Exterior products of compact quantum metric spaces

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States on operator systems

Definition

An **operator system** $X \subseteq \mathbb{L}(H)$ *is a closed unital subspace of the bounded operators on a Hilbert space* H *such that* $x^* \in X$ *for all* $x \in X$.

Definition

A **state** *on an operator system* X *is a linear positive map* $\mu : X \to \mathbb{C}$ *with* $\mu(1_X) = 1$. *The state space of* X *is denoted by* $S(X)$ *and it becomes a* **compact Hausdorff space** *when equipped with the weak-$*$-topology.*

Lipschitz seminorms

Definition

A seminorm $L : X \to [0, \infty]$ *is a* **Lipschitz seminorm** *when*

1. $L(x^*) = L(x)$ *for all* $x \in X$*;*
2. *the subspace*
$$\mathrm{dom}(L) = \{x \in X : L(x) < \infty\} \subseteq X$$
is norm-dense;
3. *for each* $r \geq 0$*, the ball*
$$\mathbb{B}_r(0, L) = \{x \in X : L(x) \leq r\}$$
is norm-closed;
4. $L(1_X) = 0$*.*

Example: Unbounded selfadjoint operators

Proposition

Let $D : \mathrm{dom}(D) \to H$ *be an unbounded selfadjoint operator and define the seminorm* $L_D : \mathbb{L}(H) \to [0, \infty]$ *by*

$$L_D(T) := \sup \big\{ |\langle D\xi, T\eta \rangle - \langle T^* \xi, D\eta \rangle| \\ : \xi, \eta \in \mathrm{dom}(D), \|\xi\|_H, \|\eta\|_H \leq 1 \big\}$$

Then $X_D := \overline{\mathrm{dom}(L_D)}$ *is an operator system and* $L_D : X_D \to [0, \infty]$ *is a Lipschitz seminorm.*

Metrics on states

Definition

Suppose that $L : X \to [0, \infty]$ *is a Lipschitz seminorm. The* **Monge-Kantorovich metric** *is the metric* $\rho_L : S(X) \times S(X) \to [0, \infty]$ *defined by*

$$\rho_L(\mu, \nu) := \sup\{|\mu(x) - \nu(x)| : x \in \mathbb{B}_1(0, L)\}.$$

Definition

The pair (X, L) *is a* **compact quantum metric space** *when the metric* ρ_L **metrizes** *the weak-$*$-topology on the state space.*

Characterization of compact quantum metric spaces

Theorem (Rieffel, Pavlović)

The following conditions are equivalent:

1. *the pair* (X, L) *is a* **compact quantum metric space***;*
2. *the subset* $q\big(\mathbb{B}_1(0, L)\big) \subseteq X/\mathbb{C}$ *is* **relatively compact** *in the quotient norm (*$q : X \to X/\mathbb{C}$ *is the quotient map).*

Finite dimensional approximation (I)

Definition

Let $\varepsilon, C > 0$ *be constants. An* (ε, C)-**approximation** *of* (X, L) *is a pair* (ι, Φ) *consisting of unital linear maps* $\iota, \Phi : X \to Z$ *into an operator system* Z *such that*

1. $C^{-1} \cdot \|x\| \leq \|\iota(x)\| \leq C \cdot \|x\|$ *and* $\|\Phi(x)\| \leq C \cdot \|x\|$ *for all* $x \in X$;
2. *the image of* Φ *is a* **finite dimensional** *subspace of* Z;
3. $\|(\iota - \Phi)(x)\| \leq \varepsilon \cdot L(x)$ *for all* $x \in X$.

Finite dimensional approximation (II)

Theorem (K.)

The following conditions are equivalent:

1. *the pair* (X, L) *is a* **compact quantum metric space***;*
2. *there exists a constant* $C > 0$ *such that*
   - $\|q(x)\|_{X/\mathbb{C}} \leq C \cdot L(x)$ *for all* $x \in X$*;*
   - *there exists an* (ε, C)-**approximation** *of* (X, L) *for all* $\varepsilon > 0$.

Spectral metric spaces

Definition

An **even unital spectral triple** (A, H, D) *consists of*

- *a unital* C^**-subalgebra* $A \subseteq \mathbb{L}(H)_0$ *of the even bounded operators on a* $\mathbb{Z}/2\mathbb{Z}$*-graded separable Hilbert space* H*;*
- *an odd unbounded selfadjoint operator* $D : \mathrm{dom}(D) \to H$

such that

1. *the* **Lipschitz algebra** $\mathrm{dom}(L_D) \cap A \subseteq A$ *is norm-dense;*
2. *the resolvent* $(i + D)^{-1} : H \to H$ *is a* **compact operator**.

Definition

An even unital spectral triple is a **spectral metric space** *when the pair* (A, L_D) *is a compact quantum metric space.*

Exterior products of spectral triples

Definition/Theorem (Baaj and Julg)

Let (A, H, D) and (B, G, E) be even unital spectral triples. The **exterior product** *is the unital spectral triple $(A \otimes_{\min} B, H \otimes_2 G, D \times E)$, where the odd unbounded selfadjoint operator $D \times E$ is defined as the closure of the unbounded symmetric operator*

$$D \otimes 1 + \gamma \otimes E : \mathrm{dom}(D) \otimes \mathrm{dom}(E) \to H \otimes_2 G,$$

with $\gamma : H \to H$ being the grading operator on H.

Exterior products of spectral metric spaces

Question

Suppose that (A, H, D) and (B, G, E) are spectral metric spaces. Is it true that $(A \otimes_{\min} B, H \otimes_2 G, D \times E)$ is again a spectral metric space?

Lipschitz operator seminorms

Definition

Let $X \subseteq \mathbb{L}(H)$ be an operator system. An **operator seminorm** *L on X is a seminorm $L_n : M_n(X) \to [0, \infty]$ for every $n \in \mathbb{N}$ such that*

1. $L_n(v \cdot x \cdot w) \leq \|v\| \cdot L_n(x) \cdot \|w\|$ *for all* $x \in M_n(X)$, $v, w \in M_n(\mathbb{C})$;
2. $L_{n+m}(x \oplus y) = \max\{L_n(x), L_m(y)\}$ *for all* $x \in M_n(X)$, $y \in M_m(X)$.

Definition

An operator seminorm L on X is a **Lipschitz operator seminorm** *when $L_n : M_n(X) \to [0, \infty]$ is a Lipschitz seminorm for all $n \in \mathbb{N}$.*

Example: Unbounded selfadjoint operators

Proposition

Let $D : \mathrm{dom}(D) \to H$ *be an unbounded selfadjoint operator. For each* $n \in \mathbb{N}$ *we define the seminorm*

$$(L_D)_n := L^{D^{\oplus n}} : M_n\big(\mathbb{L}(H)\big) \to [0, \infty]$$

where $D^{\oplus n} : \mathrm{dom}(D^{\oplus n}) \to H^{\oplus n}$ *is the unbounded selfadjoint operator given by the n-fold direct sum of* D *with itself. Then* $X_D := \overline{\mathrm{dom}\big((L_D)_1\big)}$ *is an operator system and* L_D *is a Lipschitz operator seminorm.*

Lipschitz operator seminorms on tensor products

Proposition

Let L be a Lipschitz operator seminorm on an operator system $X \subseteq \mathbb{L}(H)$ and let $Y \subseteq \mathbb{L}(G)$ be an extra operator system. For each $n \in \mathbb{N}$ we define the seminorm $(L \otimes 1)_n : M_n(X \otimes_{\min} Y) \to [0, \infty]$ by

$$(L \otimes 1)_n(z) := \sup \big\{ L_{n \cdot \dim(\text{im}(Q))}((1 \otimes Q) z (1 \otimes Q)) \\ : Q \text{ is a finite rank projection on } G \big\}.$$

Then $L \otimes 1$ is a **Lipschitz operator seminorm** *on the* **minimal tensor product** *$X \otimes_{\min} Y \subseteq \mathbb{L}(H \otimes_2 G)$. The same holds true with the roles of X and Y reversed.*

Completely bounded finite dimensional approximation

Definition

Suppose that $L : X \to [0, \infty]$ *is a Lipschitz operator seminorm and let* $\varepsilon, C > 0$ *be constants. A* **completely bounded** (ε, C)**-approximation** *of* (X, L) *is a pair* (ι, Φ) *consisting of unital linears map* $\iota, \Phi : X \to Z$ *into an operator system* Z *such that*

1. $C^{-1} \cdot \|x\| \leq \|\iota(x)\| \leq C \cdot \|x\|$ *and* $\|\Phi(x)\| \leq C \cdot \|x\|$ *for all* $x \in M_n(X)$*;*
2. *the image of* Φ *is a* **finite dimensional** *subspace of* Z*;*
3. $\|(\iota - \Phi)(x)\| \leq \varepsilon \cdot L_n(x)$ *for all* $x \in M_n(X)$*.*

Completely bounded compact quantum metric spaces

Definition

Suppose that $L : X \to [0, \infty]$ *is a Lipschitz operator seminorm. We say that* (X, L) *is a* **completely bounded compact quantum metric space** *when there exists a constant* $C > 0$ *such that*

1. $\|q_n(x)\|_{M_n(X)/M_n(\mathbb{C})} \leq C \cdot L_n(x)$ *for all* $x \in M_n(X)$*;*
2. *there exists a completely bounded* (ε, C)*-approximation of* (X, L) *for all* $\varepsilon > 0$*.*

Example: Ergodic actions of compact groups

Proposition (Rieffel, K.)

Let G be a compact group equipped with a length function $\ell : G \to [0,\infty)$ and suppose that G acts ergodically on a unital C^*-algebra A. For each $n \in \mathbb{N}$ we define

$$L_n(x) := \sup\Big\{\frac{\|g(x) - x\|}{\ell(g)} : g \in G \setminus \{e\}\Big\}$$

for all $x \in M_n(A)$. Then (A, L) is a **completely bounded** *compact quantum metric space.*

Exterior products of compact quantum metric spaces

Theorem (K.)

Suppose that (X, L) and (Y, K) are completely bounded compact quantum metric spaces. Then $(X \otimes_{\text{min}} Y, \max\{L \otimes 1, 1 \otimes K\})$ is a completely bounded compact quantum metric space.

Application to spectral metric spaces

Definition

We say that en even unital spectral triple (A, H, D) is a **completely bounded spectral metric space** *when (A, L_D) is a completely bounded compact quantum metric space.*

Theorem (K.)

Suppose that (A, H, D) and (B, G, E) are **completely bounded** *spectral metric spaces. Then the exterior product $(A \otimes_{\min} B, H \otimes_2 G, D \times E)$ is a completely bounded spectral metric space.*