The Twist of the Free Fermion

joint work with Matthias Ludewig (Universität Regensburg)

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Introduction

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- Functorial field theory for the free fermion was constructed partly (Dai-Freed 1995, Mickelsson-Scott 2001, Müller-Szabo 2018).
- Due to the chiral anomaly, we want to formulate it as a twisted field theory.

Twisted Field Theory

A *twist* is, in our case, a functor

$$T : \mathrm{Bord}^{\mathrm{Spin}}_{\langle d-1,d\rangle} \to \mathrm{sAlg}$$

from the d-dimensional spin bordism category to the bicategory sAlg of $\mathbb{Z}_2$-graded algebras, i.e.

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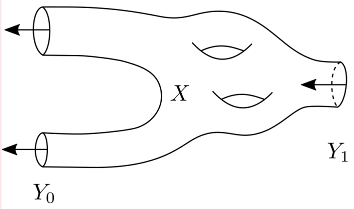
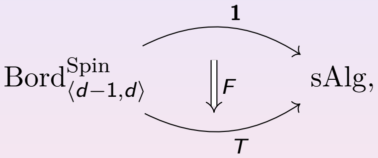
- a closed $(d-1)$-dim. spin manifold is mapped to an algebra $T(Y)$,
- a spin bordism $X : Y_1 \to Y_0$ is mapped to a $T(Y_0)$-$T(Y_1)$-bimodule $T(X)$.



Twisted Field Theory

A d-dimensional T-twisted field F is a natural transformation



where $\mathbf{1}$ is the trivial twist.

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- a closed $(d-1)$-dim. spin manifold Y is mapped to a $T(Y)$-module $F(Y)$,
- a spin bordism $X : Y_1 \to Y_0$ is mapped to a map $F(X) : T(X) \otimes_{T(Y_1)} F(Y_1) \to F(Y_0)$.

The Dirac Operator

Let X be a spin manifold and let Σ_X be the spinor bundle.
The Dirac operator acts on spinors $\varphi \in \Gamma(\Sigma_X)$ as

$$D_X\varphi = \sum_{i=1}^{n} \gamma(e_i)\nabla_{e_i}\varphi,$$

for a local orthonormal frame $(e_1, \ldots, e_n)$.

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Assumption

X is a d-dimensional spin manifold with product structure near the boundary and Y is a $(d-1)$-dimensional closed spin manifold.

Harmonic Spinors

Let X be a connected spin manifold with nonempty boundary ∂X. We set

$$\mathcal{H}_X = \ker(D_X), \quad L_X = \{\Phi_{|\partial X} \mid \Phi \in \mathcal{H}_X\}.$$

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Theorem

There is an orthogonal splitting

$$C^\infty(\Sigma_{\partial X}) = L_X \oplus \gamma(\nu) L_X,$$

where ν is the outward unit normal. In particular, L_X is a *Lagrangian* w.r.t. the symplectic structure given by $\langle \gamma(\nu) \cdot, \cdot \rangle$, where $\langle \cdot, \cdot \rangle$ is the bundle metric on $\Sigma_{\partial X}$.

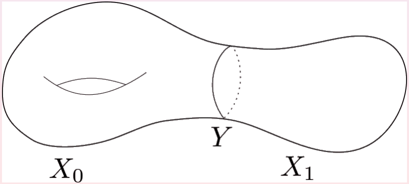
Gluing Theorem for Hamonic Spinors

Corollary

Let X be a closed d-dimensional spin manifold and $Y \subset X$ a closed hypersurface that divides X into two parts X_0, X_1. Then

$$L_{X_0} \cap L_{X_1} = K \quad \text{and} \quad L_{X_0} + L_{X_1} = (\gamma(\nu)K)^{\perp}$$

where $K = \{\Phi_{|Y} \mid \Phi \in \mathcal{H}_X\}$.



Real Structures and Clifford Algebras

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Definition (Associated Clifford Algebra)

$\mathrm{Cl}(W)$ is the Clifford algebra associated with the bilinear form $b(v, w) = \langle \bar{v}, w \rangle$, i.e.

$$v \cdot w + w \cdot v = b(v, w).$$

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$$\begin{aligned} L_{02} &= L_{01} \circ L_{12} \\ &= \{(w_0, w_2) \mid \exists w_1 \in W_1 : (w_0, w_1) \in L_{01}, (w_1, w_2) \in L_{12}\}. \end{aligned}$$

The composition of Lagrangians does not have to be a Lagrangian!

Composition of Lagrangians: An Example

Consider $l^2(\mathbb{Z})$ with the real structure $\overline{e}_n = e_{-n}$, where $e_n, n \in \mathbb{Z}$ is the n-the standard basis vector. For $\alpha \in \mathbb{R}$ consider the densely defined unbounded operator $Q_\alpha(e_n) = e^{\alpha n} e_n$.

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Then $L_\alpha = \text{graph}(Q_\alpha)$ defines a Lagrangian in $l^2(\mathbb{Z}) \oplus -l^2(\mathbb{Z})$. The composition of two such Lagrangians $L_{\alpha_1}, L_{\alpha_2}$ is the graph of $Q_{\alpha_1} Q_{\alpha_2}$.

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If α_1, α_2 have the same sign, then $L_{\alpha_1} \circ L_{\alpha_2}$ is a Lagrangian.
If α_1, α_2 have different signs, then $Q_{\alpha_1} Q_{\alpha_2}$ does not have a closed range. In particular, $L_{\alpha_1} \circ L_{\alpha_2}$ is not closed, hence it is not a Lagrangian.

Composition of Lagrangians

Proposition

Let W_0, W_1, W_2 be Hermitian vector spaces with real structures and let $L_{01} \subset W_0 \oplus -W_1$ and $L_{12} \subset W_1 \oplus -W_2$ be Lagrangians. If the map

$$\sigma : L_{01} \oplus L_{12} \to W_1,$$
$$((v_0, v_1), (w_1, w_2)) \mapsto v_1 - w_1$$

has closed range, then $L_{02} = L_{01} \circ L_{12}$ is a Lagrangian.

Lagrangian Bimodules

Let $L_{01} \subset W_0 \oplus -W_1$ be a Lagrangian. Then ΛL_{01} is a $\mathrm{Cl}(W_0 \oplus -W_1)$-left module via

$$v \cdot \xi := v \wedge \xi, \quad \overline{v} \cdot \xi := \iota(v)\xi,$$

for $v \in L_{01}$, $\xi \in \Lambda L_{01}$.

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ΛL_{01} is also a $\mathrm{Cl}(W_0)$-$\mathrm{Cl}(W_1)$-bimodule via

$$w_0 \cdot \xi := (w_0, 0) \cdot \xi, \quad \xi \cdot w_1 := (-1)^{|\xi|}(0, w_1) \cdot \xi,$$

for homogeneous elements $\xi \in \Lambda L_{01}$ and $w_i \in W_i$.

Gluing?

Let W_0, W_1, W_2 be Hermitian vector spaces with Lagrangians $L_{01} \subset W_0 \oplus -W_1$, $L_{12} \subset W_1 \oplus -W_2$ such that $L_{02} = L_{01} \circ L_{12}$ is a Langrangian.

Here, a well-defined gluing between ΛL_{01} and ΛL_{12} corresponds to a bimodule isomorphism $\Lambda L_{02} \to \Lambda L_{01} \otimes_{\mathrm{Cl}(W_1)} \Lambda L_{12}$ such that $\Omega_{02} \mapsto \Omega_{01} \otimes \Omega_{12}$, where, $\Omega_{01} := 1 \in \Lambda^0 L_{01}$ is the so-called *vacuum vector*.

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This is not possible if $K = \{w \mid (0, w) \in L_{01}, (w, 0) \in L_{12}\}$ is not empty, because for $v \in K$ with $|v| = 1$,

$$\begin{aligned}\Omega_{01} \otimes \Omega_{12} &= \Omega_{01} \cdot (\overline{v} \cdot v + v \cdot \overline{v}) \otimes \Omega_{12} \\ &= \Omega_{01} \cdot \overline{v} \otimes v \cdot \Omega_{12} + \Omega_{01} \cdot v \otimes \overline{v} \cdot \Omega_{12} \\ &= 0\end{aligned}$$

Gluing Theorem

Theorem

Let W_0, W_1, W_2 be Hermitian vector spaces with Lagrangians $L_{01} \subset W_0 \oplus -W_1$, $L_{12} \subset W_1 \oplus -W_2$.
If the map $\sigma : L_{01} \oplus L_{12} \to W_1$ has closed range and $\dim(K) = n < \infty$, then there exists a unique module isomorphism

$$\alpha : \Lambda L_{02} \otimes \Lambda^{\mathrm{top}} K \to \Lambda L_{01} \otimes_{\mathrm{Cl}(W_1)} \Lambda L_{12},$$

such that

$$\Omega_{02} \otimes u_1 \wedge \ldots \wedge u_n \mapsto \Omega_{01} \cdot u_1 \cdots u_n \otimes \Omega_{12}.$$

Coherence Theorem

Theorem

Let W_0, W_1, W_2, W_3 be Hermitian vector space with Lagrangians L_{01}, L_{12}, L_{23} Lagrangians such that for all $0 \leq i < j < k \leq 3$ we can apply the Gluing Theorem to obtain isomorphisms

$$\alpha_{ijk} : \Lambda L_{ik} \otimes \Lambda^{\mathrm{top}} K_{ijk} \to \Lambda L_{ij} \otimes_{\mathrm{Cl}(W_j)} \Lambda L_{jk},$$

where $K_{ijk} = \{w \in W_j \,|(0, w_j) \in L_{ij}, (w_j, 0) \in L_{jk}\}$. Then there exists a unique isomorphism

$$\rho : \Lambda^{\mathrm{top}}(K_{013} \oplus K_{123}) \to \Lambda^{\mathrm{top}}(K_{012} \oplus K_{023})$$

such that the following commutative diagram commutes:

Coherence Theorem

$$
\begin{array}{ccc}
\Lambda L_{03} \otimes \Lambda^{\mathrm{top}}(K_{013} \oplus K_{123}) & \xrightarrow{\mathrm{id}\otimes\rho} & \Lambda L_{03} \otimes \Lambda^{\mathrm{top}}(K_{012} \oplus K_{023}) \\
\downarrow \cong & & \downarrow \cong \\
(\Lambda L_{03} \otimes \Lambda^{\mathrm{top}} K_{013}) \otimes \Lambda^{\mathrm{top}} K_{123} & & (\Lambda L_{03} \otimes \Lambda^{\mathrm{top}} K_{023}) \otimes \Lambda^{\mathrm{top}} K_{012} \\
\alpha_{013}\otimes\mathrm{id} \downarrow & & \downarrow \alpha_{023}\otimes\mathrm{id} \\
(\Lambda L_{01} \otimes_{\mathrm{Cl}(W_1)} \Lambda L_{13}) \otimes \Lambda^{\mathrm{top}} K_{123} & & (\Lambda L_{02} \otimes_{\mathrm{Cl}(W_2)} \Lambda L_{23}) \otimes \Lambda^{\mathrm{top}} K_{012} \\
\cong \downarrow & & \downarrow \cong \\
\Lambda L_{01} \otimes_{\mathrm{Cl}(W_1)} (\Lambda L_{13} \otimes \Lambda^{\mathrm{top}} K_{123}) & & (\Lambda L_{02} \otimes \Lambda^{\mathrm{top}} K_{012}) \otimes_{\mathrm{Cl}(W_2)} \Lambda L_{23} \\
\mathrm{id}\otimes\alpha_{123} \downarrow & & \downarrow \alpha_{012}\otimes\mathrm{id} \\
\Lambda L_{01} \otimes_{\mathrm{Cl}(W_1)} (\Lambda L_{12} \otimes_{\mathrm{Cl}(W_2)} \Lambda L_{23}) & \xrightarrow[\cong]{} & (\Lambda L_{01} \otimes_{\mathrm{Cl}(W_1)} \Lambda L_{12}) \otimes_{\mathrm{Cl}(W_2)} \Lambda L_{23}
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Setting

Let Y be a $(d-1)$ dimensional spin manifold. We set $\Sigma_Y := \Sigma_{Y\times[0,1]|Y\times\{0\}}$.

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Let X be a d-dimensional spin manifold with boundary $\partial X = Y_0 \oplus -Y_1$ and set

$$\mathcal{H}_X = \ker(D_X), \quad L_X = \{\Phi_{|\partial X} \mid \Phi \in \mathcal{H}_X\}.$$

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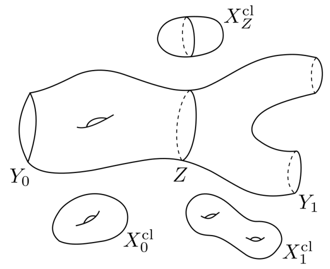
Lemma

The L^2-completion L_X^c is a Lagrangian of $\mathrm{Cl}(W_{Y_0} \oplus -W_{Y_1})$.

Splitting a Bordism

Lemma

Let X be a bordism of spin manifolds from Y_0 to Y_1 and let Z be a closed hypersurface splitting X into two parts X_0, X_1.
Then the corresponding Lagrangians L_{X_0}, L_{X_1} satisfy the assumption of the Gluing Theorem. In particular,
$K = \{\phi \mid (0,\phi) \in L_{X_0}, (\phi, 0) \in L_{X_1}\} \cong \mathcal{H}_{X_Z^{\mathrm{cl}}}$ has finite dimension.



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where X^{cl} denotes the closed part of X.

If X is a closed spin manifold, then

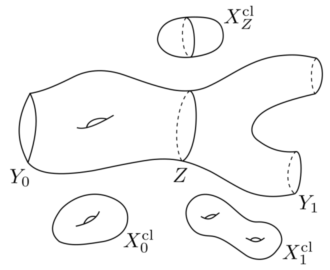
$$T(X) = \Lambda^{\mathrm{top}} \mathcal{H}_X^+ \otimes \overline{\Lambda^{\mathrm{top}} \mathcal{H}_X^-} = \overline{\mathrm{Det}}_X,$$

where Det_X is the determinant line of D_X.

Functoriality of the Twist

Let X be a bordism of spin manifolds from Y_0 to Y_1 and let Z be a closed hypersurface that splits X into two parts X_0, X_1.
The Gluing Theorem together with the restriction map to Z gives an isomorphism

$$\tau : T(X) \to T(X_0) \otimes_{T(Z)} T(X_1),$$



Coherence of the Twist

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Let X be a bordism between Y_3 and Y_0. Let Y_1, Y_2 be two closed hypersurfaces that split X into bordisms X_{ij} from Y_j to Y_i. Then,

$$\tau_{023}: T(X) \to T(X_{02}) \otimes_{T(Y_2)} T(X_{23})$$

$$\tau_{013}: T(X) \to T(X_{01}) \otimes_{T(Y_1)} T(X_{13})$$

$$\tau_{012} \otimes \mathrm{id}: T(X_{02}) \otimes_{T(Y_2)} T(X_{23}) \to T(X_{01}) \otimes_{T(Y_1)} T(X_{12}) \otimes_{T(Y_2)} T(X_{23})$$

$$\mathrm{id} \otimes \tau_{123}: T(X_{01}) \otimes_{T(Y_1)} T(X_{13}) \to T(X_{01}) \otimes_{T(Y_1)} T(X_{12}) \otimes_{T(Y_2)} T(X_{23})$$

Outlook

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- Extend the T-twisted field theory for the free fermion to the extended bordism category $\mathrm{Bord}^{\mathrm{Spin}}_{\langle d-1,d,d+1,d+2\rangle}$ including the Dai-Freed theory and Index theory for manifolds with corners.

The End

Thank you!