 \to A_0$, each $Q_\hbar$ is linear, and the next conditions $(1) - (4)$ hold:

Definition

1. $Q_\hbar(1_X) = 1_{A_\hbar}$.

2. $Q_\hbar(f^*) = Q_\hbar(f)^*$.

3. $0 \mapsto f$;
   $\hbar \mapsto Q_\hbar(f), \quad (\hbar > 0)$
defines a continuous section of the bundle.

4. For all $f, g \in \tilde{A}_0$ one has the Dirac-Groenewold-Rieffel condition:

$$\lim_{\hbar \to 0} || \frac{i}{\hbar}[Q_\hbar(f), Q_\hbar(g)] - Q_\hbar(\{f, g\})||_\hbar = 0.$$

- Consider

$$A_0 = C_0(\mathbb{R}^{2n}) \ (\hbar = 0);$$
$$A_\hbar = B_\infty(L^2(\mathbb{R}^n)) \ (\hbar > 0),$$

where $\mathbb{R}^{2n}$ is equipped with thet standard symplectic Poisson structure $\rightarrow$ fibers of a continuous bundle of C^*- algebras over $I = [0, 1]$.

- Quantization maps: for any $\hbar \in (0, 1]$ define

$$Q_\hbar : C_c(\mathbb{R}^{2n}) \rightarrow B_\infty(L^2(\mathbb{R}^n));$$
$$Q_\hbar(f) = \int_{\mathbb{R}^{2n}} \frac{d^n p d^n q}{(2\pi\hbar)^n} f(p, q) |\phi_\hbar^{(p,q)}\rangle\langle\phi_\hbar^{(p,q)}|,$$

where for each $\hbar \in I$ the operator $|\phi_\hbar^{(p,q)}\rangle\langle\phi_\hbar^{(p,q)}|$ is the projection onto the subspace spanned by the unit vector $\phi_\hbar^{(p,q)} \in L^2(\mathbb{R}^n)$, also called a Schrödinger coherent state.

- Consider

$$A'_0 = C(S^2),\ (1/N = 0);$$
$$A'_{1/N} = M_{N+1}(\mathbb{C}),\ (1/N > 0).$$

$\rightarrow$ fibers of a continuous bundle of C^*- algebras over $I = 1/\mathbb{N} \cup \{0\}$.

- Poisson structure: $\{f, g\}(x) = \sum_{a,b,c=1}^{3} \epsilon_{abc} x_c \frac{\partial f}{\partial x_a} \frac{\partial g}{\partial x_b}$ $(x \in S^2))$, with f, g restrictions of smooth functions to $S^2 \rightarrow$ dense subspace $\tilde{A}'_0 \subset A'_0$ made of polynomials in three real variables restricted to S^2.

- Quantizations maps: for any $1/N \in 1/\mathbb{N}$:

$$Q'_{1/N} : \tilde{A}'_0 \rightarrow M_{N+1}(\mathbb{C});$$
$$Q'_{1/N}(p) = \frac{N+1}{4\pi} \int_{S^2} d\mu(\Omega) p(\Omega) |\Omega_N\rangle\langle\Omega_N|.$$

$|\Omega_N\rangle\langle\Omega_N|$ is the projection onto the linear span of the vector Ω_N, called a spin coherent state.

Examples

Quantization of the algebraic state space of $M_2(\mathbb{C})$

• Consider

$$A_0 = C(S(M_2(\mathbb{C}))) \simeq C(B^3),\ (1/N = 0);$$
$$A_{1/N} = \bigotimes_{n=1}^{N} M_2(\mathbb{C}),\ (1/N > 0).$$

$\rightarrow$ fibers of a continuous bundle of C^*- algebras over $I = 1/\mathbb{N} \cup \{0\}$.

• Poisson structure on $S(M_2(\mathbb{C})) \simeq B^3$:
$\{f, g\}(x) = \sum_{a,b,c=1}^{3} \epsilon_{abc} x_c \frac{\partial f}{\partial x_a} \frac{\partial g}{\partial x_b}$ $(x \in B^3)$, with f, g restrictions of smooth functions to B^3.

• Quantizations maps are defined by (quasi)-symmetric sequences, i.e. macroscopic observables. These can start in any finite way, but their infinite tails consist of averaged observables, and therefore they asymptotically commute.

- Symmetrization operator $S_N : A_{1/N} \to A_{1/N}$, defined as the unique linear continuous extension of the following map on elementary tensors:

$$S_N(a_1 \otimes \cdots \otimes a_N) = \frac{1}{N!} \sum_{\sigma \in \mathcal{P}(N)} a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(N)}. \tag{6}$$

- For $N \geq M$ define a bounded operator $S_{M,N} : A_{1/M} \to A_{1/N}$, by linear and continuous extension of

$$S_{M,N}(b) = S_N(b \otimes \underbrace{I \otimes \cdots \otimes I}_{N-M \text{times}}), \quad b \in A_{1/M}. \tag{7}$$

• Sequences $A \ni a = (a_0, a_{1/N})_{N\in\mathbb{N}}$ are called symmetric if there exist $M \in \mathbb{N}$ and $a_{1/M} \in A_{1/M}$ such that

$$a_{1/N} = S_{M,N}(a_{1/M}) \text{ for all } N \geq M, \tag{8}$$

• They are called quasi-symmetric if $a_{1/N} = S_N(a_{1/N})$ if $N \in \mathbb{N}$, and for every $\epsilon > 0$, there is a symmetric sequence $(b_{1/N})_{N\in\mathbb{N}}$ as well as $M \in \mathbb{N}$ such that

$$\|a_{1/N} - b_{1/N}\| < \epsilon \text{ for all } N > M. \tag{9}$$

• It can be shown that the continuous cross-sections of the bundle with fibers $(A_0, A_{1/N})$ are precisely given by quasi-symmetric sequences which uniquely define this bundle (Landsman, 2017).

• Subspace $Z \subset \bigoplus_{M=0}^{\infty} M_2(\mathbb{C})^{\otimes M}$ made of symmetric tensor products $\rightarrow$ map $\chi : Z \rightarrow C(S(M_2(\mathbb{C})))$ defind by linear extension of the map

$$\chi(b_{j_1} \otimes_s \cdots \otimes_s b_{j_L})(\omega) = \omega^N(b_{j_1} \otimes_s \cdots \otimes_s b_{j_L}) = \omega(b_{j_1}) \cdots \omega(b_{j_1}),$$

where ib_1, ib_2, ib_3 form a basis of the Lie algebra of $SU(2)$, where $\omega \in S(M_2(\mathbb{C}))$ and $\omega(b_{j_i}) = x_{j_i}$ $(j_1, ..., j_L \in \{1, 2, 3\})$.

• χ is a well-defined linear injective map $\rightarrow$ $\chi(Z) \subset C(S(M_2(\mathbb{C})))$ is dense, and elements of $\chi(Z)$ are polynomials.

• Hence, each polynomial p of degree L uniquely corresponds to a polynomial of symmetric elementary tensors of the form $b_{j_1} \otimes_s \cdots \otimes_s b_{j_L}$.

• We define $\tilde{A}_0 := \chi(Z)$. For $p_L = \chi(b_{j_1} \otimes_s \cdots \otimes_s b_{j_L})$ the quantization maps $Q_{1/N} : \tilde{A}_0 \subset C(B^3) \to M_2(\mathbb{C})^{\otimes N}$ are defined as the unique continuous and linear extensions of the maps

$$Q_{1/N}(p_L) = \begin{cases} S_{L,N}(b_{j_1} \otimes_s \cdots \otimes_s b_{j_L}), & \text{if } N \geq L, \\ 0, & \text{if } N < L, \end{cases}$$
$$Q_{1/N}(1) = \underbrace{I_2 \otimes \cdots \otimes I_2}_{N \text{ times}}. \tag{10}$$

• Note that the quantization maps indeed define symmetric (hence macroscopic) observables.

Bulk-boundary asymptotic equivalence

- Existence of invariant $(N+1)$-dimensional symmetric subspace $\mathrm{Sym}^N(\mathbb{C}^2) \subset \bigotimes_{n=1}^N \mathbb{C}^2$ for operators $Q_{1/N}(p)$.

$\to Q_{1/N}(p)|_{\mathrm{Sym}^N(\mathbb{C}^2)} \in B(\mathrm{Sym}^N(\mathbb{C}^2)) \simeq M_{N+1}(\mathbb{C})$.

- This yields the following theorem relating both quantization maps

Theorem (Moretti, van de Ven, 2020)

For any polynomial $p \in \tilde{A}_0$ (the complex vector space of polynomials in three real variables on the closed unit ball B^3), one has

$$||Q'_{1/N}(p|_{S^2}) - Q_{1/N}(p)|_{Sym^N(\mathbb{C}^2)}||_N \to 0, \text{ as } N \to \infty, \tag{11}$$

the (operator) norm being the one on $B(Sym^N(\mathbb{C}^2))$.

• Consider collection of N two-level atoms corresponding to a spin chain of N sites described by a mean-field Hamiltonian H_N.

• Example: quantum Curie-Weiss spin Hamiltonian defined on $\mathcal{H}_N = \bigotimes_{n=1}^N \mathbb{C}^2$:

$$H_N \equiv H_N^{CW} = -\frac{J}{2N}\sum_{i,j=1}^N \sigma_3(i)\sigma_3(j) - B\sum_{i=1}^N \sigma_1(i), \qquad (12)$$

with B magnetic field and J a coupling constant .

Applications

mean-field theories

- H_N typically leaves the subspace $\mathrm{Sym}^N(\mathbb{C}^2) \subset \bigotimes_{n=1}^N \mathbb{C}^2$ invariant.

- $(H_N)_N$ defines a quasi-symmetric sequence $\rightarrow$ relation with SDQ of $S(M_2(\mathbb{C})) \simeq B^3$:

$$\lim_{N\to\infty} ||H_N - Q_{1/N}(h)||_N = 0, \tag{13}$$

for some polynomial $h \in C(B^3)$ (called the classical CW model).

- By the theorem $\lim_{N\to\infty} ||H_N|_{\mathrm{Sym}^N(\mathbb{C}^2)} - Q'_{1/N}(h|_{S^2})||_N = 0$, $\rightarrow$ the restricted mean-field spin system is represented by quantization of the Bloch sphere in the semiclassical limit $1/\hbar := N \to \infty$.

• Quantization theory $\to$ existence of classical limit of algebraic states with respect to quantum mechanical observables, i.e. does

$$\omega_0^{(0)}(f) := \lim_{\hbar \to 0} \omega_\hbar(Q_\hbar(f)), \quad (f \in C_0(X)); \tag{14}$$

exists as a state $\omega_0^{(0)}$ on $A_0 = C_0(X)$? Here, X plays the role of the classical phase space.

• Which states admit a classical limit? (Think e.g. of pure (vector) states, local Gibbs states).

• Characterizing the limiting states on $C_0(X)$.

• 1-dimensional Schrodinger operator $h_\hbar = -\hbar^2 \frac{d^2}{dx^2} + V(x)$, with V a double well potential, $h_\hbar \psi_\hbar^{(0)} = \lambda_\hbar^{(0)} \psi_\hbar^{(0)}$ where $\lambda_\hbar^{(0)}$ minimal.

• One can show that the Berezin quantization on $\mathbb{R}^2$ induces the existence of the classical limit on $C_0(\mathbb{R}^2)$:

$$\lim_{\hbar \to 0} \langle \psi_\hbar^{(0)}, Q_\hbar(f) \psi_\hbar^{(0)} \rangle = \frac{1}{2}(\omega_+^{(0)}(f) + \omega_-^{(0)}(f)). \tag{15}$$

where $\omega_\pm^{(0)}$ are Dirac measures localized in the minima of both wells (Lansdman 2017).

• We consider mean-field theories. Recall the CW model

$$H_N^{CW} = -\frac{J}{2N}\sum_{i,j=1}^{N}\sigma_3(i)\sigma_3(j) - B\sum_{i=1}^{N}\sigma_1(i), \tag{16}$$

• Existence of a unique (up to phase) ground state eigenvector $\Psi_N^{(0)}$. The vector state

$$\omega_{1/N}^{(0)}(\cdot) = \langle\Psi_N^{(0)}, \cdot\Psi_N^{(0)}\rangle, \tag{17}$$

converges w.r.t. macroscopic observables:

$$\omega_0^{(0)}(f) := \lim_{N\to\infty}\omega_{1/N}(Q_{1/N}(f)), \quad (f \in C(S(M_2(\mathbb{C})));\tag{18}$$

defines a state on the algebra $C(S(M_2(\mathbb{C})))$ (van de Ven, 2020).

Applications
Classical limit: Example 2

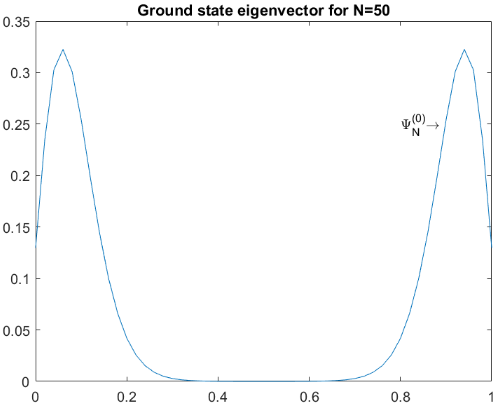
• Also in this case $\omega_0^{(0)}(f) = \frac{1}{2}(\omega_+^{(0)}(f) + \omega_-^{(0)}(f))$, where $\omega_\pm^{(0)}$ are Dirac measures corresponding to the minima of the classical CW model h^{CW},

$$h^{CW}(x, y, z) = -\frac{1}{2}(x^2 + Bz), \quad ((x, y, z) \in B^3). \tag{19}$$

• Note: parameter $1/N$ now plays the role of the usual semi-classical parameter $\hbar$.

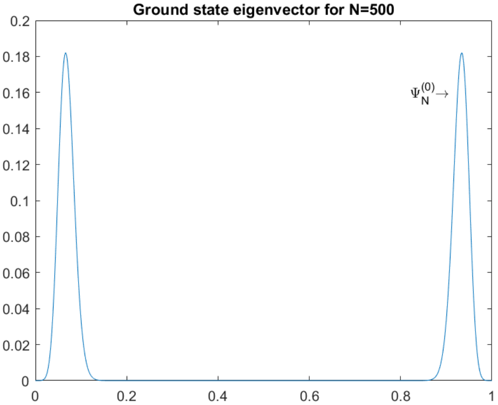
• Existence of spontaneous symmetry breaking (SSB) in the classcial limit: pure ground states are not invariant, whilst invariant ground states are not pure.

Applications
SSB: Example 2



• The (pure) ground state eigenvector $\Psi_N^{(0)}$ of the quantum Curie-Weiss model is invariant under $\mathbb{Z}_2$- reflexion symmetry for any N.

Applications
SSB: Example 2



• In the limit $N \to \infty$ the ground state eigenvector $\Psi_N^{(0)}$ 'decomposes' into two parts corresponding to the invariant (but not pure) state $\frac{1}{2}(\omega_+^{(0)}(f) + \omega_-^{(0)}(f))$.

Poisson bracket

$$\{f,g\}(x) = \sum_{a,b,c=1}^{n} C^{c}_{a,b} x_c \frac{\partial f(x)}{\partial x_a} \frac{\partial g(x)}{\partial x_b},$$

with structure constants coming from the Lie- algebra of $SU(k)$.

SSB

• The non-degenerate states $(\psi_N^{(0)}, \psi_N^{(1)})$ converge (in algebraic sense) to mixed classical states, i.e.,

$$\lim_{N\to\infty} \psi_N^{(0)} = \lim_{N\to\infty} \psi_N^{(1)} = \omega_0^{(0)},$$

where $\omega_0^{(0)} = \frac{1}{2}(\omega_0^+ + \omega_0^-)$.

• In contrast, the localized pure ground states

$$\psi_N^{\pm} = \frac{1}{\sqrt{2}}(\psi_N^{(0)} + \psi_N^{(1)}),$$

converge (in algebraic sense) to pure classical states, i.e.,

$$\lim_{N\to\infty} \psi_N^{\pm} = \omega_0^{\pm}.$$

Continuous bundle of C^*-algebras

Definition

Let I be a locally compact Hausdorff space. A continuous bundle of C^*-algebras over I consists of a C^*-algebra A, a collection of C^*-algebras $(A_\hbar)_{\hbar \in I}$ with norms $||\cdot||_\hbar$, and surjective homomorphisms $\varphi_\hbar : A \to A_\hbar$ for each $\hbar \in I$, such that

1. The function $\hbar \mapsto ||\varphi_\hbar(a)||_\hbar$ is in $C_0(I)$ for all $a \in A$.

2.The norm for any $a \in A$ is given by

$$||a|| = \sup_{\hbar \in I} ||\varphi_\hbar(a)||_\hbar. \tag{20}$$

3. For any $f \in C_0(I)$ and $a \in A$, there is an element $fa \in A$ such that for each $\hbar \in I$,

$$\varphi_\hbar(fa) = f(\hbar)\varphi_\hbar(a). \tag{21}$$

• A continuous (cross-) section of the bundle in question is a map $\hbar \mapsto a(\hbar) \in A_\hbar$, $(\hbar \in I)$, for which there exists an $a \in A$ such that $a(\hbar) = \varphi_\hbar(a)$ for each $\hbar \in I$.

Algebraic ground states and SSB

Definition

Let A be a C^*-algebra with time evolution, i.e., a continuous homomorphism $\alpha : \mathbb{R} \to \mathrm{Aut}(A)$. A *ground state* of (A, α) is a state ω on A such that:

1. ω is time independent, i.e., $\omega(\alpha_t(a)) = \omega(a)\ \forall a \in A\ \forall t \in \mathbb{R}$.
2. The generator h_ω of the ensuing continuous unitary representation

$$t \mapsto u_t = e^{ith_\omega} \tag{22}$$

of $\mathbb{R}$ on $\mathcal{H}_\omega$ has positive spectrum, i.e., $\sigma(h_\omega) \subset \mathbb{R}_+$, or equivalently $\langle \psi, h_\omega \psi \rangle \geq 0$ $(\psi \in D(h_\omega))$.

• The set of ground states forms a compact convex subset of $S(A)$, and we denote this set by $S_0(A)$. We moreover assume that pure ground states are pure states as well as ground states.

Definition

Suppose we have a C^*-algebra A, a time evolution α, a group G, and a homomorphism $\gamma : G \to \mathrm{Aut}(A)$, which is a symmetry of the dynamics α in that

$$\alpha_t \circ \gamma_g = \gamma_g \circ \alpha_t \quad (g \in G, t \in \mathbb{R}). \tag{23}$$

The G-symmetry is said to be spontaneously broken (at temperature $T = 0$) if

$$(\partial_e S_0(A))^G = \emptyset, \tag{24}$$

- Here $\mathscr{S}^G = \{\omega \in \mathscr{S} | \ \omega \circ \gamma_g = \omega \ \forall g \in G\}$, defined for any subset $\mathscr{S} \in S(A)$, is the set of G- invariant states in $\mathscr{S}$. (24) means that there are no G-invariant pure ground states. This means also that if spontaneous symmetry breaking occurs, then invariant ground states are not pure.